Week 7: Multiple Regression

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October 24, 26, 2016

[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Danny Hidalgo.

Where We've Been and Where We're Going...

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  - regression with two variables
  - omitted variables, multicollinearity, interactions

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  - then ... regression in social science
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

Why Matrices and Vectors?

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We are going to review the key points quite quickly just to refresh the basics.

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- Generic entry: a_{ik} where this is the entry in row i and column k

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$$\mathbf{X} = \begin{bmatrix} 1 & \text{exports}_1 & \text{age}_1 & \text{male}_1 \\ 1 & \text{exports}_2 & \text{age}_2 & \text{male}_2 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & \text{exports}_n & \text{age}_n & \text{male}_n \end{bmatrix}$$

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$$\mathbf{y} = \left[\begin{array}{c} y_1 \\ y_2 \\ \vdots \\ y_n \end{array} \right]$$

- Convention: we'll assume that a vector is column vector and vectors will be written with lowercase bold lettering (**b**)

Vector Examples

Vector Examples

One common vector that we will work with are individual variables, such as the dependent variable, which we will represent as **y**:

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

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$$\mathbf{Q} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \\ q_{31} & q_{32} \end{bmatrix} \quad \mathbf{Q}' = \begin{bmatrix} q_{11} & q_{21} & q_{31} \\ q_{12} & q_{22} & q_{32} \end{bmatrix}$$

If $\mathbf{A}$ is $j \times k$, then $\mathbf{A}'$ will be $k \times j$.

Transposing Vectors

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Transposing will turn a $k \times 1$ column vector into a $1 \times k$ row vector and vice versa:

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Addition and Subtraction

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$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

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&= \mathbf{C}
\end{aligned}
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$$\alpha \mathbf{A} = \alpha \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \alpha \times a_{11} & \alpha \times a_{12} \\ \alpha \times a_{21} & \alpha \times a_{22} \end{bmatrix}$$

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- We can write this as:

$$
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \beta_0 + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \beta_1 + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \beta_2 + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}
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- Outcome is a linear combination of the the $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ vectors

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- And the left-hand side here only uses scalars times vectors, which is easy!

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- We can also write this at the individual level, where $\mathbf{x}_i'$ is the ith row of $\mathbf{X}$:

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- Thus, each column of $\mathbf{C}$ is a linear combination of the columns of $\mathbf{A}$.

Special Multiplications

- The inner product of a two column vectors **a** and **b** (of equal dimension, $K \times 1$):

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- Special case of above: $\mathbf{a}'$ is a matrix with K columns and just 1 row, so the "columns" of $\mathbf{a}'$ are just scalars.

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$$\hat{\mathbf{u}}'\hat{\mathbf{u}} = \widehat{u}_1\widehat{u}_1 + \widehat{u}_2\widehat{u}_2 + \cdots + \widehat{u}_n\widehat{u}_n = \sum_{i=1}^{n} \widehat{u}_i^2$$

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- The identity matrix multiplied by any matrix returns the matrix: $\mathbf{AI} = \mathbf{A}$.

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- We've learned about matrix multiplication, but what about matrix "division"?

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- Need a matrix version of this: $\frac{1}{a}$.

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- Just like variation in X led us to be able to divide by the variance in simple OLS

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$$\mathbb{E}[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} \mathbb{E}[u_1|\mathbf{X}] \\ \mathbb{E}[u_2|\mathbf{X}] \\ \vdots \\ \mathbb{E}[u_n|\mathbf{X}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

OLS is Unbiased

Under matrix assumptions 1-4, OLS is unbiased for $\boldsymbol{\beta}$:

$$\mathbb{E}[\widehat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$$

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Unbiasedness of $\hat{\beta}$

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So, yes!

A Much Shorter Proof of Unbiasedness of $\hat{\beta}$

A shorter but perhaps less informative proof of unbiasedness,

A Much Shorter Proof of Unbiasedness of $\hat{\boldsymbol{\beta}}$

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$$\begin{aligned} E[\hat{\boldsymbol{\beta}}] &= E[\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}] \text{ (definition of the estimator)} \\ &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}\boldsymbol{\beta} \text{ (expectation of y)} \\ &= \boldsymbol{\beta} \end{aligned}$$

Variance-Covariance Matrix

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- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

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- This matrix is symmetric since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$

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$$\text{var}[\mathbf{u}] = \sigma_u^2 \mathbf{I}_n = \begin{bmatrix} \sigma_u^2 & 0 & 0 & \ldots & 0 \\ 0 & \sigma_u^2 & 0 & \ldots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \ldots & \sigma_u^2 \end{bmatrix}$$

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- In less matrix notation:
  - $\text{var}(u_i) = \sigma_u^2$ for all i (constant variance)
  - $\text{cov}(u_i, u_j) = 0$ for all $i \neq j$ (implied by iid)

Sampling Variance for OLS Estimates

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$$\text{var}[\widehat{\beta}] = \sigma_u^2(\mathbf{X}'\mathbf{X})^{-1}$$

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	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\cdots$	$\widehat{\beta}_K$
$\widehat{\beta}_0$	$\text{var}[\widehat{\beta}_0]$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\cdots$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_K]$
$\widehat{\beta}_1$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$\text{var}[\widehat{\beta}_1]$	$\text{cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$\cdots$	$\text{cov}[\widehat{\beta}_1, \widehat{\beta}_K]$
$\widehat{\beta}_2$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\text{cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$\text{var}[\widehat{\beta}_2]$	$\cdots$	$\text{cov}[\widehat{\beta}_2, \widehat{\beta}_K]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\widehat{\beta}_K$	$\text{cov}[\widehat{\beta}_0, \widehat{\beta}_K]$	$\text{cov}[\widehat{\beta}_K, \widehat{\beta}_1]$	$\text{cov}[\widehat{\beta}_K, \widehat{\beta}_2]$	$\cdots$	$\text{var}[\widehat{\beta}_K]$

Sampling Distribution for $\widehat{\beta}_j$

Under the first four assumptions,

$$\hat{\beta}_j | X \sim N\left(\beta_j, SE(\hat{\beta}_j)^2\right)$$

$$SE(\hat{\beta}_j)^2 = \frac{1}{1 - R_j^2} \frac{\sigma_u^2}{\sum_{i=1}^n (x_{ij} - \bar{x}_j)^2}$$

where R_j^2 is from the regression of x_j on all other explanatory variables.

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- Under assumption 1-5 in large samples:

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- Here, the estimated SEs come from:

$$\widehat{\text{var}}[\widehat{\boldsymbol{\beta}}] = \widehat{\sigma}_u^2(\mathbf{X}'\mathbf{X})^{-1}$$
$$\widehat{\sigma}_u^2 = \frac{\widehat{\mathbf{u}}'\widehat{\mathbf{u}}}{n-(k+1)}$$

Properties of the OLS Estimator: Summary

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Theorem

Under Assumptions 1–6, the $(k+1) \times 1$ vector of OLS estimators $\hat{\boldsymbol{\beta}}$, conditional on $\mathbf{X}$, follows a multivariate normal distribution with mean $\boldsymbol{\beta}$ and variance-covariance matrix $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$:

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- *Each element of* $\hat{\boldsymbol{\beta}}$ *(i.e.* $\hat{\beta}_0, ..., \hat{\beta}_{k+1}$*) is normally distributed, and* $\hat{\boldsymbol{\beta}}$ *is an unbiased estimator of* $\boldsymbol{\beta}$ *as* $E[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$

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- *With a large sample, $\hat{\boldsymbol{\beta}}$ approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of $\mathbf{u}$.*

Implications of the Variance-Covariance Matrix

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- In a practical sense, this means that our uncertainty about coefficients is correlated across variables.
- Let's go to the board and discuss!

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Let's work through an example.

Sample

Suppose that a week before the 2012 election, you contacted a sample of $n = 625$ potential Florida voters, randomly selected (with replacement) from the population of $N = 11,900,000$ on the public voters register, to ask whether they planned to vote.

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Suppose also,

- voters register is completely up to date
- all potential voters can be contacted, will respond honestly to your questions, and will not change their minds about voting

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Table: Sample

i	1	2	3	4	...	625	$\bar{y}_{625}$
y_i	1	1	0	1	...	0	.68

Sample versus Population

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After election day, we found that in fact 71% of the registered voters turned out to vote.

Table: Population

j	1	2	3	4	...	...	...	...	11.9 mil	$\bar{y}_{11.9mil}$
y_j	0	1	0	1	...	...	...	...	1	.71

Sampling Distribution

Table: Sampling Distribution of $\overline{Y}_{625}$

i	1		2		...	625		
s	J_1	Y_1	J_2	Y_2	...	J_{625}	Y_{625}	$\overline{Y}_{625}$
1	9562350	1	8763351	1	...	1294801	0	.68
2	5331704	0	4533839	1	...	3342359	1	.70
3	5129936	0	10981600	0	...	4096184	1	.75
4	803605	0	7036389	1	...	803605	0	.73
5	148567	0	3833847	1	...	4769869	1	.69
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
1 mil	4163458	0	8384613	1	...	377981	1	.74
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
f.	Be(.71)		Be(.71)			Be(.71)		$\frac{Bin(625,.71)}{625}$

The Sampling Distribution in R

```
# Resample the number of voters 1,000,000
# times and store these 1,000,000
# numbers in a vector.

sumY_vec <- rbinom(1000000, size=625, prob=.71)
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# Plot a histogram

hist(Ybar_vec)
```

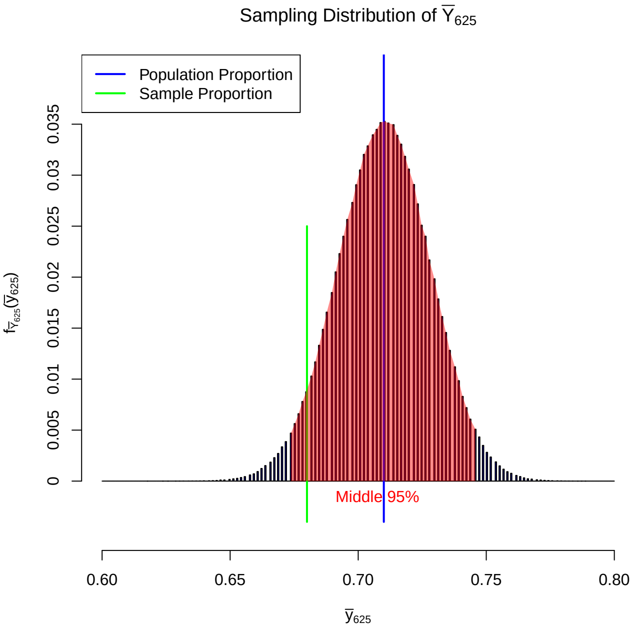
Sampling Distribution of $\overline{Y}_{625}$
Population Proportion
Sample Proportion
$f_{\overline{Y}_{625}}(\overline{y}_{625})$
0
0.005
0.01
0.015
0.02
0.025
0.03
0.035
Middle 95%
0.60
0.65
0.70
0.75
0.80
$\overline{y}_{625}$

Bootstrapping

At the time of our sample, we don't observe the population or population proportion (.71), so we cannot construct the sampling distribution.

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However, we can take repeated random samples with replacement of size 625 from the sample of size 625.

Table: Sample

i	1	2	3	4	$\dots$	625	$\bar{y}_{625}$
y_i	1	1	0	1	$\dots$	0	.68

Bootstrapping

At the time of our sample, we don't observe the population or population proportion (.71), so we cannot construct the sampling distribution.

However, we can take repeated random samples with replacement of size 625 from the sample of size 625.

Table: Sample

i	1	2	3	4	...	625	$\bar{y}_{625}$
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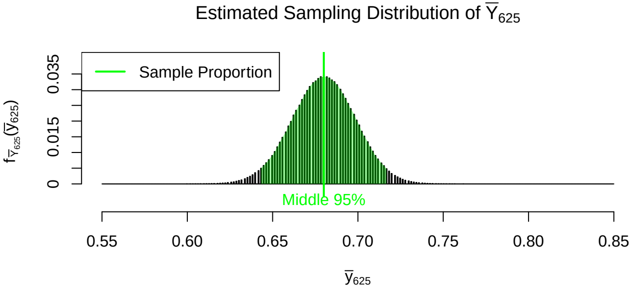
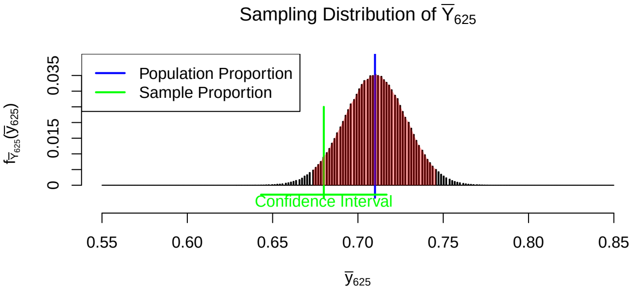
The is equivalent to replacing .71 with .68 in the R code.

```
sumY_vec <- rbinom(1000000, size=625, prob=.68)
```



Estimated Sampling Distribution of $\overline{Y}_{625}$
Sample Proportion
Middle 95%
0.55
0.60
0.65
0.70
0.75
0.80
0.85
$\overline{y}_{625}$
$f_{\overline{Y}_{625}}(\overline{y}_{625})$
0
0.015
0.035



Sampling Distribution of $\overline{Y}_{625}$
Population Proportion
Sample Proportion
Confidence Interval
0.55
0.60
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3) Repeat M times, form distribution of statistics
4) Calculate confidence interval by identifying $\alpha/2$ and $1-\alpha/2$ value of statistic. (percentile method)

Confidence Intervals, via Bootstrap

Suppose we draw 20 realizations of

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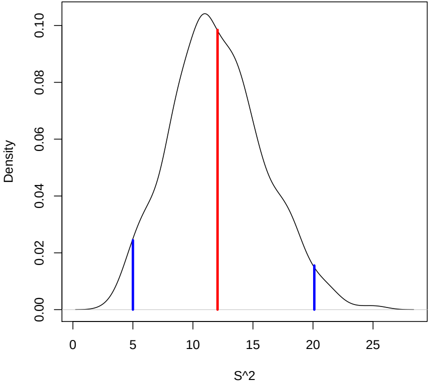

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  - Estimates are constructed by the plug-in principle, which says that the parameter $\theta = t(F)$ is estimated by $\hat{\theta} = t(\hat{F})$. (i.e. we plug in the ECDF for the CDF)
  - Why does this work? Sampling distribution entirely determined by the CDF and n, WLLN says the ECDF will look more and more like the CDF as n gets large.

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Fox Chapter 21 has a nice section on the bootstrap, Aronow and Miller (2016) covers the theory well.

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- We will just preview this stuff now, but I'm happy to answer questions for those who want to engage it more.

Gradient

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Definition (Gradient)

We can define the column vector of partial derivatives

$$\frac{\partial v(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \partial v/\partial u_1 \\ \partial v/\partial u_2 \\ \vdots \\ \partial v/\partial u_n \end{bmatrix}$$

This vector of partial derivatives is called the gradient.

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Theorem (differentiation of linear functions)

Given a linear function $v(\mathbf{u}) = \mathbf{c}'\mathbf{u}$ *of an* $(n \times 1)$ *vector* $\mathbf{u}$, *the derivative of* $v(\mathbf{u})$ *w.r.t.* $\mathbf{u}$ *is given by*

$$\frac{\partial v}{\partial \mathbf{u}} = \mathbf{c}$$

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Hence,

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Definition (Hessian)

The $(k+1) \times (k+1)$ matrix of second-order partial derivatives of $v = f(\mathbf{u})$ is called the Hessian matrix and denoted

$$\frac{\partial v^2}{\partial \mathbf{u} \partial \mathbf{u}'} = \begin{bmatrix} \frac{\partial v^2}{\partial u_1 \partial u_1} & \cdots & \frac{\partial v^2}{\partial u_1 \partial u_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial v^2}{\partial u_{k+1} \partial u_1} & \cdots & \frac{\partial v^2}{\partial u_{k+1} \partial u_{k+1}} \end{bmatrix}$$

Hessian

Suppose v is a scalar-valued function $v = f(\mathbf{u})$ of a $(k+1) \times 1$ column vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_{k+1} \end{bmatrix}'$

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The above rules are used to derive the optimal estimators in the appendix slides.

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- We showed how to estimate the coefficients and get the variance covariance matrix
- We discussed the bootstrap as an alternative strategy for estimating the sampling distribution
- Appendix contains numerous additional topics worth knowing:
  - Systems of Equations
  - Details on the variance/covariance interpretation of estimator
  - Derivation for the estimator
  - Proof of consistency

Fun With Weights

Aronow, Peter M., and Cyrus Samii. "Does Regression Produce Representative Estimates of Causal Effects?." *American Journal of Political Science* (2015).[2]

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- Imagine we care about the possibly heterogeneous causal effect of a treatment D and we control for some covariates X?
- We can express the regression as a weighting over individual observation treatment effects where the weight depends only on X.
- Useful technology for understanding what our models are identifying off of by showing us our effective sample.

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How this works

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$$\hat{\beta} \xrightarrow{p} \frac{E[w_i \tau_i]}{E[w_i]} \text{ where } w_i = (D_i - E[D_i|X])^2 \, ,$$

so that $\hat{\beta}$ converges to a reweighted causal effect. As $E[w_i|X_i] = \text{Var}[D_i|X_i]$, we obtain an average causal effect reweighted by conditional variance of the treatment.

Estimation

A simple, consistent plug-in estimator of w_i is available: $\hat{w}_i = \tilde{D}_i^2$ where $\tilde{D}_i$ is the residualized treatment. (the proof is connected to the partialing out strategy we showed last week)

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Easily implemented in R:

```
wts <- (d - predict(lm(d~x)))^2
```

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Application

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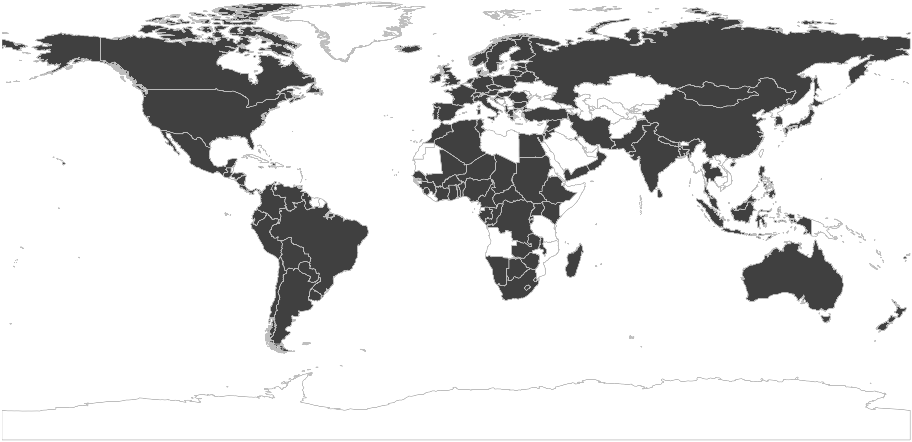
The nominal sample: 114 countries from 1970 to 1997.

Jensen estimates that a 1 unit increase in polity score corresponds to a 0.020 increase in net FDI inflows as a percentage of GDP ($p < 0.001$).

Nominal and Effective Samples



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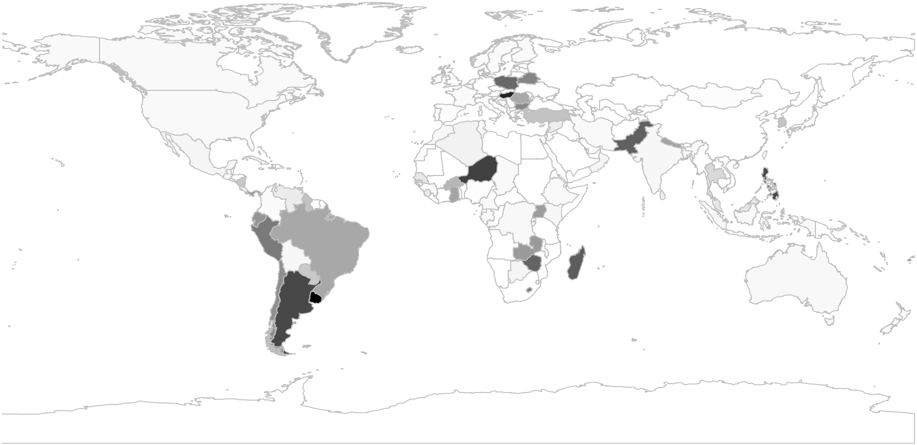

Nominal and Effective Samples

Over 50% of the weight goes to just 12 (out of 114) countries.

Broader Implications

When causal effects are heterogeneous, we can draw a distinction between "internally valid" and "externally valid" estimates of an Average Causal Effect (ACE). (See, e.g., Cook and Campbell 1979)

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  - randomized (lab, field, survey) experiments, instrumental variables, regression discontinuity designs, other natural experiments
- "Externally valid": perhaps unreliable estimates of ACEs, but for the population of interest
  - large-N analyses, representative surveys

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- When a treatment is "as-if" randomly assigned conditional on covariates, regression distorts the sample by implicitly applying weights.
- The *effective sample* (upon which causal effects are estimated) may have radically different properties than the nominal sample.
- When there is an underlying natural experiment in the data, a properly specified regression model may reproduce the internally valid estimate associated with the natural experiment.

Solving Systems of Equations Using Matrices

Matrices are very useful to solve linear systems of equations, such as the first order conditions for our least squares estimates.

Here is an example with three equations and three unknowns:

$$\begin{aligned} x + 2y + z &= 3 \\ 3x - y - 3z &= -1 \\ 2x + 3y + z &= 4 \end{aligned}$$

How would one go about solving this?
There are various techniques, including substitution, and multiplying equations by constants and adding them to get single variables to cancel.

Solving Systems of Equations Using Matrices

An easier way is to use matrix algebra. Note that the system of equations

$$\begin{aligned} x + 2y + z &= 3 \\ 3x - y - 3z &= -1 \\ 2x + 3y + z &= 4 \end{aligned}$$

can be written as follows:

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 4 \end{bmatrix} \iff \mathbf{Au} = \mathbf{b}$$

How do we solve this for $\mathbf{u} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$? Let's look again at the scalar case first.

Solving Equations with Inverses (scalar case)

Let's go back to the scalar world of 8th grade algebra. How would you solve the following for u?

$$au = b$$

We multiply both sides of by the reciprocal $1/a$ (the inverse of a) and get:

$$\frac{1}{a} a\, u = \frac{1}{a} b$$

$$u = \frac{b}{a}$$

(Note that this technique only works if $a \neq 0$. If $a = 0$, then there are either an infinite number of solutions for u (when $b = 0$), or no solutions for u (when $b \neq 0$).)

So to solve our multiple equation problem in the matrix case we need a matrix equivalent of the inverse. This equivalent is the inverse matrix. The inverse of $\mathbf{A}$ is written as $\mathbf{A}^{-1}$.

Inverse of a Matrix

The inverse $\mathbf{A}^{-1}$ of $\mathbf{A}$ has the property that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$ where $\mathbf{I}$ is the identity matrix.

- The inverse $\mathbf{A}^{-1}$ exists only if $\mathbf{A}$ is invertible or nonsingular (more on this soon)
- The inverse is unique if it exists and then the linear system has a unique solution.
- There are various methods for finding/computing the inverse of a matrix

The inverse matrix allows us to solve linear systems of equations.

$$\begin{aligned} \mathbf{A}\mathbf{u} &= \mathbf{b} \\ \mathbf{A}^{-1}\mathbf{A}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{I}\mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \\ \mathbf{u} &= \mathbf{A}^{-1}\mathbf{b} \end{aligned}$$

Given $\mathbf{A}$ we find that $\mathbf{A^{-1}}$ is:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & -1 & -3 \\ 2 & 3 & 1 \end{bmatrix} ; \mathbf{A^{-1}} = \begin{bmatrix} 8 & 1 & -5 \\ -9 & -1 & 6 \\ 11 & 1 & -7 \end{bmatrix}$$

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So the solution vector is $x = 3$, $y = -2$, and $z = 4$. Verifying:

$$\begin{aligned} x + 2y + z &= 3 + 2 \cdot -2 + 4 &= 3 \\ 3x - y - 3z &= 3 \cdot 3 - -2 - 3 \cdot 4 &= -1 \\ 2x + 3y + z &= 2 \cdot 3 + 3 \cdot -2 + 4 &= 4 \end{aligned}$$

Computationally, this method is very convenient. We "just" compute the inverse, and perform a single matrix multiplication.

Singularity of a Matrix

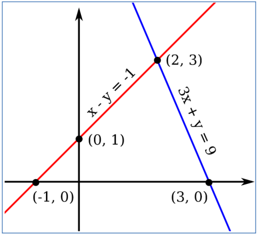
If the inverse of **A** exists, then the linear system has a unique (non-trivial) solution. If it exists, we say that **A** is nonsingular or invertible (these statements are equivalent).

A must be square to be invertible, but not all square matrices are invertible. More precisely, a square matrix **A** is invertible iff its column vectors (or equivalently its row vectors) are linearly independent.

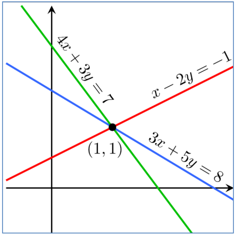
The column rank of a matrix **A** is the largest number of linearly independent columns of **A**. If the rank of **A** equals the number of columns of **A**, then we say that **A** has full column rank. This implies that all its column vectors are linearly independent.

If a column of **A** is a linear combination of the other columns, there are either no solutions to the system of equations or infinitely many solutions to the system of equations. The system is said to be underdetermined.

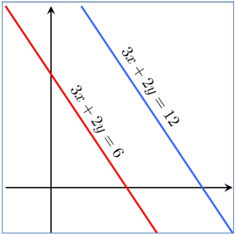
Geometric Example in 2D



Unique Solution



Redundant Equation



No Solution

$$A = \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 4 & 3 \\ 1 & -2 \\ 3 & 5 \end{bmatrix} \quad A = \begin{bmatrix} 3 & 2 \\ 3 & 2 \end{bmatrix}$$

Why do we care about invertibility?

We have seen that OLS regression is defined by a system of linear equations

$$\hat{\mathbf{y}} = \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \\ \vdots \\ \hat{y}_n \end{bmatrix} = \mathbf{X}\hat{\boldsymbol{\beta}} = \begin{bmatrix} 1\hat{\beta}_0 + x_{11}\hat{\beta}_1 + x_{12}\hat{\beta}_2 + \cdots + x_{1k}\hat{\beta}_k \\ 1\hat{\beta}_0 + x_{21}\hat{\beta}_1 + x_{22}\hat{\beta}_2 + \cdots + x_{2k}\hat{\beta}_k \\ \vdots \\ 1\hat{\beta}_0 + x_{n1}\hat{\beta}_1 + x_{n2}\hat{\beta}_2 + \cdots + x_{nk}\hat{\beta}_k \end{bmatrix}$$

with our data matrix

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & x_{12} & \dots & x_{1k} \\ 1 & x_{21} & x_{22} & \dots & x_{2k} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n1} & x_{n2} & \dots & x_{nk} \end{bmatrix}$$

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We have also learned that $\hat{\boldsymbol{\beta}}$ is obtained by solving normal equations, a linear system of equations.

It turns out that to solve for $\hat{\boldsymbol{\beta}}$, we need to invert $\mathbf{X}'\mathbf{X}$, a $(k+1) \times (k+1)$ matrix.

Some Non-invertible Explanatory Data Matrices

$\mathbf{X}'\mathbf{X}$ is invertible iff $\mathbf{X}$ is full column rank (see Wooldridge D.4), so the collection of predictors need to be linearly independent (no perfect collinearity).

Some example of $\mathbf{X}$ that are not full column rank:

$$\mathbf{X} = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 3 & -3 \\ 1 & 4 & -4 \\ 1 & 5 & -5 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 54 & 54,000 \\ 1 & 37 & 37,000 \\ 1 & 89 & 89,000 \\ 1 & 72 & 72,000 \end{bmatrix}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Covariance/variance interpretation of matrix OLS

$$\mathbf{X}'\mathbf{y} = \sum_{i=1}^{n} \begin{bmatrix} y_i \\ y_i x_{i1} \\ y_i x_{i2} \\ \vdots \\ y_i x_{iK} \end{bmatrix} \approx \begin{bmatrix} n\overline{y} \\ \widehat{\text{cov}}(y_i, x_{i1}) \\ \widehat{\text{cov}}(y_i, x_{i2}) \\ \vdots \\ \widehat{\text{cov}}(y_i, x_{iK}) \end{bmatrix}$$

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} \begin{bmatrix} 1 & x_{i1} & x_{i2} & \cdots & x_{iK} \\ x_{i1} & x_{i1}^2 & x_{i2}x_{i1} & \cdots & x_{i1}x_{iK} \\ x_{i2} & x_{i1}x_{i2} & x_{i2}^2 & \cdots & x_{i2}x_{iK} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{iK} & x_{i1}x_{iK} & x_{i2}x_{iK} & \cdots & x_{iK}x_{iK} \end{bmatrix} \approx \begin{bmatrix} n & n\overline{x}_1 & n\overline{x}_2 \\ n\overline{x}_1 & \widehat{\text{var}}(x_{i1}) & \widehat{\text{cov}}(x_{i1}, x_{i2}) \\ n\overline{x}_2 & \widehat{\text{cov}}(x_{i2}, x_{i1}) & \widehat{\text{var}}(x_{i2}) \\ \vdots & \vdots & \vdots \\ n\overline{x}_K & \widehat{\text{cov}}(x_{iK}, x_{i1}) & \widehat{\text{cov}}(x_{iK}, x_{i2}) \end{bmatrix}$$

Derivatives with respect to $\tilde{\boldsymbol{\beta}}$

$$\begin{aligned} S(\tilde{\boldsymbol{\beta}}, \mathbf{X}, \mathbf{y}) &= (\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}) \\ &= \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\tilde{\boldsymbol{\beta}} \end{aligned}$$

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- The first term does not contain $\tilde{\boldsymbol{\beta}}$
- The second term is an example of rule I from the derivative section
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And while we are at it the Hessian is:

$$
\frac{\partial^2 S(\tilde{\boldsymbol{\beta}}, \mathbf{X}, \mathbf{y})}{\partial \tilde{\boldsymbol{\beta}} \partial \tilde{\boldsymbol{\beta}}'} = 2\mathbf{X}'\mathbf{X}
$$

Solving for $\hat{\boldsymbol{\beta}}$

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$$\begin{aligned} \mathbf{0} &= -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} \\ -2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} &= -2\mathbf{X}'\mathbf{y} \\ \mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} &= \mathbf{X}'\mathbf{y} \end{aligned}$$

Solving for $\hat{\boldsymbol{\beta}}$

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\end{aligned}$$

Solving for $\hat{\boldsymbol{\beta}}$

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\end{aligned}$$

Note that we implicitly assumed that $\mathbf{X}'\mathbf{X}$ is invertible.

Variance-Covariance Matrix of Random Vectors

Let's unpack the homoskedasticity assumption $V[\mathbf{u}|\mathbf{X}] = \sigma^2 \mathbf{I}_n$.

Definition (variance-covariance matrix)

For a $(n \times 1)$ random vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \dots & u_n \end{bmatrix}'$, its variance-covariance matrix, denoted $V[\mathbf{u}]$ or also $\Sigma_{\mathbf{u}}$, is defined as:

$$V[\mathbf{u}] = \Sigma_{\mathbf{u}} = \begin{bmatrix} \sigma_1^2 & \sigma_{12}^2 & \dots & \sigma_{1n}^2 \\ \sigma_{21}^2 & \sigma_2^2 & \dots & \sigma_{2n}^2 \\ \vdots & & \dots & \\ \sigma_{n1}^2 & \sigma_{n2}^2 & \dots & \sigma_n^2 \end{bmatrix}$$

where $\sigma_j^2 = V[u_j]$ and $\sigma_{ij}^2 = Cov[u_i, u_j]$.

Notice that this matrix is always symmetric.

Homoskedasticity in Matrix Notation

If $V[u_i] = \sigma^2$ for all $i = 1, ..., n$ and the units are independent then $V[\mathbf{u}] = \sigma^2 \mathbf{I}_n$.

Homoskedasticity in Matrix Notation

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More visually:

$$V[\mathbf{u}] = \sigma^2 \mathbf{I}_n = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

Homoskedasticity in Matrix Notation

If $V[u_i] = \sigma^2$ for all $i = 1, ..., n$ and the units are independent then $V[\mathbf{u}] = \sigma^2 \mathbf{I}_n$.
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$$V[\mathbf{u}] = \sigma^2 \mathbf{I}_n = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

So homoskedasticity $V[\mathbf{u}|\mathbf{X}] = \sigma^2 \mathbf{I}_n$ implies that:

1. $V[u_i|\mathbf{X}] = \sigma^2$ for all i (the variance of the errors u_i does not depend on $\mathbf{X}$ and is constant across observations)
2. $Cov[u_i, u_j|\mathbf{X}] = 0$ for all $i \neq j$ (the errors are uncorrelated across observations). This holds under our random sampling assumption.

Estimation of the Error Variance

Given our vector of regression error terms $\mathbf{u}$, what is $E[\mathbf{u}\mathbf{u}']$?

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$$E[\mathbf{u}\mathbf{u}'] = \begin{bmatrix} E[u_1^2] & E[u_1 u_2] & \dots & E[u_1 u_n] \\ E[u_2 u_1] & E[u_2^2] & \dots & E[u_2 u_n] \\ \vdots & \vdots & \ddots & \vdots \\ E[u_n u_1] & E[u_n u_2] & \dots & E[u_n^2] \end{bmatrix} =$$

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Recall $E[u_i] = 0$ for all i. So $V[u_i] = E[u_i^2] - (E[u_i])^2 = E[u_i^2]$ and by independence $E[u_iu_j] = E[u_i] \cdot E[u_j] = 0$

$$Var(\mathbf{u}) = E[\mathbf{u}\mathbf{u}'] = \sigma^2\mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

Variance of Linear Function of Random Vector

Definition (Variance of Linear Transformation of Random Vector)

Recall that for a linear transformation of a random variable X we have $V[aX + b] = a^2 V[X]$ with constants a and b.

There is an analogous rule for linear functions of random vectors. Let $v(\mathbf{u}) = \mathbf{Au} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[v(\mathbf{u})] = V[\mathbf{Au} + \mathbf{B}] = \mathbf{A} V[\mathbf{u}]\mathbf{A}' = \mathbf{A}\Sigma_{\mathbf{u}}\mathbf{A}'$$

Conditional Variance of $\hat{\beta}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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V[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= V[\boldsymbol{\beta}|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\
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\end{aligned}
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&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' V[\mathbf{u}|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
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&= \sigma^2\mathbf{I}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}
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\end{aligned}$$

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$\hat{\beta} = \beta + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\beta}|\mathbf{X}] = \beta + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \beta$ so the OLS estimator is a linear function of the errors. Thus:

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V[\hat{\beta}|\mathbf{X}] &= V[\beta|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\
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&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}$$

This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\beta}$.

Conditional Variance of $\hat{\boldsymbol{\beta}}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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V[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= V[\boldsymbol{\beta}|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\
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\end{aligned}$$

This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\boldsymbol{\beta}}$.

To estimate $V[\hat{\boldsymbol{\beta}}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$\hat{\sigma}^2 = \frac{SSR}{n-(k+1)} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$$

Variance-covariance matrix of $\hat{\beta}$

The variance-covariance matrix of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} =$$

	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\cdots$	$\widehat{\beta}_k$
$\widehat{\beta}_0$	$V[\widehat{\beta}_0]$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k]$
$\widehat{\beta}_1$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$V[\widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_k]$
$\widehat{\beta}_2$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$V[\widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_2, \widehat{\beta}_k]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\widehat{\beta}_k$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k]$	$\text{Cov}[\widehat{\beta}_k, \widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_k, \widehat{\beta}_2]$	$\cdots$	$V[\widehat{\beta}_k]$

Consistency of $\hat{\beta}$

To show consistency, we rewrite the OLS estimator in terms of sample means so that we can apply LLN.

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First, note that a matrix cross product can be written as a sum of vector products:

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i \quad \text{and} \quad \mathbf{X}'\mathbf{y} = \sum_{i=1}^{n} \mathbf{x}_i' y_i$$

where $\mathbf{x}_i$ is the $1 \times (k+1)$ row vector of predictor values for unit i.

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where $\mathbf{x}_i$ is the $1 \times (k+1)$ row vector of predictor values for unit i.

Now we can rewrite the OLS estimator as,

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= \left(\sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum_{i=1}^{n} \mathbf{x}_i' y_i\right) \\
&= \left(\sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum_{i=1}^{n} \mathbf{x}_i'(\mathbf{x}_i\boldsymbol{\beta} + u_i)\right) \\
&= \boldsymbol{\beta} + \left(\sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum_{i=1}^{n} \mathbf{x}_i' u_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\frac{1}{n}\sum_{i=1}^{n} \mathbf{x}_i' u_i\right)
\end{aligned}$$

Consistency of $\hat{\beta}$

Now let's apply the LLN to the sample means:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right) \xrightarrow{p} E[\mathbf{x}_i'\mathbf{x}_i], \text{ a } (k+1)\times(k+1) \text{ nonsingular matrix.}$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'u_i\right) \xrightarrow{p} E[\mathbf{x}_i'u_i] = 0, \text{ by the zero cond. mean assumption.}$$

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Therefore, we have

$$\begin{aligned} \text{plim}(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + \left(E[\mathbf{x}_i'\mathbf{x}_i]\right)^{-1}\cdot 0 \\ &= \boldsymbol{\beta}. \end{aligned}$$

Consistency of $\hat{\boldsymbol{\beta}}$

Now let's apply the LLN to the sample means:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right) \xrightarrow{p} E[\mathbf{x}_i'\mathbf{x}_i], \text{ a } (k+1)\times(k+1) \text{ nonsingular matrix.}$$
$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i' u_i\right) \xrightarrow{p} E[\mathbf{x}_i' u_i] = 0, \text{ by the zero cond. mean assumption.}$$

Therefore, we have

$$\begin{aligned} \text{plim}(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + \left(E[\mathbf{x}_i'\mathbf{x}_i]\right)^{-1}\cdot 0 \\ &= \boldsymbol{\beta}. \end{aligned}$$

We can also show the asymptotic normality of $\hat{\boldsymbol{\beta}}$ using a similar argument but with the CLT.

References

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Where We've Been and Where We're Going...

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  - regression with two variables
  - omitted variables, multicollinearity, interactions

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  - then ... regression in social science
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

Running Example: Chilean Referendum on Pinochet

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- Plebiscite was held on October 5, 1988. The No side won with 56% of the vote, with 44% voting Yes.
- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

R Code

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> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

The t-Value for Multiple Linear Regression

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$$\hat{SE}(\hat{\beta}_j) = \sqrt{\widehat{V}(\hat{\beta}_j)} = \sqrt{\widehat{V}(\hat{\boldsymbol{\beta}})_{(j,j)}} = \sqrt{\hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}_{(j,j)}}$$

where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.

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where $\mathbf{A}_{(j,j)}$ is the (j, j) element of matrix $\mathbf{A}$.
That is, take the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and square root the diagonal element corresponding to j.

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We can pull out the variance-covariance matrix $\hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$ in R from the `lm()` object:

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We can pull out the variance-covariance matrix $\hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$ in R from the lm() object:

R Code

```
> V <- vcov(fit)
> V
              (Intercept)           fem          educ           age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem         -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ        -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age         -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
 (Intercept)          fem         educ          age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

Sampling Distribution of the t-Value

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Theorem (Small-Sample Distribution of the t-Value)

Under Assumptions 1–6, for any sample size n the t-value has the t distribution with $(n-k-1)$ degrees of freedom:

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Under Assumptions 1–5, as $n \to \infty$ the distribution of the t-value approaches the standard normal distribution:

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- $t_{n-k-1} \to \mathcal{N}(0,1)$ as $n \to \infty$, so the difference disappears when n large.
- In practice people often just use t_{n-k-1} to be on the conservative side.

Using the t-Value as a Test Statistic

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3. Decide whether the realized value of T in our data is unusual given the known distribution of the test statistic.
4. Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

To construct confidence intervals, there is again no difference compared to the case of $k = 1$, except that we need to use t_{n-k-1} instead of t_{n-2}

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So we also know the probability that the value of our test statistics falls into a given interval:

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We rearrange:

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We rearrange:

$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\, ,\, \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

R Code

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---
```

R Code

```
> confint(fit)
                   2.5 %      97.5 %
(Intercept)  0.303407780  0.50504909
fem          0.089493169  0.18251357
educ        -0.087954435 -0.03356629
age          0.002147755  0.00540954
```

Testing Hypothesis About a Linear Combination of β_j

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> fit <- lm(REALGDPCAP ~ Region, data = D)
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Coefficients:
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RegionAfrica      -2552.8     1204.5  -2.119   0.0372 *
RegionAsia          148.9     1149.8   0.129   0.8973
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- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0: \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

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- What would be an appropriate test statistic for this hypothesis?

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- Let's consider a t-value:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})}$$

We will reject H_0 if T is sufficiently different from zero.

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- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About A Linear Combination of β_j

- Our test statistic:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \;\sim\; t_{n-k-1}$$

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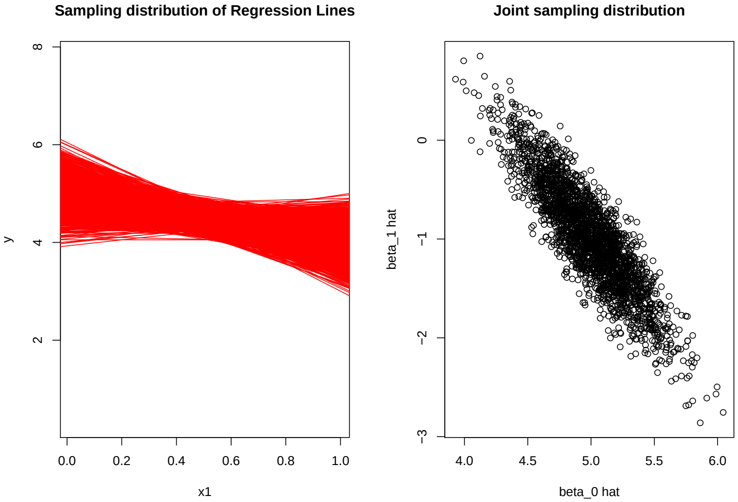
- Since the estimates of the coefficients are correlated, we need the covariance term.

Joint Normality: Simulation

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:



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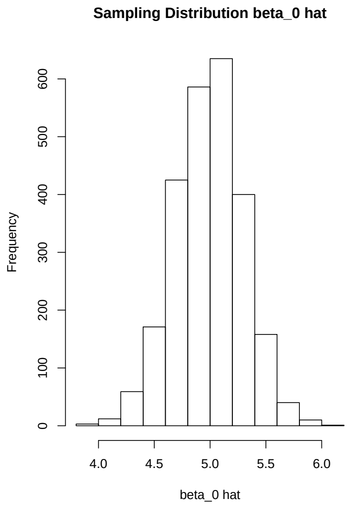

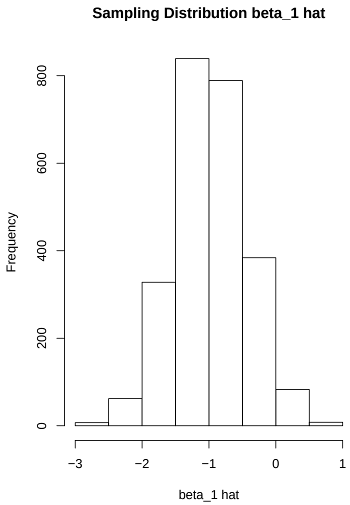

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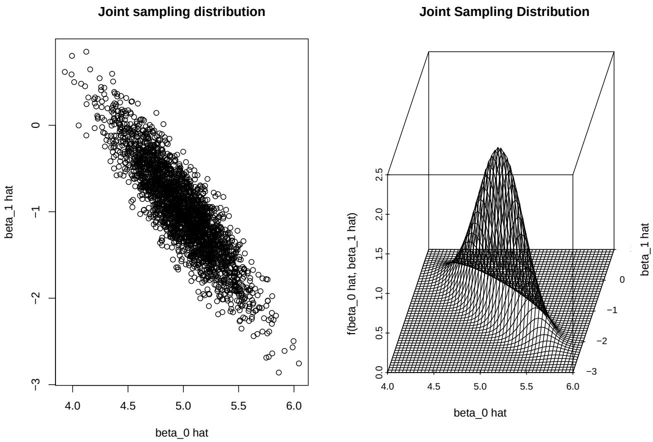

The variance-covariance matrix of the estimators is:

	$\widehat{\beta}_0$	$\widehat{\beta}_1$
$\widehat{\beta}_0$	.08	$-.11$
$\widehat{\beta}_1$	$-.11$	.24

Example: GDP per capita on Regions

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
               (Intercept) RegionAfrica RegionAsia RegionLatAmerica
(Intercept)       613769.9    -613769.9  -613769.9        -613769.9
RegionAfrica     -613769.9    1450728.8   613769.9         613769.9
RegionAsia       -613769.9     613769.9  1321965.9         613769.9
RegionLatAmerica -613769.9     613769.9   613769.9        1014054.6
RegionOecd       -613769.9     613769.9   613769.9         613769.9
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> se <- sqrt(V[4,4] + V[3,3] - 2*V[3,4])
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[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
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```

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We cannot reject the null that the difference in average GDP resulted from chance.

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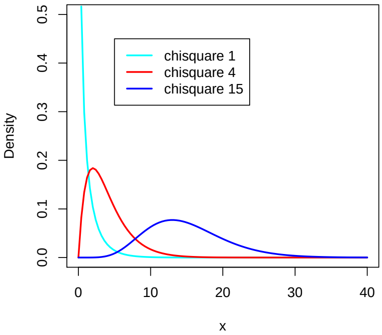

Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

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The F distribution arises as a ratio of two independent chi-squared distributed random variables:

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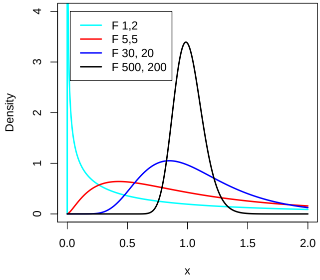

In R: `df()`, `pf()`, `rf()`

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- Under Assumptions 1–5, $qF_0 \overset{a.}{\sim} \chi^2_q$ as $n \to \infty$ (see next section).

Unrestricted Model (UR)

R Code

```
> fit.UR <- lm(vote1 ~ fem + educ + age + fem:age + fem:educ, data = Chile
> summary(fit.UR)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,        Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

R Code

```
> fit.R <- lm(vote1 ~ educ + age, data = Chile)
> summary(fit.R)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4878039  0.0497550   9.804  < 2e-16 ***
educ        -0.0662022  0.0139615  -4.742 2.30e-06 ***
age          0.0035783  0.0008385   4.267 2.09e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4921 on 1700 degrees of freedom
Multiple R-squared:  0.03275,       Adjusted R-squared:  0.03161
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

R Code
```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

> DFdenom <- df.residual(fit.UR)  # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

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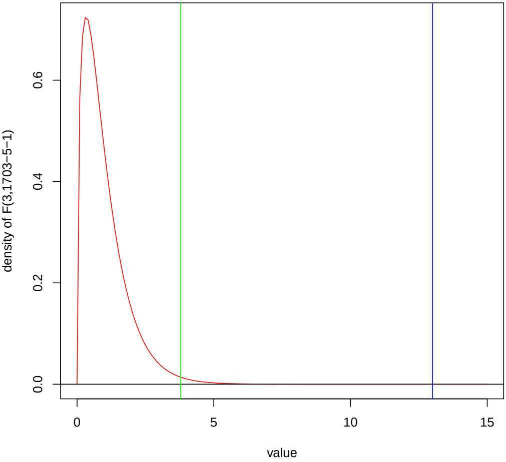
Given above, what do we conclude?

$F_0 = 13$ is greater than the critical value for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



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- This is called the omnibus test and is routinely reported by statistical software.

Omnibus Test in R

R Code

```
> summary(fit.UR)
~~~~~

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,        Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

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$$H_0 : \beta_1 = \beta_2 = \beta_3$$

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- What question are we asking?

  $\rightarrow$ Are the effects of X_1, X_2 and X_3 different from each other?

- How many restrictions?

  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)

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  $\rightarrow$ Are the effects of X_1, X_2 and X_3 different from each other?
- How many restrictions?
  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)
- How do we fit the restricted model?
  $\rightarrow$ The null hypothesis implies that the model can be written as:
  $$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3) + ... + \beta_k X_k + u$$
  So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:
  $$Y = \beta_0 + \beta_1 X^* + ... + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
~~~~~

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1899.9      914.9   2.077   0.0410 *
Asia          2701.7     1243.0   2.173   0.0327 *
LatAmerica    2281.5     1112.3   2.051   0.0435 *
Transit       2552.8     1204.5   2.119   0.0372 *
Oecd         12224.2     1112.3  10.990   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,        Adjusted R-squared: 0.6951
F-statistic: 48.88 on 4 and 80 DF,  p-value: < 2.2e-16
```

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

R Code

```
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)

> SSR.UR2 <- sum(resid(fit.UR2)^2)
> SSR.R2 <- sum(resid(fit.R2)^2)

> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

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So, what do we conclude?
The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

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- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})/(SSR_R/(n-k-1))$ and do the F test.

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- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n - k - 1)}$$

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- F tests only work for testing nested models, i.e. the restricted model must be a special case of the unrestricted model.

  For example F tests cannot be used to test

$$Y = \beta_0 + \beta_1 X_1 \qquad\qquad + \beta_3 X_3 + u$$

  against

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \qquad\quad + u$$

Some More Notes on F Tests

- Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

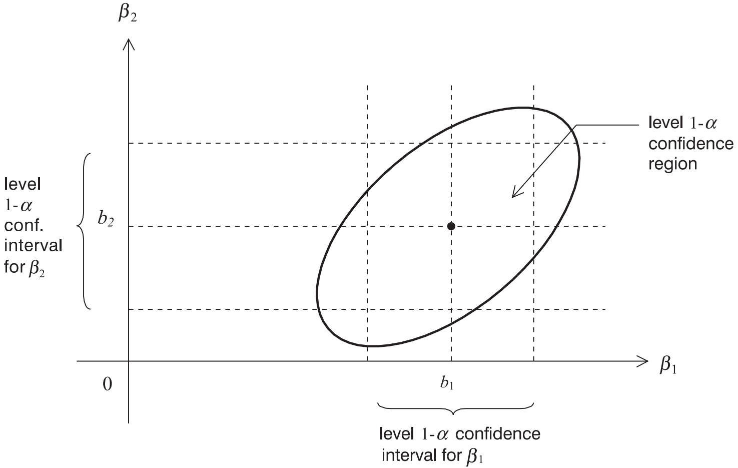

Figure 1.5: t- versus F-Tests

Limitation of the F Formula

Consider the following null hypothesis:

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To compute the F value, we need to fit the restricted model. How?

- Some restrictions are difficult to impose when fitting the model.
- Even when we can, the procedure will be ad hoc and require some creativity.
- Is there a general solution?

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$$\beta_1 = \beta_2 = \beta_3 = 3 \Leftrightarrow \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

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- It turns out that the following Wald statistic can be used:

$$W = \left(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}\right)' \cdot \left[\hat{\sigma}^2 \mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right]^{-1} \cdot \left(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}\right)$$

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  - The middle component is the variance of $\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$. This standardizes the sum of squares to have variance one.
- We know $\hat{\boldsymbol{\beta}}$ is approximately normal $\Rightarrow \mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$ should also be normal $\Longrightarrow W$ should therefore be ...

Wald Statistic

- Let's consider testing $H_0 : \mathbf{R}\boldsymbol{\beta} = \mathbf{r}$, a set of q linear restrictions.
- If H_0 is true, $\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$ should be zero except for sampling variability.
- To formally evaluate the statistical significance of the deviation from zero, we must transform $\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$ to a statistic that can be compared to a reference distribution.
- It turns out that the following Wald statistic can be used:

$$W = \left(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}\right)' \cdot \left[\hat{\sigma}^2 \mathbf{R}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{R}'\right]^{-1} \cdot \left(\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}\right)$$

- Looks complicated? Let's figure out why this makes sense:
  - The first and last components give the sum of squares of the components of $\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$. This summarizes its deviation from zero.
  - The middle component is the variance of $\mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$. This standardizes the sum of squares to have variance one.
- We know $\hat{\boldsymbol{\beta}}$ is approximately normal $\Rightarrow \mathbf{R}\hat{\boldsymbol{\beta}} - \mathbf{r}$ should also be normal $\Longrightarrow W$ should therefore be ... χ^2 distributed!

Sampling Distribution of the Wald Statistic

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Theorem (Large-Sample Distribution of the Wald Statistic)

Under Assumptions 1–5, as $n \to \infty$ the distribution of the Wald statistic approaches the chi square distribution with q degrees of freedom:

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- $q\mathcal{F}_{q,n-k-1} \xrightarrow{d} \chi^2_q$ as $n \to \infty$, so the difference disappears when n large.

```
> pf(3.1, 2, 500,lower.tail=F) [1] 0.04591619

> pchisq(2*3.1, 2,lower.tail=F) [1] 0.0450492

> pf(3.1, 2, 50000,lower.tail=F) [1] 0.04505786
```

Testing General Linear Hypotheses in R

In R, the `linearHypothesis()` function in the car package does the Wald test for general linear hypotheses.

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> R <- matrix(c(0,1,-1,0,0, 0,1,0,-1,0), nrow = 2, byrow = T)
> r <- c(0,0)
> linearHypothesis(fit.UR2, R, r)
Linear hypothesis test

Hypothesis:
Asia - LatAmerica = 0
Asia - Transit = 0

Model 1: restricted model
Model 2: REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd

  Res.Df       RSS Df Sum of Sq      F Pr(>F)
1     82 738141635
2     80 736523836  2   1617798 0.0879  0.916
```

Next Week (of Classes)

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- Linear Regression in the Social Sciences
- Reading:

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- Reading:
  - Healy and Moody (2014) "Data Visualization in Sociology" *Annual Review of Sociology*
  - Morgan and Winship (2015) Chapter 1: Causality and Empirical Research in the Social Sciences
  - Morgan and Winship (2015) Chapter 13.1: Objections to Adoption of the Counterfactual Approach
  - Optional: Morgan and Winship (2015) Chapter 2-3 (Potential Outcomes and Causal Graphs)
  - Optional: Hernán and Robins (2016) Chapter 1: A definition of a causal effect.

Fun Without Weights

The Robust Beauty of Improper Linear Models in Decision Making

ROBYN M. DAWES *University of Oregon*

ABSTRACT: *Proper linear models are those in which predictor variables are given weights in such a way that the resulting linear composite optimally predicts some criterion of interest; examples of proper linear models are standard regression analysis, discriminant function analysis, and ridge regression analysis. Research summarized in Paul Meehl's book on clinical versus statistical prediction—and a plethora of research stimulated in part by that book—all indicates that when a numerical criterion variable (e.g., graduate grade point average) is to be predicted from numerical predictor variables, proper linear models outperform clinical intuition. Improper linear models are those in which the weights of the predictor variables are obtained by some nonoptimal method; for example, they may be obtained on the basis of intuition, derived from simulating a clinical judge's predictions, or set to be equal. This article presents evidence that even such improper linear models are superior to clinical intuition when predicting a numerical criterion from numerical predictors. In fact, unit (i.e., equal) weighting is quite robust for making such predictions. The article discusses, in some detail, the application of unit weights to decide what bullet the Denver Police Department should use. Finally, the article considers commonly raised technical, psychological, and ethical*

A *proper linear model* is one in which the weights given to the predictor variables are chosen in such a way as to optimize the relationship between the prediction and the criterion. Simple regression analysis is the most common example of a proper linear model; the predictor variables are weighted in such a way as to maximize the correlation between the subsequent weighted composite and the actual criterion. Discriminant function analysis is another example of a proper linear model; weights are given to the predictor variables in such a way that the resulting linear composites maximize the discrepancy between two or more groups. Ridge regression analysis, another example (Darlington, 1978; Marquardt & Snee, 1975), attempts to assign weights in such a way that the linear composites correlate maximally with the criterion of interest in a new set of data.

Thus, there are many types of proper linear models and they have been used in a variety of contexts. One example (Dawes, 1971) was presented in this Journal; it involved the prediction

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- Dawes argues that even improper linear models (those where weights are set by hand or set to be equal), outperform clinical intuition.
- Equal weight models are argued to be quite robust for these predictions

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- Correlation of faculty ratings with average rating of admissions committee was .19
- Standardized and equally weighted improper linear model, correlated at .48

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- Einhorn (1972) study of doctors coding biopsies of patients with Hodgkin's disease and then rated severity. Their rating of severity was essentially uncorrelated with survival times, but the variables they coded predicted outcomes using a regression model.

Other Examples

TABLE 1

Correlations Between Predictions and Criterion Values

Example	Average validity of judge	Average validity of judge model	Average validity of random model	Validity of equal weighting model	Cross-validity of regression analysis	Validity of optimal linear model
Prediction of neurosis vs. psychosis	.28	.31	.30	.34	.46	.46
Illinois students' predictions of GPA	.33	.50	.51	.60	.57	.69
Oregon students' predictions of GPA	.37	.43	.51	.60	.57	.69
Prediction of later faculty ratings at Oregon	.19	.25	.39	.48	.38	.54
Yntema & Torgerson's (1961) experiment	.84	.89	.84	.97	—	.97

Note. GPA = grade point average.

Column descriptions:

C1) average of human judges

C2) model based on human judges

C3) randomly chosen weights preserving signs

C4) equal weighting

C5) cross-validated weights

C6) unattainable optimal linear model

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- People are good at picking out relevant information, but terrible at integrating it.
- The difficulty arises in part because people in general lack a strong reference to the distribution of the predictors.
- Linear models are robust to deviations from the optimal weights (see also Waller 2008 on "Fungible Weights in Multiple Regression")

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- This all applies because predictors well chosen and the sample size is small (so the weight optimization isn't great)
- It is a fascinating paper!