Week 6: Linear Regression with Two Regressors

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Princeton

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[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller.

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

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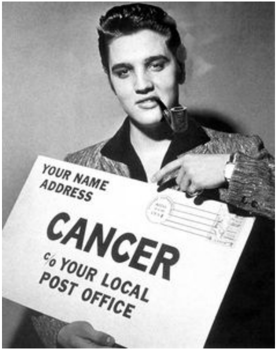

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Example 2: Berkeley Graduate Admissions

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- What about the conditional relationship within departments?

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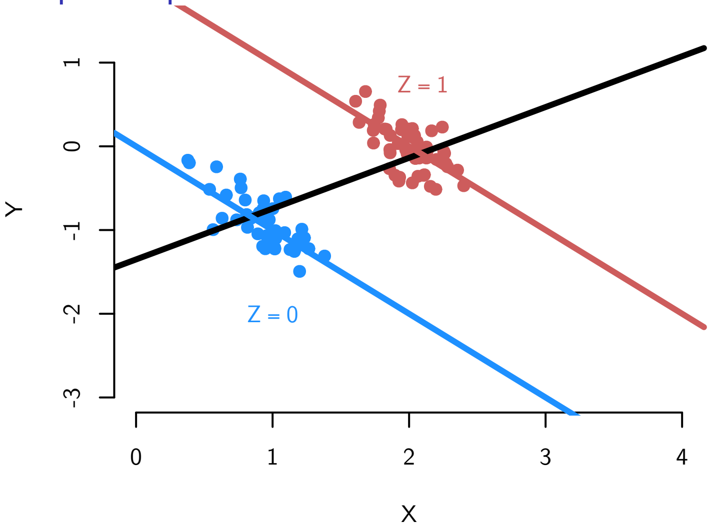
- Marginal relationships (admissions and gender) $\neq$ conditional relationship given third variable (department)

Simpson's paradox



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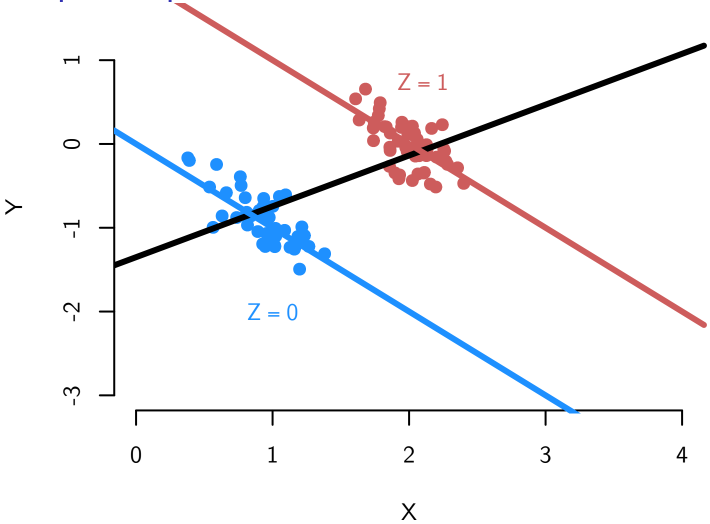

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Instance of a more general problem called the ecological inference fallacy

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- β's are the population parameters we want to estimate

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- The rest of this lecture is designed to explain what is meant by "controlling for another variable" with linear regression.

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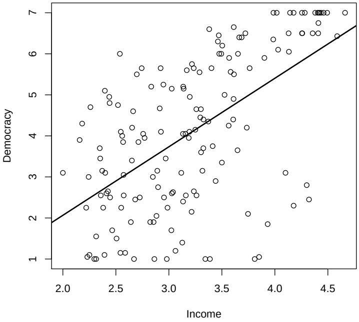
$$\widehat{Demo} = -1.26 + 1.6\, Log(GDP)$$



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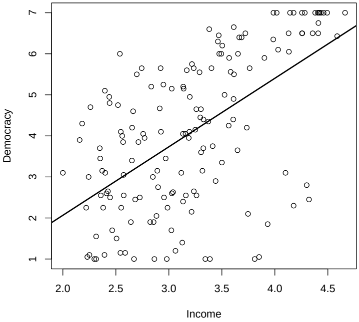

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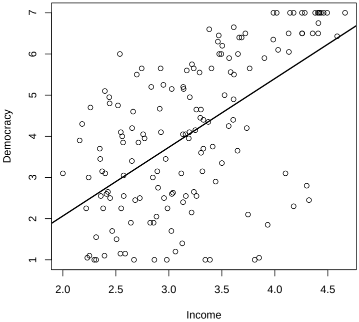

Interpretation: A one percent increase in GDP is associated with a .016 point increase in democracy.

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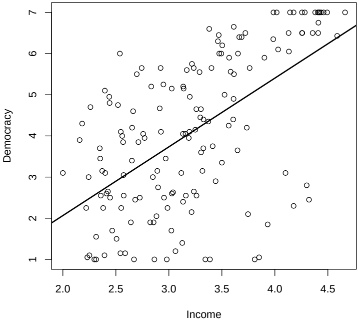

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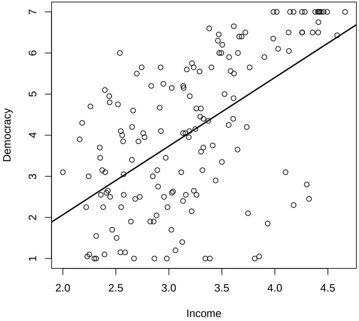

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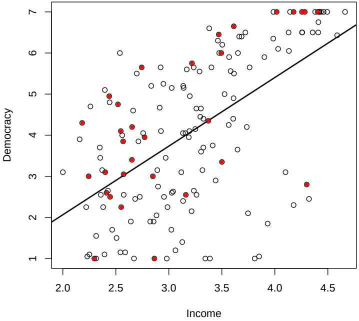

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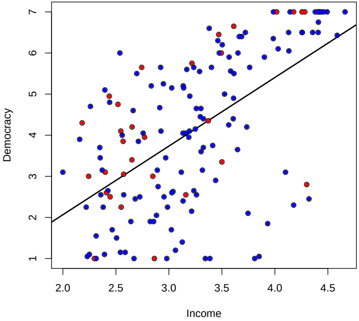
  - Non-colony countries tend to have lower levels of democracy



Adding a Covariate

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$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{1i} + \widehat{\beta}_2 x_{2i}$$

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In words:

$$\widehat{Democracy} = \widehat{\beta}_0 + \widehat{\beta}_1 \, Log(GDP) + \widehat{\beta}_2 \, Colony$$

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What does this mean?

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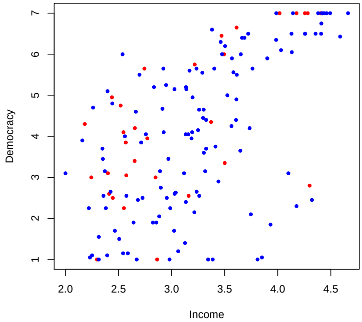
What does this mean? We are fitting two lines with the same slope but different intercepts.

Regression of Democracy on Income

From R, we obtain estimates $\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$:

```
Coefficients:
             Estimate
(Intercept)  -1.5060
GDP90LGN      1.7059
BRITCOL       0.5881
```



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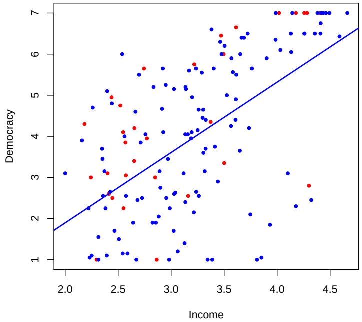

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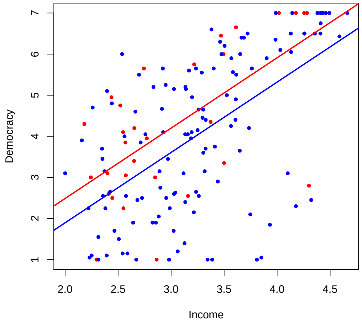
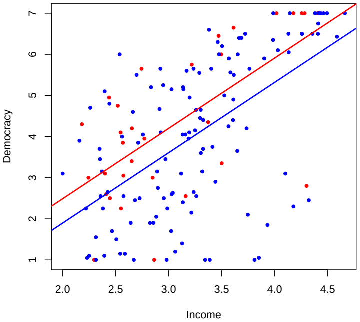
$$\widehat{y} = -.92 + 1.7\, x_1$$



Regression of Democracy on Income

Our prediction equation is:
$\widehat{y} = -1.5 + 1.7\, x_1 + .58\, x_2$

Where do these quantities appear on the graph?



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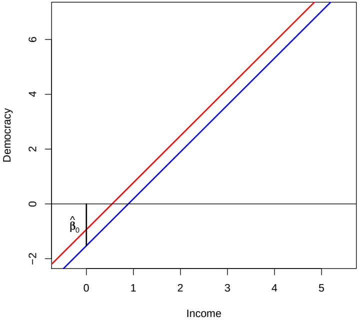

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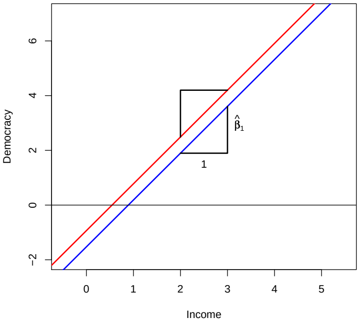

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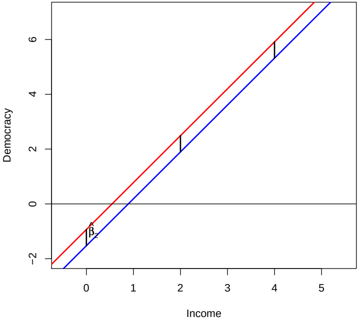
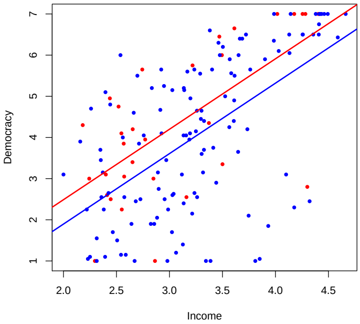


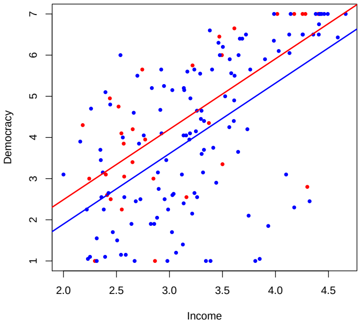
Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.



Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.
- This is easy to represent graphically in two dimensions because we can use colors to distinguish the two groups in the data.



Regression of Democracy on Income

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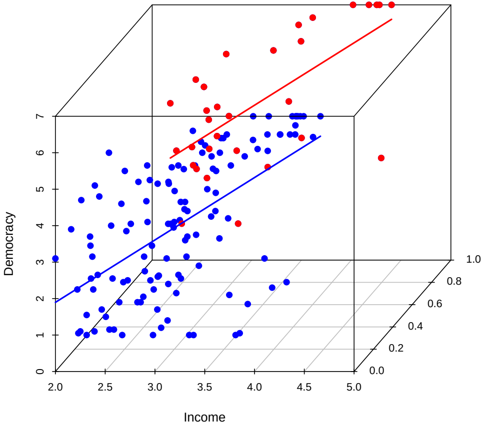

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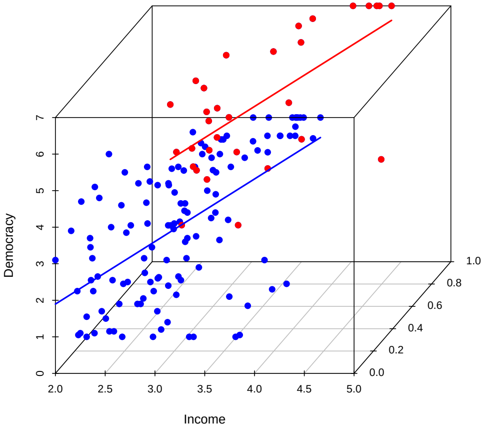

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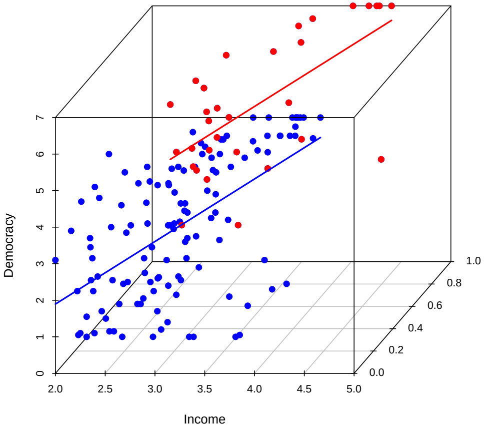
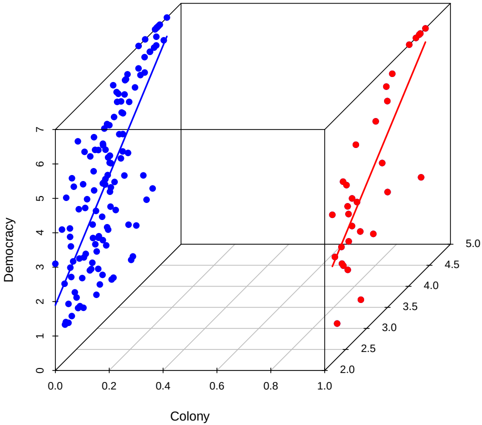
- We can try to represent this using a 3D scatterplot.
- In this view, we are looking at the data from the Income side; the two regression lines are drawn in the appropriate locations.



Regression of Democracy on Income

- We can also look at the 3D scatterplot from the British colony side.



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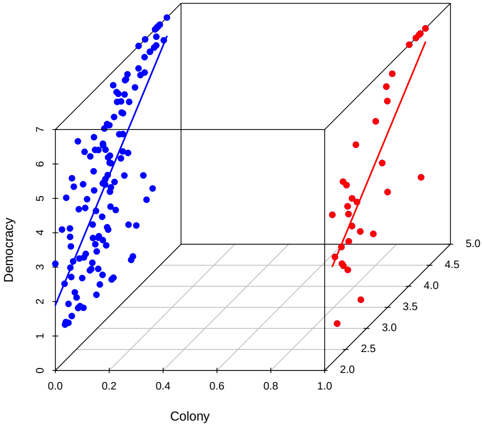
- We can also look at the 3D scatterplot from the British colony side.
- While the British colonial status variable is either 0 or 1, there is nothing in the prediction equation that requires this to be the case.



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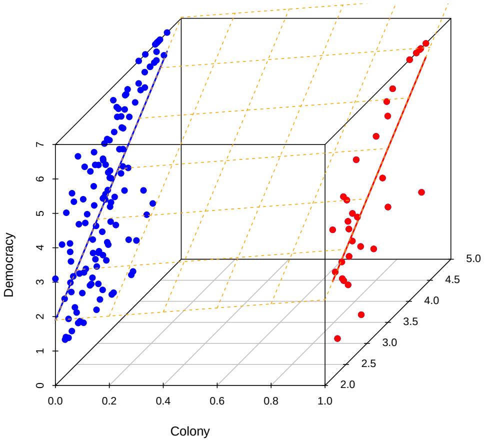
- In fact, the prediction equation defines a regression plane that connects the lines when $x_2 = 0$ and $x_2 = 1$.



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- For example, we might want to use:
  - X_1 Income and X_2 Ethnic Heterogeneity
  - Y Democracy

$$\widehat{\text{Democracy}} = \hat{\beta}_0 + \hat{\beta}_1 \text{Income} + \hat{\beta}_2 \text{Ethnic Heterogeneity}$$

Regression of Democracy on Income

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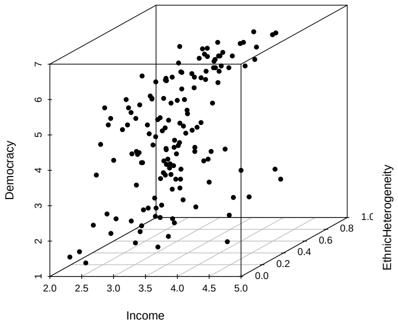

Regression of Democracy on Income

- We can plot the points in a 3D scatterplot.
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  - $\widehat{\beta}_0 = -.71$
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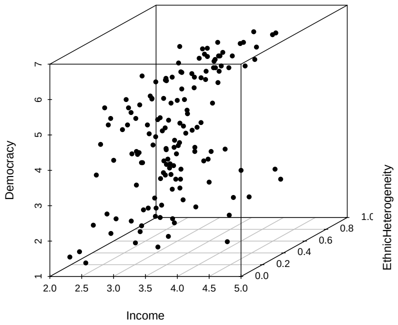
How does this look graphically?



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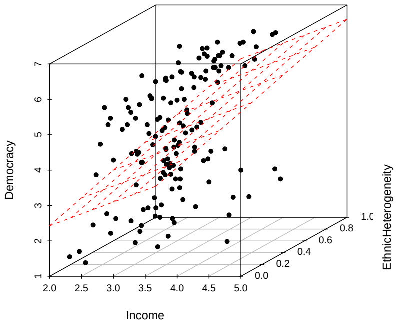

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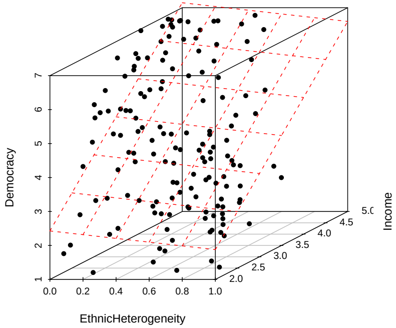

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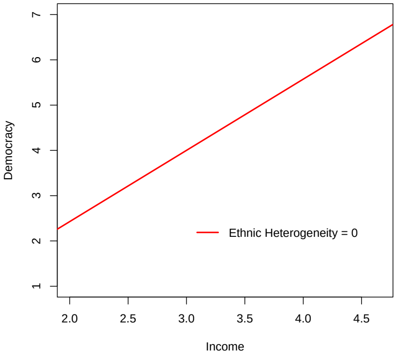

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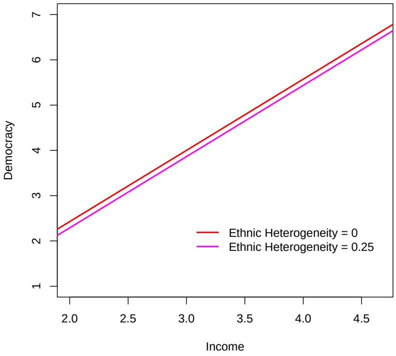

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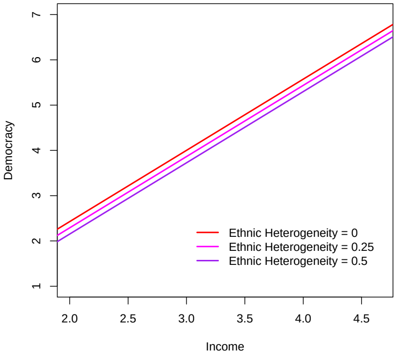

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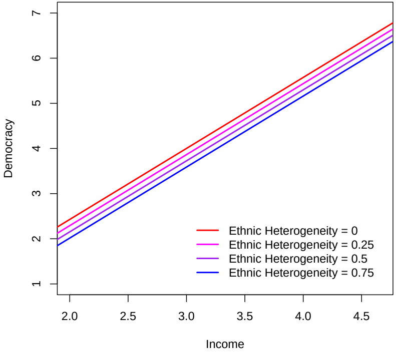

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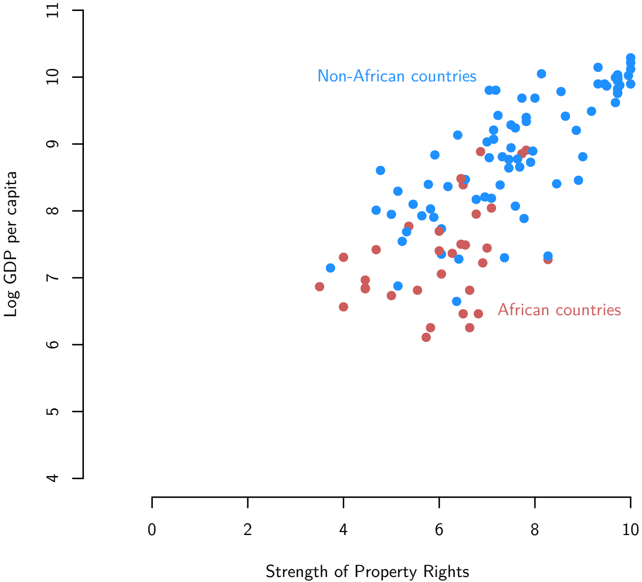
  Predicted difference is thus: 1.8 or $(3.5 - 2.5)\widehat{\beta}_1 + (.06 - .5)\widehat{\beta}_2$

AJR Example


Log GDP per capita
4
5
6
7
8
9
10
11
Non-African countries
African countries
0
2
4
6
8
10
Strength of Property Rights

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- New model:
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AJR model

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.65556    0.31344  18.043  < 2e-16 ***
## avexpr       0.42416    0.03971  10.681  < 2e-16 ***
## africa      -0.87844    0.14707  -5.973 3.03e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6253 on 108 degrees of freedom
##   (52 observations deleted due to missingness)
## Multiple R-squared:  0.7078, Adjusted R-squared:  0.7024
## F-statistic: 130.8 on 2 and 108 DF,  p-value: < 2.2e-16
```

Two lines in one regression

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- Two different intercepts, same slope

Example interpretation of the coefficients

- Let's review what we've seen so far:

	Intercept for X_i	Slope for X_i
Non-African country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
African country ($Z_i = 1$)	$\widehat{\beta}_0 + \widehat{\beta}_2$	$\widehat{\beta}_1$

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- In this example, we have:

$$\widehat{Y}_i = 5.656 + 0.424 \times X_i - 0.878 \times Z_i$$

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  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)

Example interpretation of the coefficients

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	Intercept for X_i	Slope for X_i
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- We can read these as:
  - $\widehat{\beta}_0$: average log income for non-African country ($Z_i = 0$) with property rights measured at 0 is 5.656
  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)
  - $\widehat{\beta}_2$: there is a -0.878 average difference in log income per capita between African and non-African counties **conditional on** property rights

General interpretation of the coefficients

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

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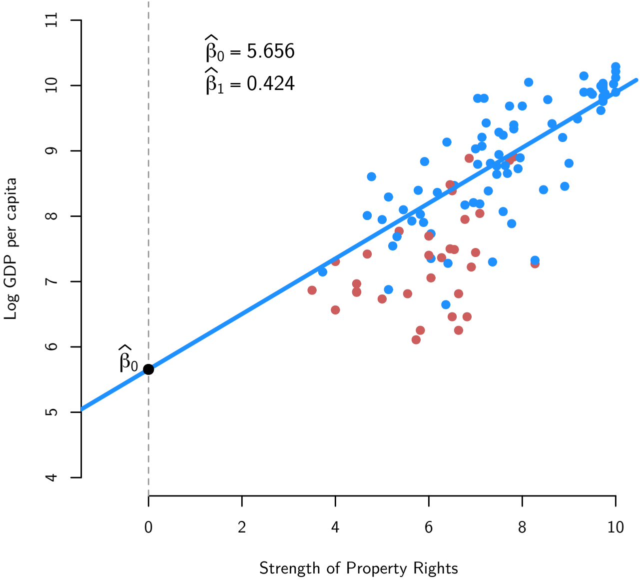
- $\widehat{\beta}_1$: A one-unit change in X_i is associated with a $\widehat{\beta}_1$-unit change in Y_i conditional on Z_i
- $\widehat{\beta}_2$: average difference in Y_i between $Z_i = 1$ group and $Z_i = 0$ group conditional on X_i

Adding a binary variable, visually



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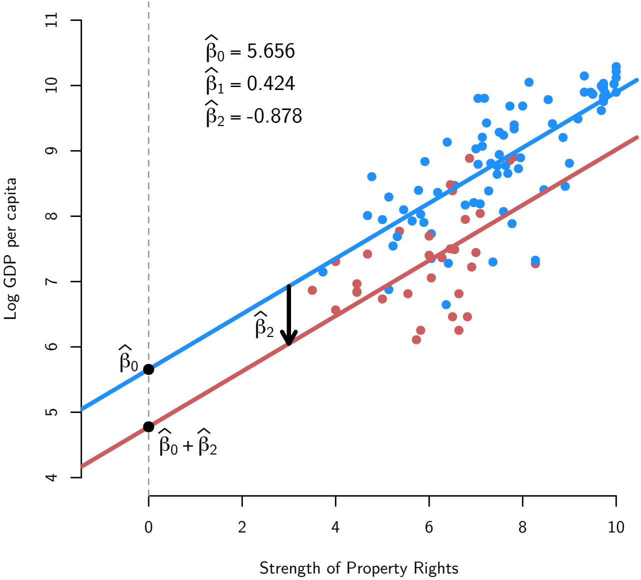

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$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i$$

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- Z_i: mean temperature in country i (continuous)
- Concern: geography is confounding the effect
    - geography might affect political institutions
    - geography might affect average incomes (through diseases like malaria)
- New model:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

AJR model, revisited

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  6.80627    0.75184   9.053 1.27e-12 ***
## avexpr       0.40568    0.06397   6.342 3.94e-08 ***
## meantemp    -0.06025    0.01940  -3.105  0.00296 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6435 on 57 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.6155, Adjusted R-squared:  0.602
## F-statistic: 45.62 on 2 and 57 DF,  p-value: 1.481e-12
```

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^\circ\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$

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- In this example we have:

$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

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- $\widehat{\beta}_2$: A one-degree increase in mean temperature is associated with a -0.06 change in average log incomes conditional on strength of property rights

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i for a fixed value of Z_i.

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Fitted values and residuals

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- To answer this, we first need to redefine some terms from simple linear regression.
- Fitted values for $i = 1, \dots, n$:

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- Residuals for $i = 1, \dots, n$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i$$

Least squares is still least squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?

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- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?
- Minimize the sum of the squared residuals, just like before:

$$(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2) = \underset{b_0, b_1, b_2}{\arg\min} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i - b_2 Z_i)^2$$

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- The calculus is the same as last week, with 3 partial derivatives instead of 2

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- The calculus is the same as last week, with 3 partial derivatives instead of 2
- Let's start with a simple recipe and then rigorously show that it holds

OLS estimator recipe using two steps

- "Partialling out" OLS recipe:

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  2. Calculate residuals from this regression:
  $$\widehat{r}_{xz,i} = X_i - \widehat{X}_i$$

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  2. Calculate residuals from this regression:
  $$\widehat{r}_{xz,i} = X_i - \widehat{X}_i$$
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  $$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 \widehat{r}_{xz,i}$$
- Estimate of $\widehat{\beta}_1$ will be the same as running:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

Regression property rights on mean temperature

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  9.95678    0.82015  12.140  < 2e-16 ***
## meantemp    -0.14900    0.03469  -4.295 6.73e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.321 on 58 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.2413, Adjusted R-squared:  0.2282
## F-statistic: 18.45 on 1 and 58 DF,  p-value: 6.733e-05
```

Regression of log income on the residuals

```
## (Intercept)  avexpr.res
##   8.0542783   0.4056757

## (Intercept)      avexpr    meantemp
##  6.80627375  0.40567575 -0.06024937
```

Residual/partial regression plot

Residual/partial regression plot

Useful for plotting the conditional relationship between property rights and income given temperature:

Residual/partial regression plot

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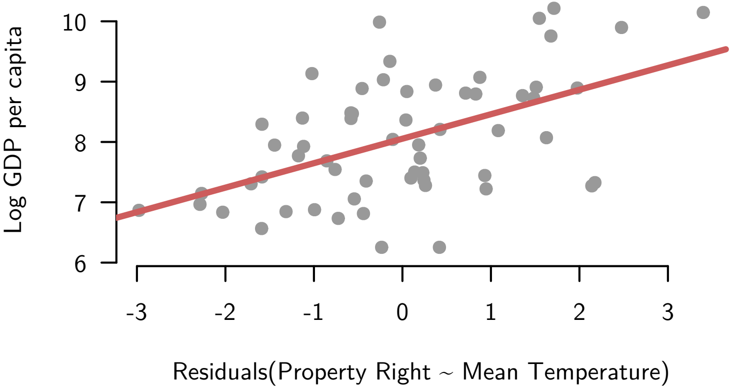

Deriving the Linear Least Squares Estimator

- In simple regression, we chose $(\widehat{\beta}_0, \widehat{\beta}_1)$ to minimize the sum of the squared residuals

Deriving the Linear Least Squares Estimator

- In simple regression, we chose $(\widehat{\beta}_0, \widehat{\beta}_1)$ to minimize the sum of the squared residuals
- We use the same principle for picking $(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2)$ for regression with two regressors (x_i and z_i):

$$\begin{aligned}(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) &= \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} \widehat{u}_i^2 = \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \\ &= \underset{\tilde{\beta}_0,\tilde{\beta}_1,\tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - x_i\tilde{\beta}_1 - z_i\tilde{\beta}_2)^2\end{aligned}$$

Deriving the Linear Least Squares Estimator

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$$
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\end{aligned}
$$

- (The same works more generally for k regressors, but this is done more easily with matrices as we will see next week)

Deriving the Linear Least Squares Estimator

We want to minimize the following quantitity with respect to $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)$:

$$S(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2) = \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - \tilde{\beta}_1 x_i - \tilde{\beta}_2 z_i)^2$$

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1. Take the partial derivatives of S with respect to $\tilde{\beta}_0, \tilde{\beta}_1$ and $\tilde{\beta}_2$.
2. Set each of the partial derivatives to 0 to obtain the first order conditions.
3. Substitute $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ for $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2$ and solve for $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ to obtain the OLS estimator.

First Order Conditions

Setting the partial derivatives equal to zero leads to a system of 3 linear equations in 3 unknowns: $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$

$$
\begin{aligned}
\frac{\partial S}{\partial \tilde{\beta}_0} &= \sum_{i=1}^{n} (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
\frac{\partial S}{\partial \tilde{\beta}_1} &= \sum_{i=1}^{n} x_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
\frac{\partial S}{\partial \tilde{\beta}_2} &= \sum_{i=1}^{n} z_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0
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When will this linear system have a unique solution?

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\end{aligned}$$

When will this linear system have a unique solution?

- More observations than predictors (i.e. $n > 2$)
- x and z are linearly independent, i.e.,
  - neither x nor z is a constant
  - x is not a linear function of z (or vice versa)
- Wooldridge calls this assumption no perfect collinearity

The OLS Estimator

The OLS estimator for $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ can be written as

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} - \hat{\beta}_2 \bar{z} \\
\hat{\beta}_1 &= \frac{Cov(x,y)Var(z) - Cov(z,y)Cov(x,z)}{Var(x)Var(z) - Cov(x,z)^2} \\
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Condition fails if:

1. If x or z is a constant ($\Rightarrow Var(x)Var(z) = Cov(x,z) = 0$)
2. One explanatory variable is an exact linear function of another ($\Rightarrow\ Cor(x,z) = 1\ \Rightarrow\ Var(x)Var(z) = Cov(x,z)^2$)

"Partialling Out" Interpretation of the OLS Estimator

Assume $Y = \beta_0 + \beta_1 X + \beta_2 Z + u$. Another way to write the OLS estimator is:

$$\hat{\beta}_1 = \frac{\sum_i^n \hat{r}_{xz,i}\, y_i}{\sum_i^n \hat{r}_{xz,i}^2}$$

where $\hat{r}_{xz,i}$ are the residuals from the regression of X on Z:

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- That is, same as the simple regresson of Y on X alone.

Origin of the Partial Out Recipe

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- Can use same equation with k explanatory variables; $\hat{r}_{xz}$ will then come from a regression of X on all the other explanatory variables.

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4. Zero conditional mean error
$$\mathbb{E}[u_i | X_i, Z_i] = 0$$

New assumption

Assumption 3: No perfect collinearity

(1) No explanatory variable is constant in the sample and (2) there are no exactly linear relationships among the explanatory variables.

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- What's the correlation between Z_i and X_i? 1!

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- These two variables are perfectly collinear.
- What about the following:
  - $X_i =$ income
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- Do we have to worry about collinearity here?
- No! Because while Z_i is a deterministic function of X_i, it is not a linear function of X_i.

R and perfect collinearity

- R, and all other packages, will drop one of the variables if there is perfect collinearity:

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```
##
## Coefficients: (1 not defined because of singularities)
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.71638    0.08991  96.941  < 2e-16 ***
## africa      -1.36119    0.16306  -8.348 4.87e-14 ***
## nonafrica         NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9125 on 146 degrees of freedom
##   (15 observations deleted due to missingness)
## Multiple R-squared:  0.3231, Adjusted R-squared:  0.3184
## F-statistic: 69.68 on 1 and 146 DF,  p-value: 4.87e-14
```

Perfect collinearity example (II)

- Another example:

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  - X_i = mean temperature in Celsius

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  - $Z_i = 1.8X_i + 32$ (mean temperature in Fahrenheit)

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```
## (Intercept)    meantemp  meantemp.f
##  10.8454999  -0.1206948          NA
```

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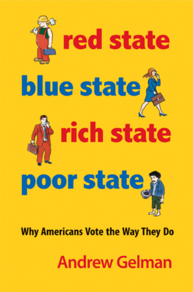
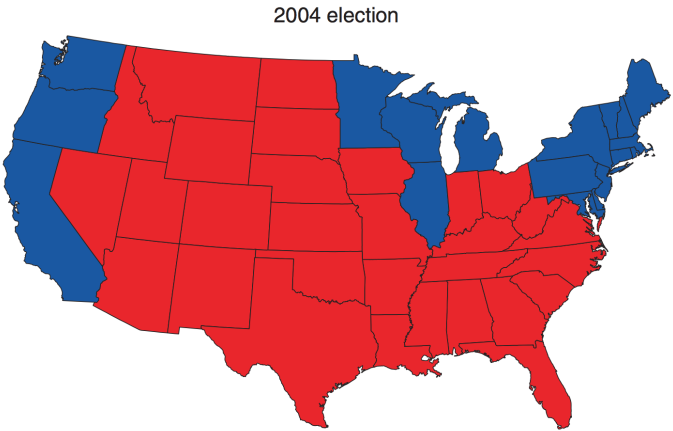
- $\rightsquigarrow$ small adjustments to the critical values and the t-values for our hypothesis tests and confidence intervals.

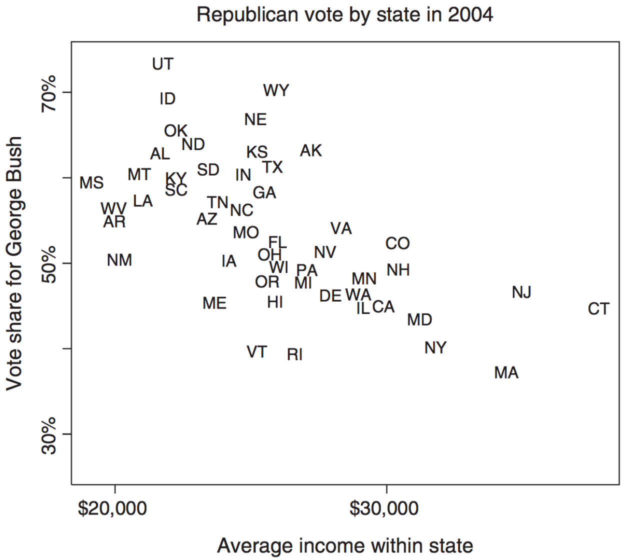
Red State Blue State



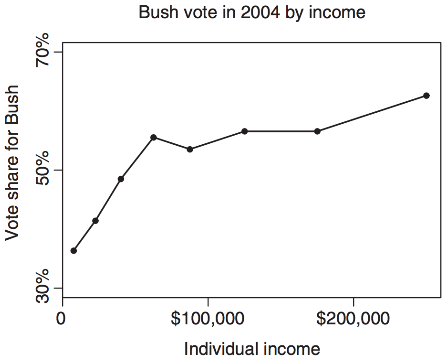
Red and Blue States

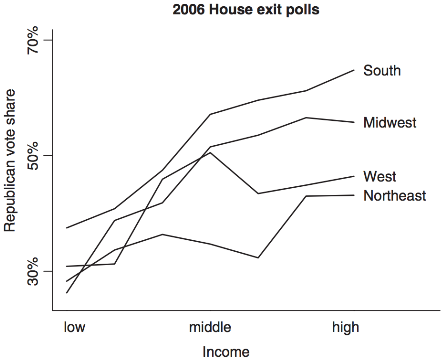


Rich States are More Democratic

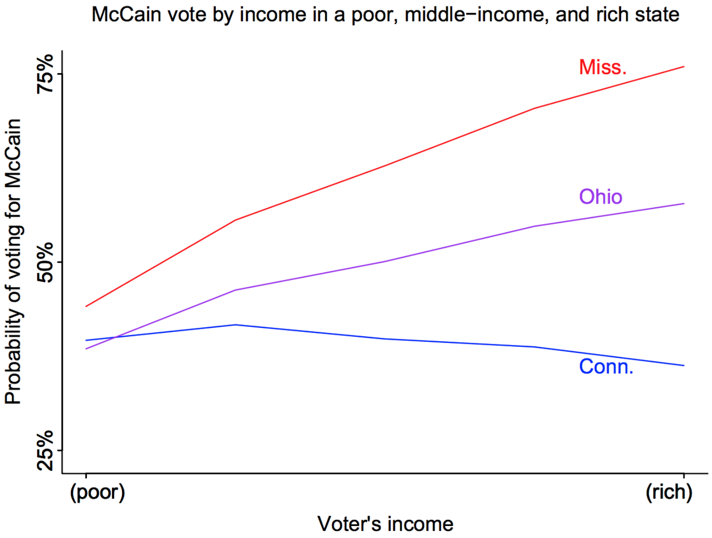


But Rich People are More Republican

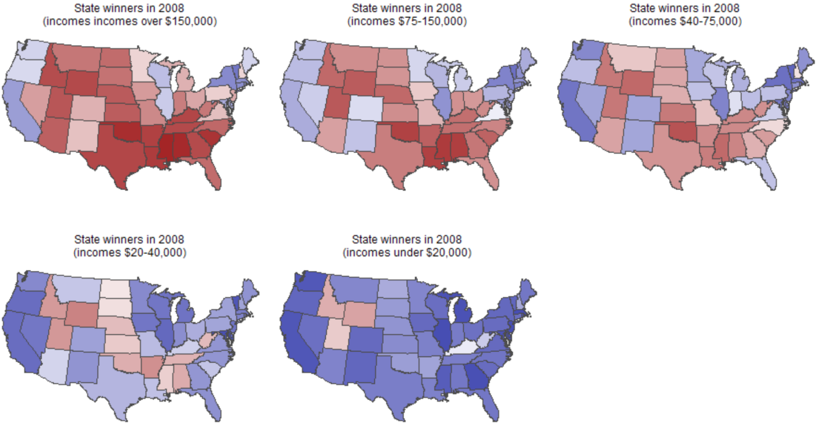




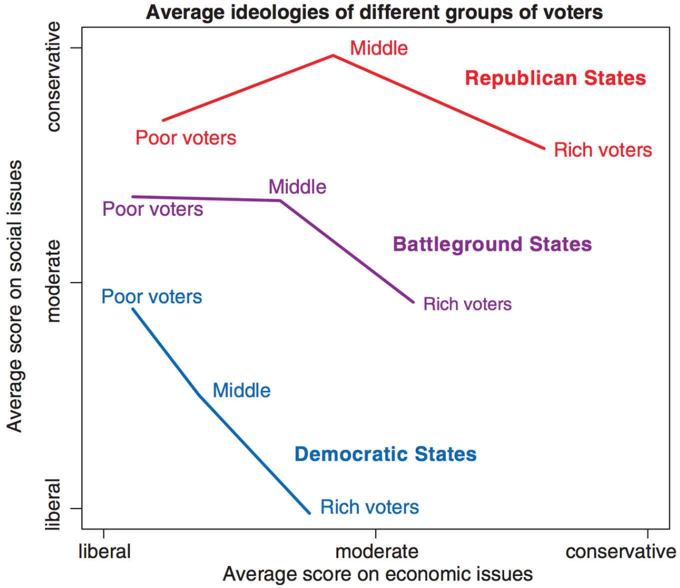
Paradox Resolved



If Only Rich People Voted, it Would Be a Landslide



A Possible Explanation



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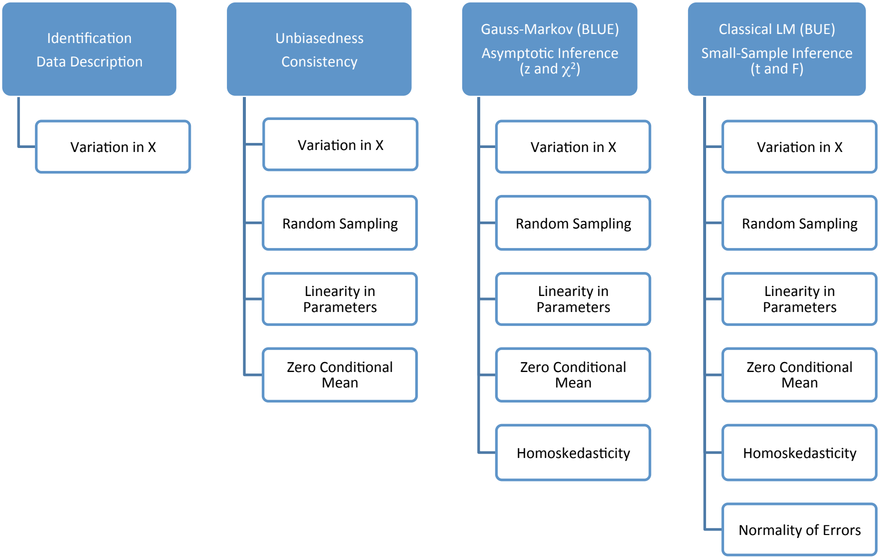
Questions?

Remember This?



Unbiasedness revisited

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True Population Model:

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Answer: $\tilde{\beta}_1$ is upward biased since being a strong republican is positively correlated with both watching fox news and voting republican. We have $\beta_1 < \tilde{\beta}_1$.

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Answer: The negative coefficient $\tilde{\beta}_1$ is downward biased compared to the true β_1 so $\beta_1 > \tilde{\beta}_1$. Being hospitalized is negatively correlated with health, and health is positively correlated with survival.

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A. If $\tilde{\delta} = 0$ or $\hat{\beta}_2 = 0$.

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We take expectations to see what the bias will be:

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$$\text{Bias}[\tilde{\beta}_1 \mid X] = E[\tilde{\beta}_1 \mid X] - \beta_1 = \beta_2 \cdot \tilde{\delta}$$

So the bias depends on the relationship between x_2 and x_1, our $\tilde{\delta}$, and the relationship between x_2 and y, our β_2.

Omitted Variable Bias: Simple Case

We take expectations to see what the bias will be:

$$
\begin{aligned}
\tilde{\beta}_1 &= \hat{\beta}_1 + \hat{\beta}_2 \cdot \tilde{\delta} \\
E[\tilde{\beta}_1 \mid X] &= E[\hat{\beta}_1 + \hat{\beta}_2 \cdot \tilde{\delta} \mid X] \\
&= E[\hat{\beta}_1 \mid X] + E[\hat{\beta}_2 \mid X] \cdot \tilde{\delta} \text{ ($\tilde{\delta}$ nonrandom given x)} \\
&= \beta_1 + \beta_2 \cdot \tilde{\delta} \text{ (given assumptions 1-4)}
\end{aligned}
$$

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So the bias depends on the relationship between x_2 and x_1, our $\tilde{\delta}$, and the relationship between x_2 and y, our β_2.

Any variable that is correlated with an included X and the outcome Y is called a confounder.

Omitted Variable Bias: Simple Case

Direction of the bias of $\tilde{\beta}_1$ compared to β_1 is given by:

	$\text{cov}(X_1, X_2) > 0$	$\text{cov}(X_1, X_2) < 0$	$\text{cov}(X_1, X_2) = 0$
$\beta_2 > 0$	Positive bias	Negative Bias	No bias
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Further points:

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- In the more general case with more than two covariates the bias is more difficult to discern. It depends on all the pairwise correlations.

Including an Irrelevant Variable: Simple Case

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True Population Model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u \quad \text{where} \quad \beta_2 = 0$$

and Assumptions I–IV hold.

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Q: Which statement is correct?

1. $\beta_1 > \tilde{\beta}_1$
2. $\beta_1 < \tilde{\beta}_1$
3. $\beta_1 = \tilde{\beta}_1$
4. Can't tell

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and thus including the irrelevant variable does not generally affect the unbiasedness. The sampling distribution of $\hat{\beta}_2$ will be centered about zero.

Sampling variance for simple linear regression

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- What happens with perfect collinearity? $R_1^2 = 1$ and the variances are infinite.

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- Given the symmetry, it will also increase $\text{var}(\widehat{\beta}_2)$ as well.

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- Low variation in an independent variable (here, $\widehat{r}_{xz,i}$) $\rightsquigarrow$ high SEs
- Basically, there is less residual variation left in X_i after "partialling out" the effect of Z_i

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In R, `vif()` in the `car` package.

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- If X_1 and X_2 are almost the same, why would you want a unique β_1 and a unique β_2? Think about how you would interpret that?
- Relax, you got way more important things to worry about!
- If possible, get more data
- Drop one of the variables, or combine them
- Or maybe linear regression is not the right tool

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  - E.g. does the effect of education differ by gender?

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- Let's regress GDP on this dummy variable and a constant:
  $Y = \beta_0 + \beta_1 D + u$

Example: GDP per capita on Electoral System

R Code

```
> summary(lm(REALGDPCAP ~ MAJORITARIAN, data = D))

Call:
lm(formula = REALGDPCAP ~ MAJORITARIAN, data = D)

Residuals:
   Min     1Q Median     3Q    Max
 -5982  -4592  -2112   4293  13685

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    7097.7      763.2    9.30 1.64e-14 ***
MAJORITARIAN  -1053.8     1224.9   -0.86    0.392
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5504 on 83 degrees of freedom
Multiple R-squared: 0.008838,       Adjusted R-squared: -0.003104
F-statistic: 0.7401 on 1 and 83 DF,  p-value: 0.3921
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```
> gdp.pro <- D$REALGDPCAP[D$MAJORITARIAN == 0]
> summary(gdp.pro)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   1116    2709    5102    7098   10670   20780

> gdp.maj <- D$REALGDPCAP[D$MAJORITARIAN == 1]
> summary(gdp.maj)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  530.2  1431.0  3404.0  6044.0 11770.0 18840.0
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So this is just like a difference in means two sample t-test!

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> gdp.pro <- D$REALGDPCAP[D$MAJORITARIAN == 0]
> summary(gdp.pro)
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
   1116    2709    5102    7098   10670   20780

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  530.2  1431.0  3404.0  6044.0 11770.0 18840.0
```

So this is just like a difference in means two sample t-test!

Example: GDP per capita on Electoral System

R Code

```
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    7097.7      763.2    9.30 1.64e-14 ***
MAJORITARIAN  -1053.8     1224.9   -0.86    0.392
```

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Dummy Variables for Multiple Categories

- More generally, let's say X measures which of m categories each unit i belongs to. E.g. the type of electoral system or region of country i is given by:
  - $X_i \in \{Proportional, Majoritarian\}$ so $m = 2$
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- The omitted category is our baseline case (also called a reference category) against which we compare the conditional means of Y for the other $m - 1$ categories.

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Our regression equation is:

$$Y = \beta_0 + \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + u$$

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- Interactions often confuses researchers and mistakes in use and interpretation occur frequently (even in top journals)

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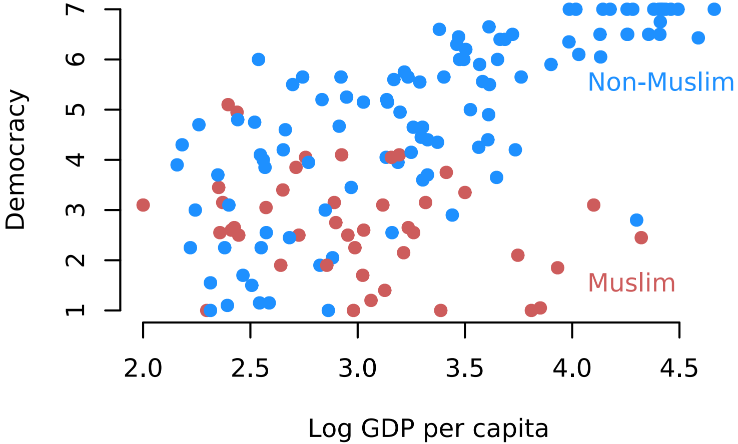
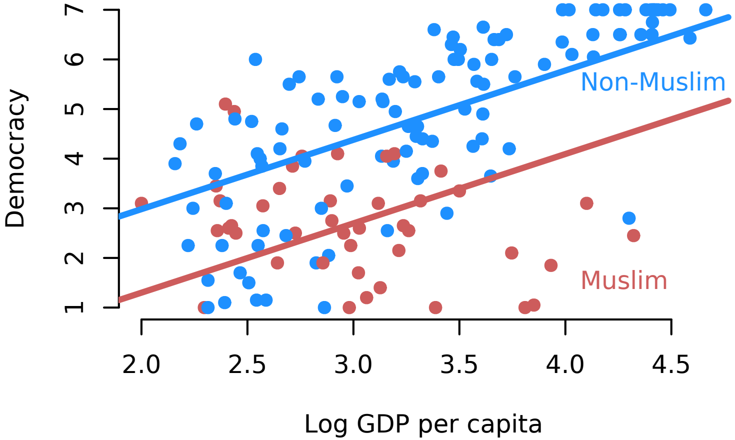
- We measure democracy with a Freedom House score, 1 (less free) to 7 (more free)

Let's see the data



Fish argues that Muslim countries are less likely to be democratic no matter their economic development

Controlling for Religion Additively



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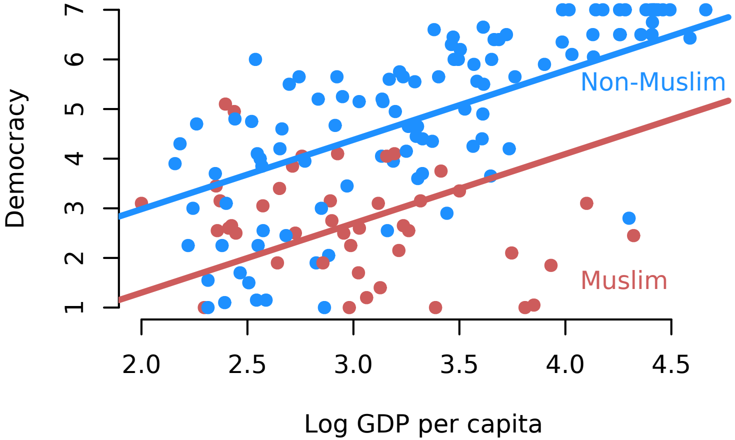

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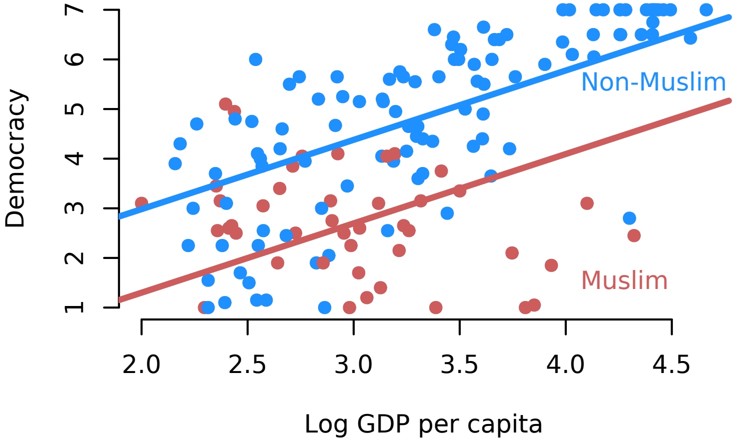

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Can we allow for different slopes for each group?

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- This covariate is called an interaction term and it is the product of the two marginal variables of interest: $income_i \times muslim_i$
- Here is the model with the interaction term:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

Two lines in one regression

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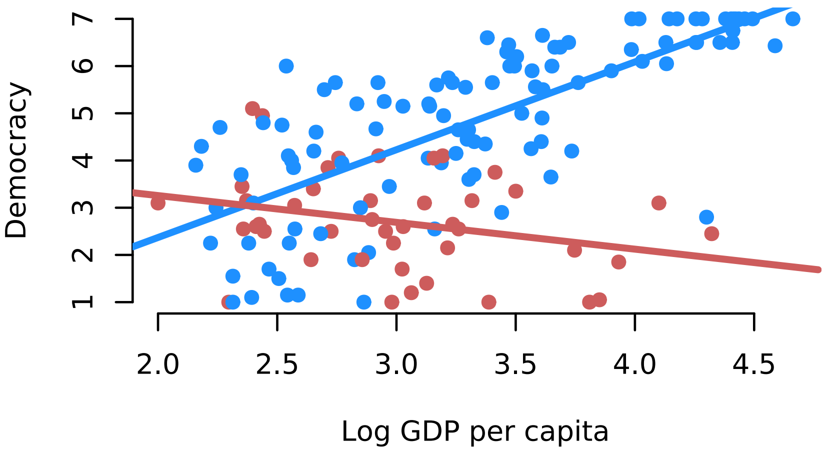

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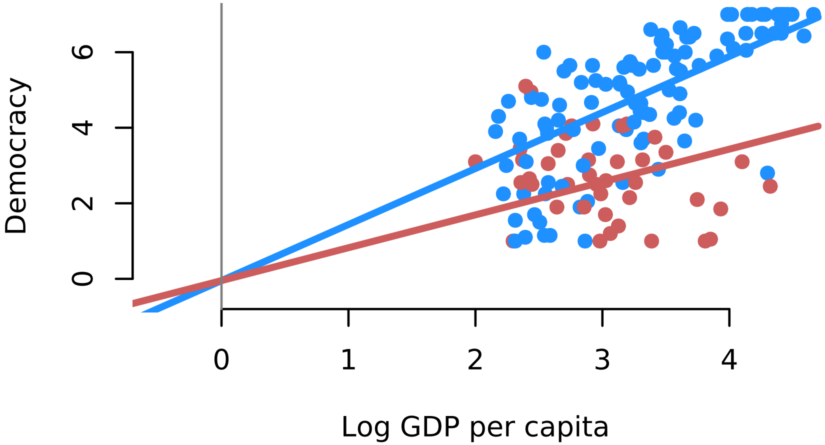

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$Z_i = 5$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 5$	$\widehat{\beta}_1 + \widehat{\beta}_3 \times 5$

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i when $Z_i = 0$.

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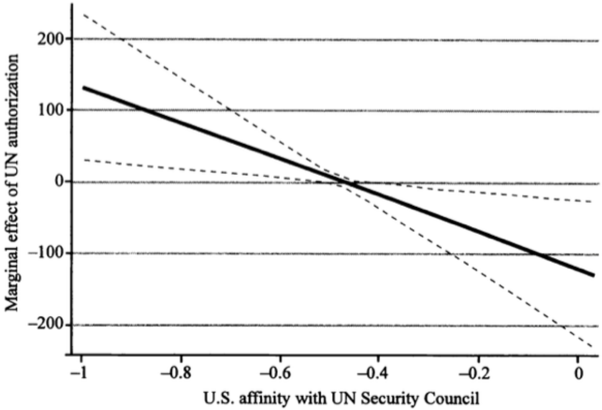
We will talk about checking these assumptions in a few weeks.

Example: Common Support

Chapman 2009 analysis

example and reanalysis from Hainmueller, Mummolo, Xu 2016



Note: Dashed lines give 95 percent confidence interval.

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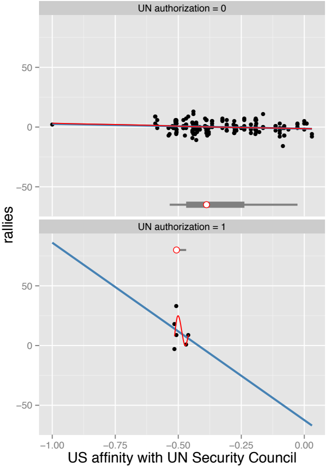

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Further Reading: Brambor, Clark, and Golder. 2006. Understanding Interaction Models: Improving Empirical Analyses. *Political Analysis* 14 (1): 63-82.

Hainmueller, Mummolo, Xu. 2016. How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice. *Working Paper*

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$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2 + u$$
$$Y = \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u$$
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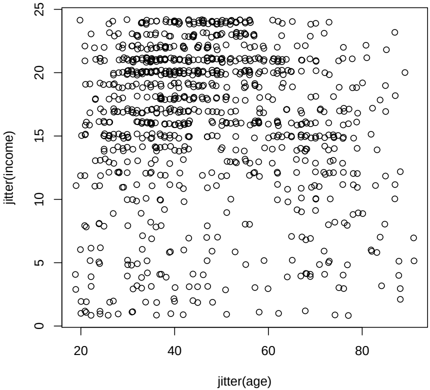
\end{aligned}
$$

- This is called a second order polynomial in X_1
- A third order polynomial is given by: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + u$

Polynomial Example: Income and Age

- Let's look at data from the U.S. and examine the relationship between Y: income and X: age



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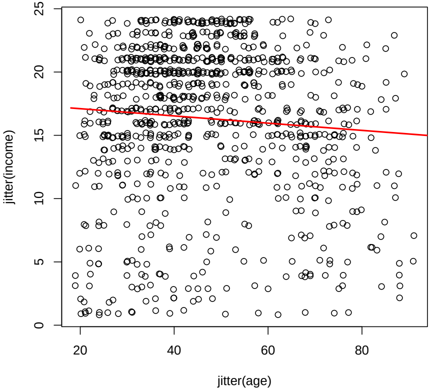

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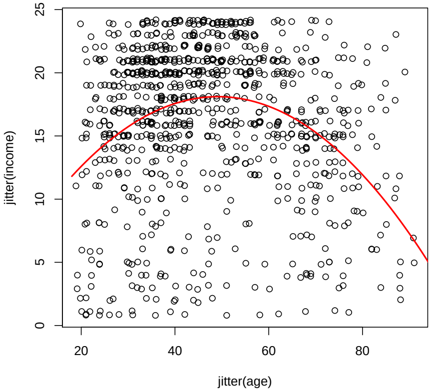
- Let's look at data from the U.S. and examine the relationship between Y: income and X: age
- We see that a simple linear specification does not fit the data very well: $Y = \beta_0 + \beta_1 X_1 + u$
- A second order polynomial in age fits the data a lot better: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$



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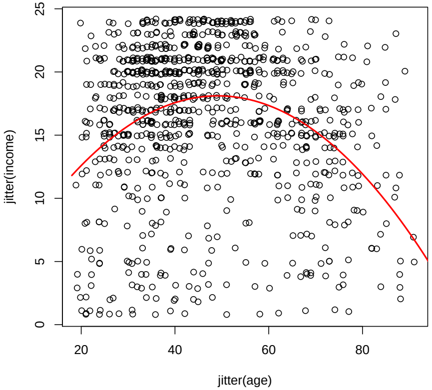

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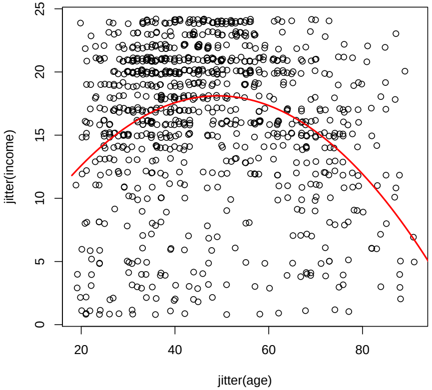
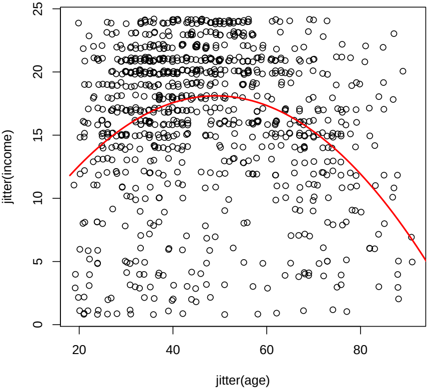


Polynomial Example: Income and Age

- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$
- Is β_1 the marginal effect of age on income?
- No! The marginal effect of age depends on the level of age: $\frac{\partial Y}{\partial X_1} = \widehat{\beta}_1 + 2\,\widehat{\beta}_2\,X_1$
  Here the effect of age changes monotonically from positive to negative with income.



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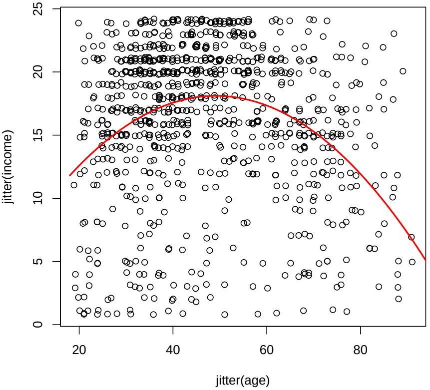
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.



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- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$
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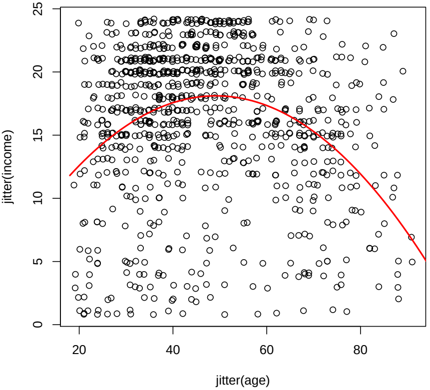
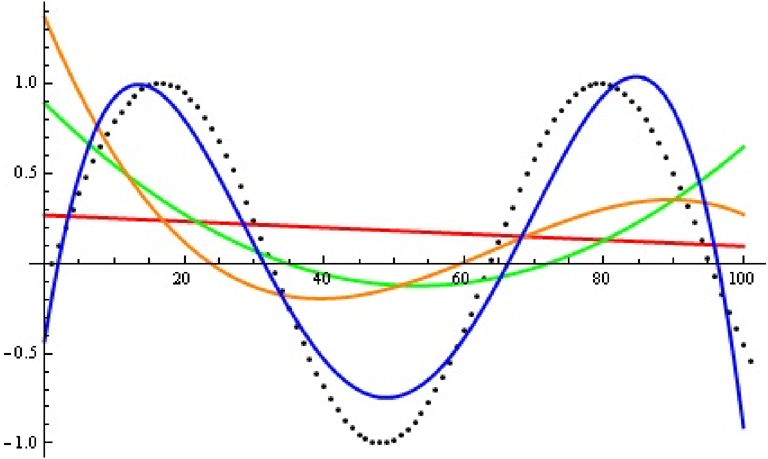
- No! The marginal effect of age depends on the level of age: $\frac{\partial Y}{\partial X_1} = \widehat{\beta}_1 + 2\,\widehat{\beta}_2\, X_1$ Here the effect of age changes monotonically from positive to negative with income.
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $|\frac{\beta_1}{2\beta_2}|$. Here turning point is at $X_1 = 50$.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

Conclusion

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Conclusion

In this brave new world with 2 independent variables:

1. β's have slightly different interpretations
2. OLS still minimizing the sum of the squared residuals
3. Small adjustments to OLS assumptions and inference
4. Adding or omitting variables in a regression can affect the bias and the variance of OLS
5. We can optionally consider interactions, but must take care to interpret them correctly

Next Week

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- OLS in its full glory

Next Week

- OLS in its full glory
- Reading:
  - Practice up on matrices
  - Fox Chapter 9.1-9.4 (skip 9.1.1-9.1.2) Linear Models in Matrix Form
  - Aronow and Miller 4.1.2-4.1.4 Regression with Matrix Algebra
  - Optional: Fox Chapter 10 Geometry of Regression
  - Optional: Imai Chapter 4.3-4.3.3
  - Optional: Angrist and Pischke Chapter 3.1 Regression Fundamentals

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Breznau (2015) "The Missing Main Effect of Welfare State Regimes: A Replication of 'Social Policy Responsiveness in Developed Democracies."' *Sociological Science*.

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- Democracy empowers the masses, and that empowerment helps define social outcomes
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- but...they leave out a main effect.

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- They mean center so the 0 represents the average over the entire sample

What Happens?

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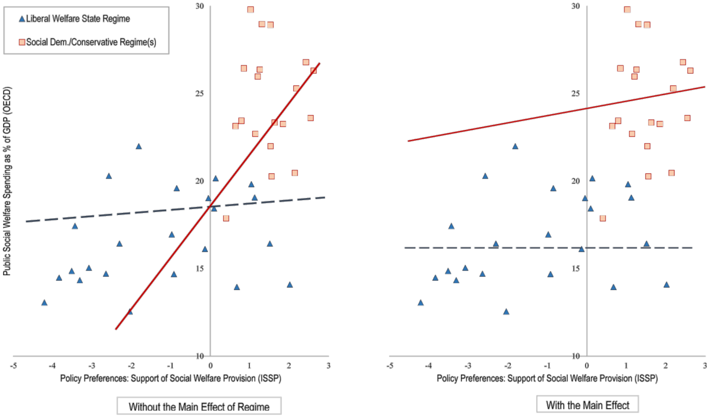

Figure 1: Predicted Regression Lines for the Effect of Policy Preferences on Social Welfare Spending, without and with the Main Effect of Regime

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<drops mic>

References

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