Week 9: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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Princeton

November 14, 16 and 21, 2016

[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Kevin Quinn.

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- Long Run
  - regression $\rightarrow$ diagnostics $\rightarrow$ causal inference

Questions?

Argument for Next Three Classes

Residuals are important. Look at them.

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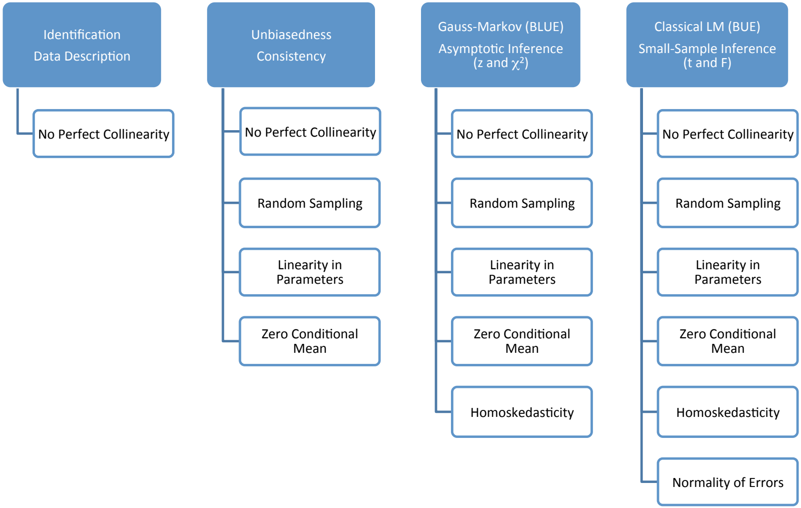

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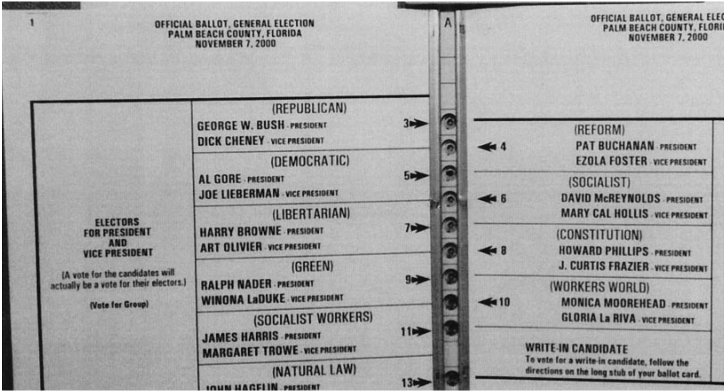
Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL's 25 electoral votes and the presidency.

FIGURE 1. The Palm Beach County Bufferfly Ballot



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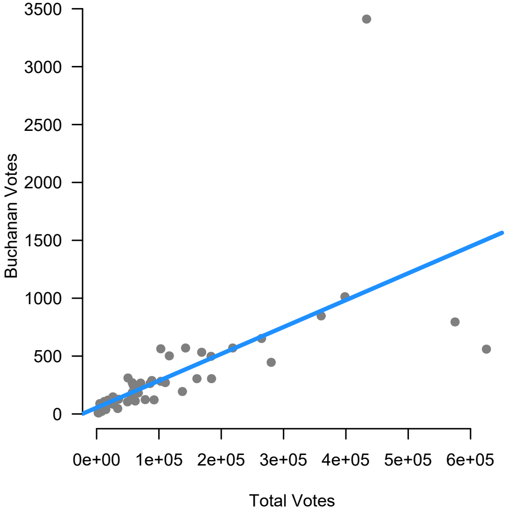

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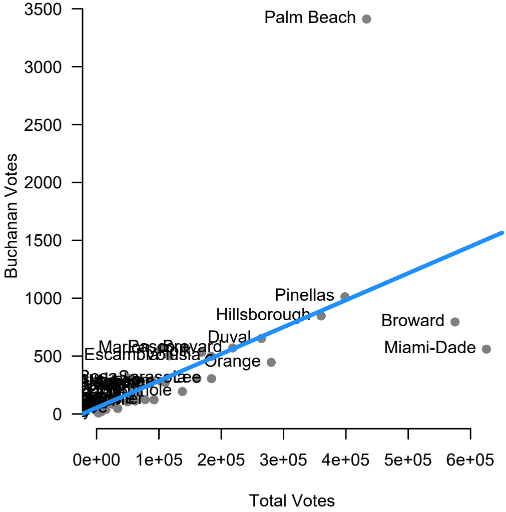

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.

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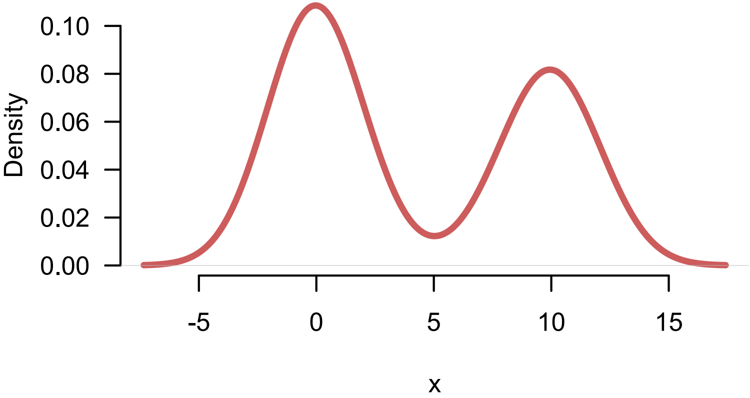
- The plausibility depends on the X chosen by the researcher.

Example: Is this a violation?



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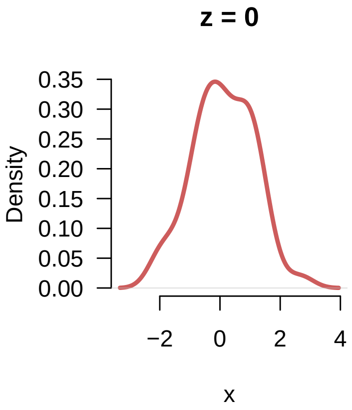

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Solution: Carefully investigate the residuals numerically and graphically.

To understand the relationship between residuals and errors, we need to derive the distribution of the residuals.

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  - $\mathbf{H}$ is idempotent: $\mathbf{H}\mathbf{H} = \mathbf{H}$

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- Note that the residual is a function of all of the errors

Distribution of the residuals

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$$\text{Var}[\hat{\mathbf{u}}] = \sigma_u^2(\mathbf{I} - \mathbf{H})$$

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The variance of the ith residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the ith diagonal element of the matrix $\mathbf{H}$ (called the hat value).

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These properties can obscure the true patterns in the error distribution, and thus are inconvenient for our diagnostics.

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}'_i$ are not independent. This makes the distribution of $\hat{u}'_i$ nonstandard.

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- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

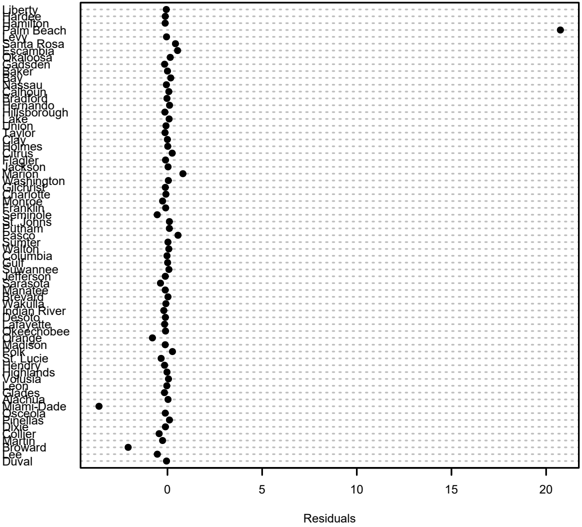
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min       1Q   Median       3Q      Max
-947.05  -41.74  -19.47   20.20 2350.54

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.423e+01  4.914e+01   1.104    0.274
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared: 0.4628,        Adjusted R-squared: 0.4545
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

> residuals              <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals  <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals



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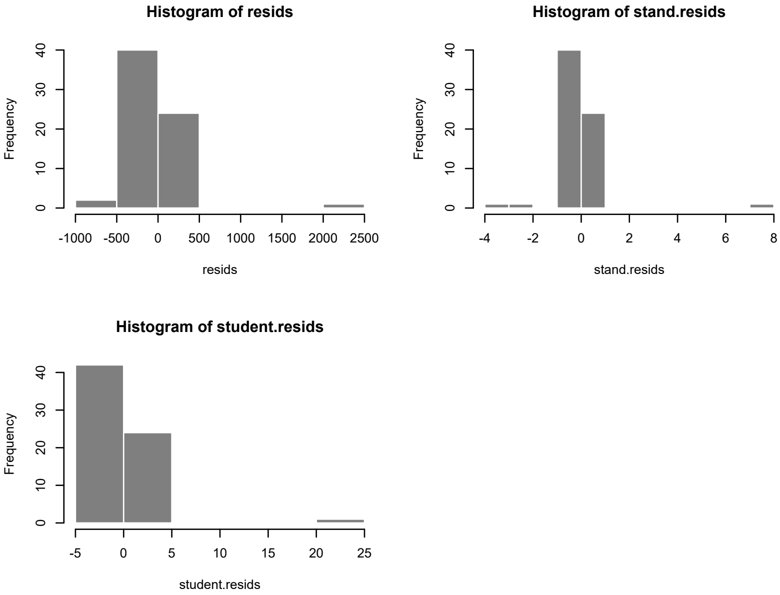

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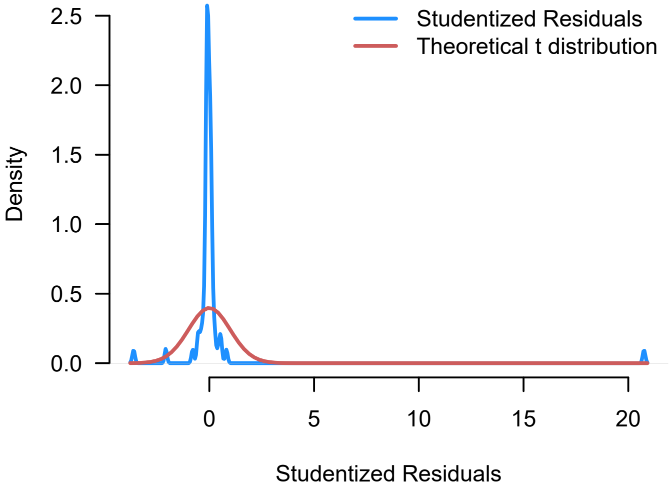

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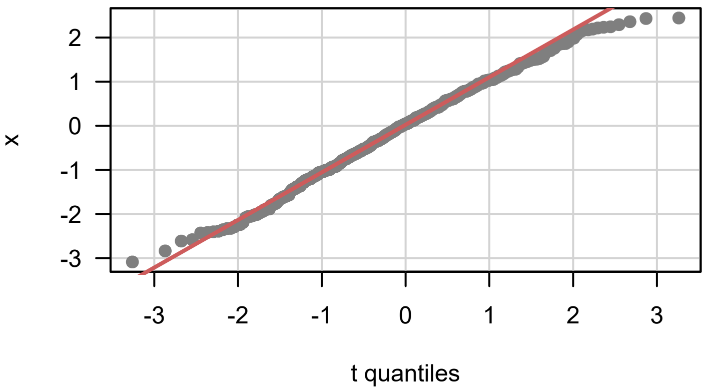
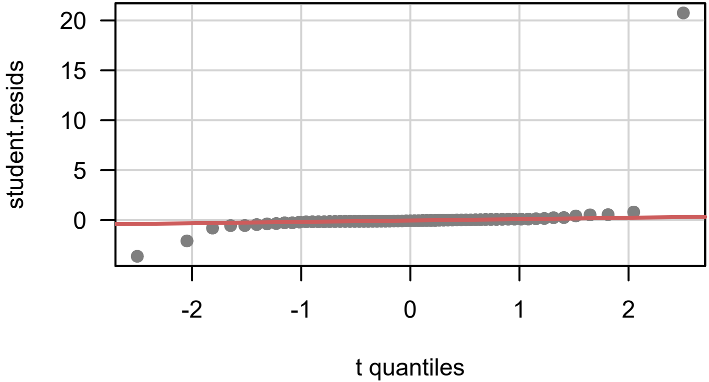
- For example, one point is the (m_x, m_y) where m_x is the median of the x distribution and m_y is the median for the y distribution
- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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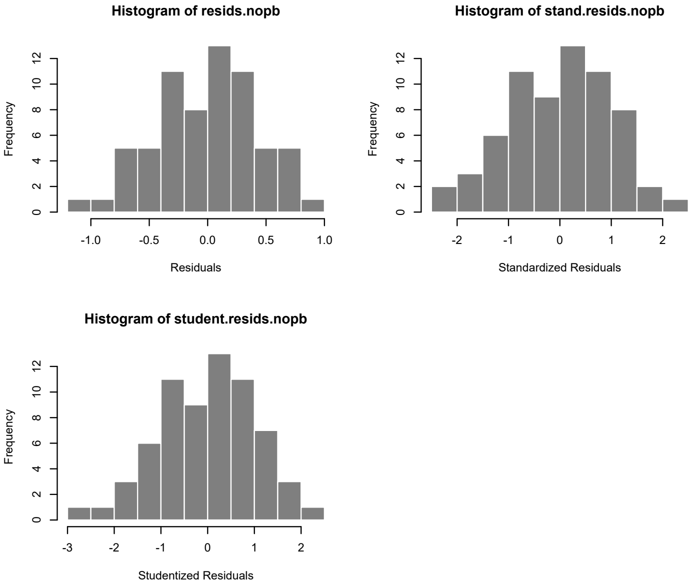
- Consider other causes (next two classes)

Buchanan revisited

Let's delete Palm Beach and also use log transformations for both variables

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -2.48597    0.37889  -6.561 1.09e-08 ***
## log(edaytotal)  0.70311    0.03621  19.417  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4362 on 64 degrees of freedom
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526
## F-statistic:   377 on 1 and 64 DF,  p-value: < 2.2e-16
```

Buchanan revisited



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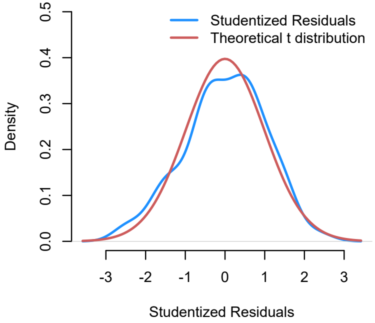

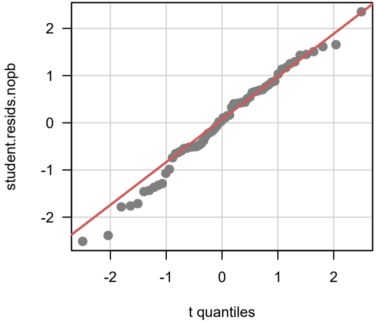

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- Jensen's inequality gives us information on this relation: $f(E[X]) \leq E[f(X)]$ for any convex function $f()$
- The results will in general be consistent which ensures that the bias decreases in sample size.

The trouble with Norway

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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814	-.192	-.278	.137
	(4.7)	(2.0)	(2.4)	(2.9)
Norway Obs Excluded	.641	-.068	-.138	.054
	(4.8)	(0.9)	(1.5)	(1.3)

Creative curve fitting with Norway

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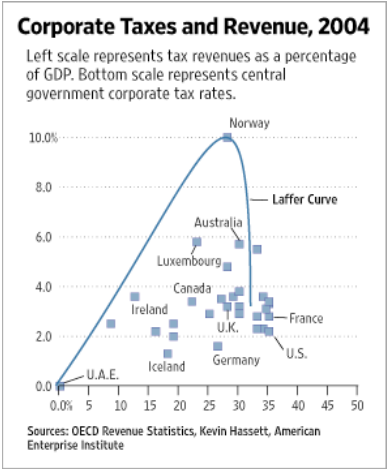

The Most Important Lesson: Check Your Data

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"Do not attempt to build a model on a set of poor data! In human surveys, one often finds 14-inch men, 1000-pound women, students with 'no' lungs, and so on. In manufacturing data, one can find 10,000 pounds of material in a 100 pound capacity barrel, and similar obvious errors.

All the planning, and training in the world will not eliminate these sorts of problems. In our decades of experience with 'messy data,' we have yet to find a large data set completely free of such quality problems."

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Always Carefully Examine the Data First!!

1. Examine summary statistics: `summary(data)`
2. Scatterplot matrix for densities and bivariate relationships:
   E.g. `scatterplotMatrix(data)` from `car` library.
3. Further conditional plots for multivariate data:
   E.g. use the `lattice` library or `ggplot2`

Three types of extreme values

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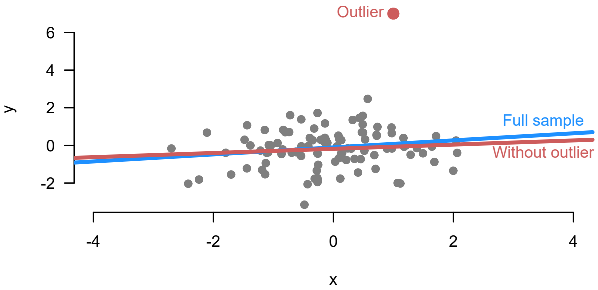
- Can be a violation of iid (not identically distributed)

Outlier definition



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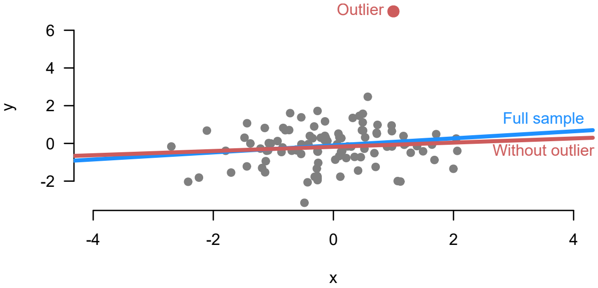

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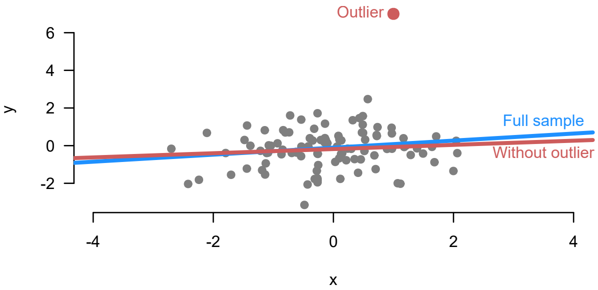

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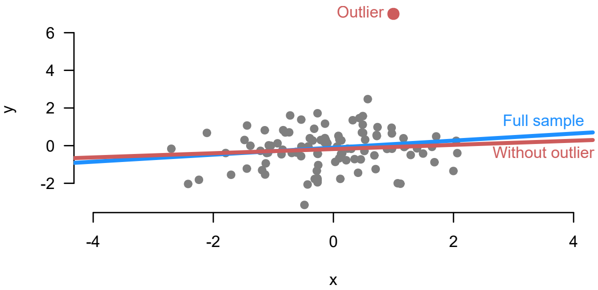

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- No bias if typical in the x's

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- $\widehat{\sigma} > \widehat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\widehat{u}_i' < \widehat{u}_i^*$

Cutoff rules for outliers

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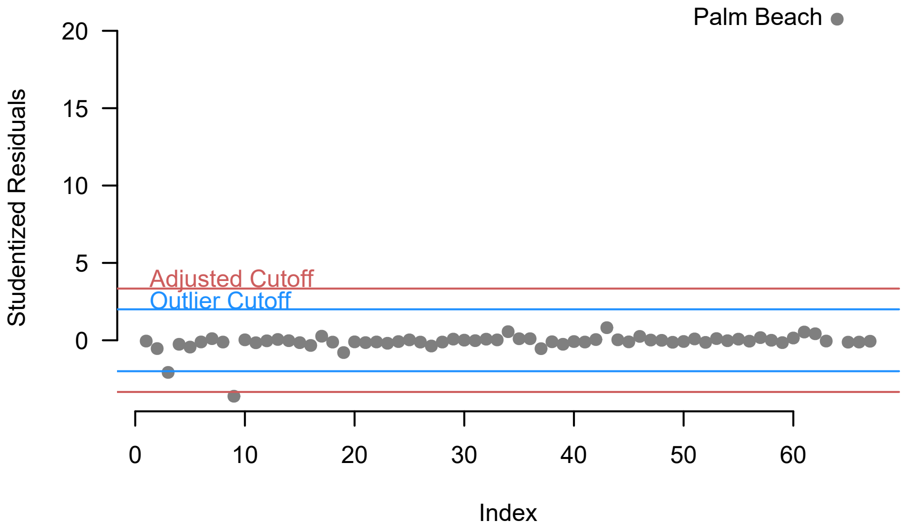
Buchanan outliers



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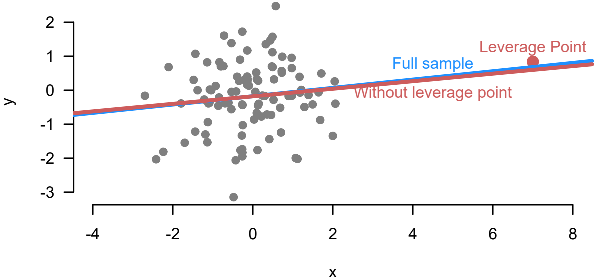
- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

Leverage point definition



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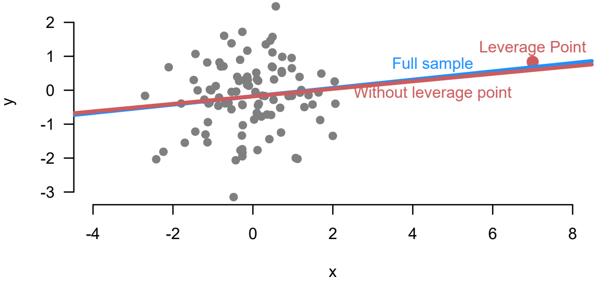

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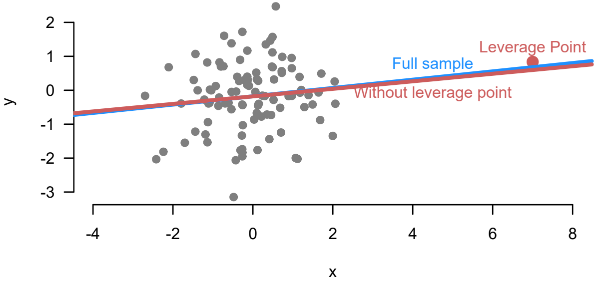

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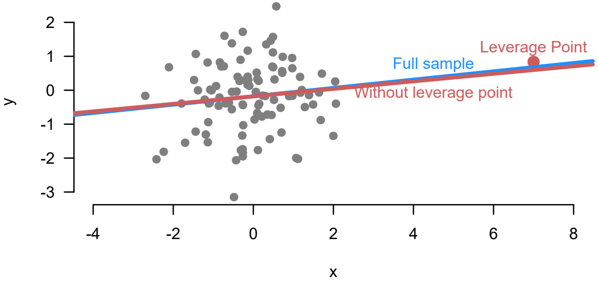

- Values that are extreme in the x direction
- That is, values far from the center of the covariate distribution
- Decrease SEs (more X variation)
- No bias if typical in y dimension

Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix $\mathbf{H}$:

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}$$

$\mathbf{H}$ is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

$$\hat{y}_i = \mathbf{h}_i'\mathbf{y} = \begin{bmatrix} h_{i,1} & h_{i,2} & \cdots & h_{i,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^{n} h_{i,j} y_j$$

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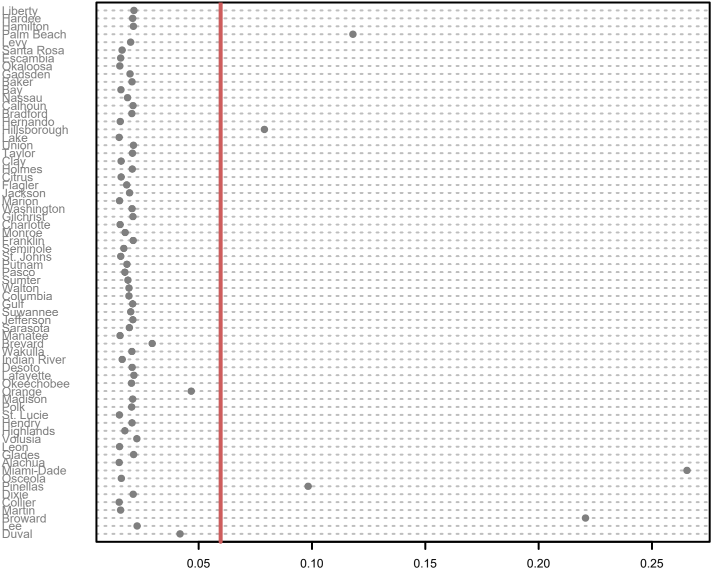
Buchanan hats



Influence points

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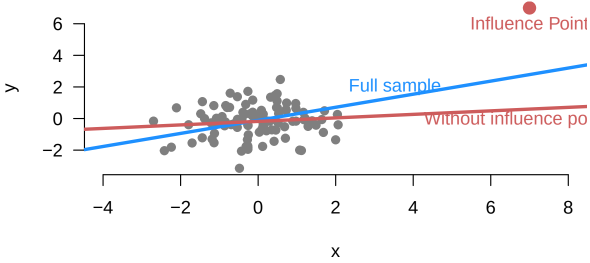

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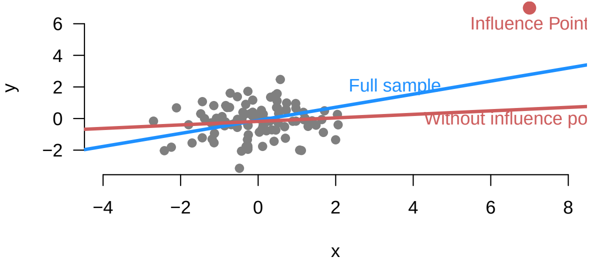

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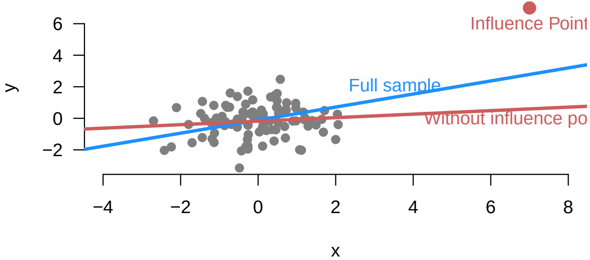

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- Causes the regression line to move toward it (bias?)

Detecting Influence Points/Bad Leverage Points

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- D_{ij} is called the DFbeta, which measures the influence of observation i on the estimated coefficient for the jth explanatory variable.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

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- In R: `dfbetas(model)`

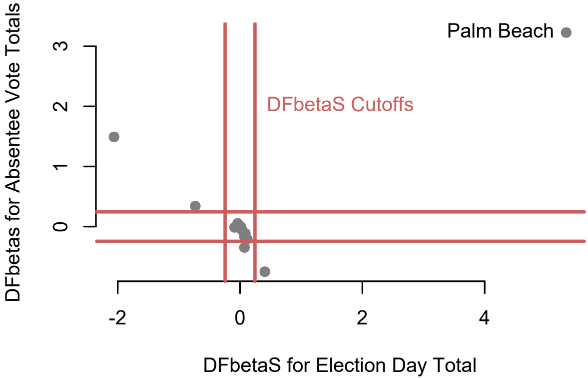
Buchanan influence

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal     1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
##   (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

```
##     (Intercept)     edaytotal  absnbuchanan
## 1  0.3454475146  0.4050504921 -0.7505222758
## 2 -0.0234266617 -0.0241000045 -0.0131672181
## 3  0.0650795039 -0.7319311820  0.3401669862
## 4 -0.0333980968  0.0133802934 -0.0087505576
## 5 -0.0397626659 -0.0073746223  0.0096551713
## 6 -0.0009277798  0.0001505476  0.0002210247
```

Buchanan influence



- Palm Beach county moves each of the coefficients by more than 3 standard errors!

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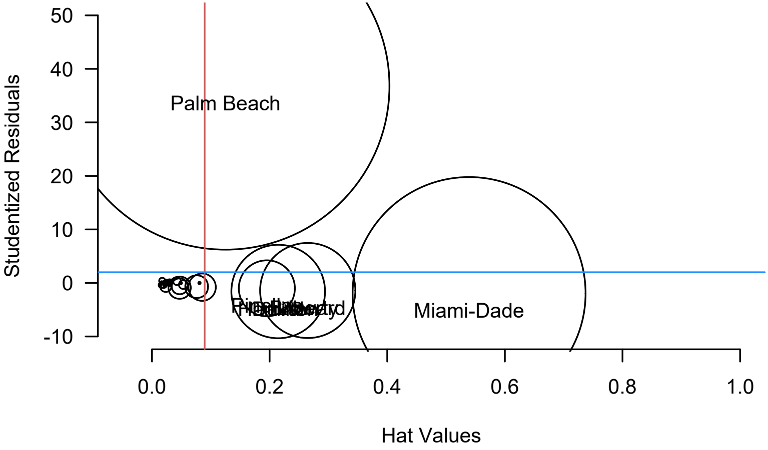
Influence Plot Buchanan



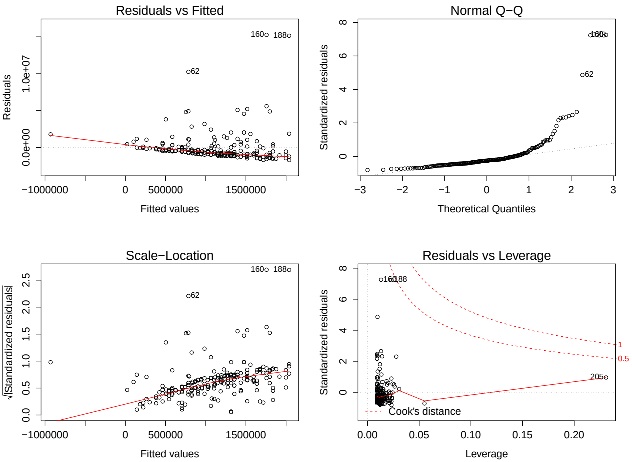
Code for Influence Plot

```
mod3 <- lm(edaybuchanan ~ edaytotal + absnbuchanan, data = flvote)
symbols(y = rstudent(mod3), x = hatvalues(mod3),
            circles = sqrt(cooks.distance(mod3)),
            ylab = "Studentized Residuals",
            xlab = "Hat Values", xlim = c(-0.05, 1),
            ylim = c(-10, 50), las = 1, bty = "n")
cutoffstud <- 2
cutoffhat <- 2 * (3)/nrow(flvote)
abline(v = cutoffhat, col = "indianred")
abline(h = cutoffstud, col = "dodgerblue")
filter <- rstudent(mod3) > cutoffstud | hatvalues(mod3) > cutoffhat
text(y = rstudent(mod3)[filter],
       x = hatvalues(mod3)[filter],
       flvote$county[filter], pos = 1)
```

A Quick Function for Standard Diagnostic Plots

R Code

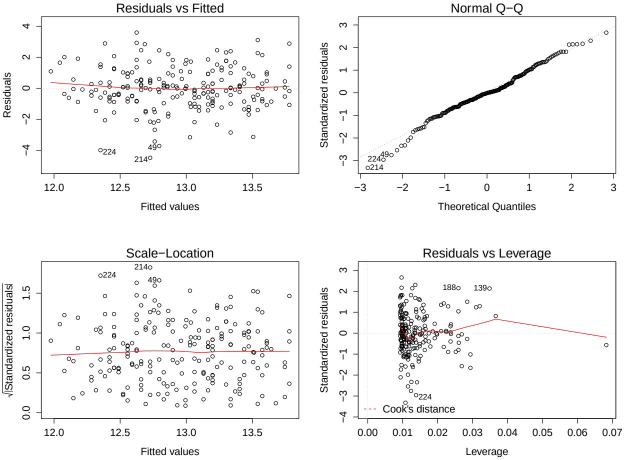
```
> par(mfrow=c(2,2))
> plot(mod1)
```



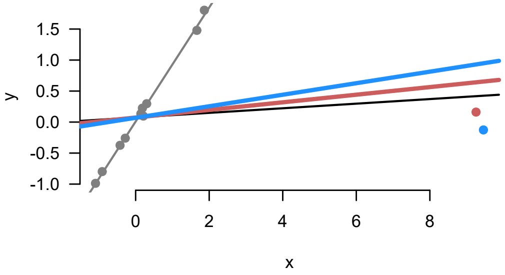
The Improved Model

R Code

```
> par(mfrow=c(2,2))
> plot(mod2)
```



Limitations of the standard tools



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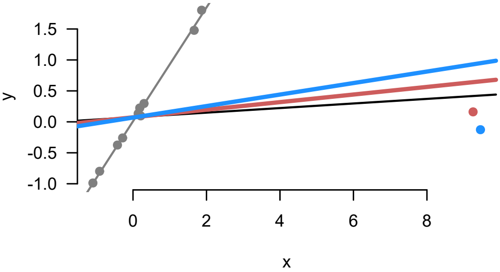

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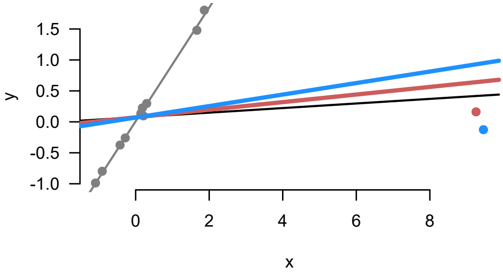

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point
- Neither of the “leave-one-out” approaches helps recover the line

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But What About Gauss-Markov and BLUE?

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- With normality assumption we get Best Unbiased Estimator (BUE) which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).

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Quotes from Rainey and Baissa (2015) presentation

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"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators" - Wooldridge (2013)

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"We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator"
- Gujarati (2004)

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- We can measure sensitivity with the influence function which measures change in estimator based on corruption in one datapoint.

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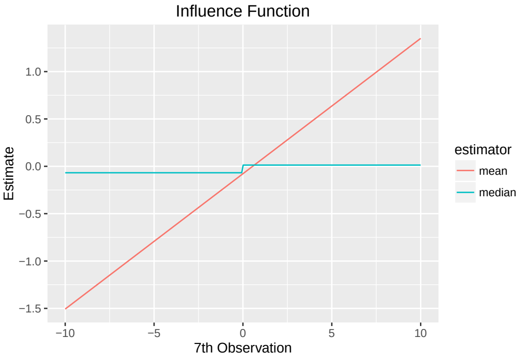

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- The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.

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- Other objectives include the Huber objective and Tukey's biweight objective which have different properties.
- Calculating robust M estimators often requires an iterative procedure and a careful initialization.

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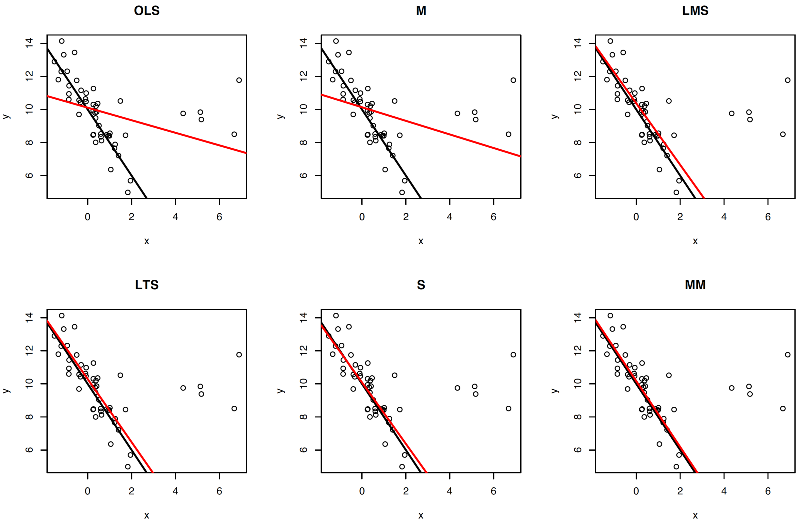
  - MM-estimator: with Huber's loss is what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for M-estimators but I would bootstrap it if possible as the asymptotics kick in slowly.

```
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 +  rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
```

Simulation Results



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- In most cases I personally would start with OLS, do diagnostics and then consider a robust alternative. If I don't have time for diagnostics, maybe robust is better from the outset.
- I highly recommend Baissa and Rainey (2016) "When BLUE is Not Best: Non-Normal Errors and the Linear Model" for more on this topic.
- The Fox textbook Chapter 19 is also quite good on this and points out to the key references

Appendix: Characterizing Estimator Robustness (formally)

Definition (Breakdown Point)

The breakdown point of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82)

Definition (Influence Function)

Let $F_p = (1-p)F + p\delta_{\mathbf{z}_0}$ where F is a probability measure, $\delta_{\mathbf{z}_0}$ is the point mass at $\mathbf{z}_0 \in \mathbb{R}^k$, and $p \in (0,1)$.
Let $T(\cdot)$ be a statistical functional. The influence function of T is

$$IF(\mathbf{z}_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}$$

The influence function is a function of $\mathbf{z}_0$ given T and F. It describes how T changes with small amounts of contamination at $\mathbf{z}_0$ (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).

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An S-estimator for the regression model is defined as the values of $\hat{\boldsymbol{\beta}}_S$ and s that minimize s subject to the constraint:

$$\frac{1}{n}\sum_{i=1}^{n}\rho\left(\frac{y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}_S}{s}\right) \geq K$$

where K is user-defined constant (typically set to 0.5) and $\rho : \mathbb{R} \to [0, 1]$ is a function with the following properties (Davies, 1990, p. 1653):

1. $\rho(0) = 1$
2. $\rho(u) = \rho(-u), u \in \mathbb{R}$
3. $\rho : \mathbb{R}_+ \to [0, 1]$ is nonincreasing, continuous at 0, and continuous on the left
4. for some $c > 0$, $\rho(u) > 0$ if $|u| < c$ and $\rho(u) = 0$ if $|u| > c$

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Good properties, but costly to compute (usually impossible to compute exactly).

Measurement Error

“It seems as if measurement error has been pushed into the role of the unwanted child whose existence we would rather deny. Maybe because measurement error is common, insipid, and unsophisticated. Unlike the hidden confounder challenging our intellect, to discover measurement error is a ‘no-brainer’ - it simply lurks everywhere. Our epidemiological fingerprints are contaminated with measurement error. Everything we observe, we observe with error. Since observation is our business, we would probably rather deny that what we observe is imprecise and maybe even inaccurate, but time has come to unveil the secret: measurement error is threatening our profession.”

Karen Michals (2001)

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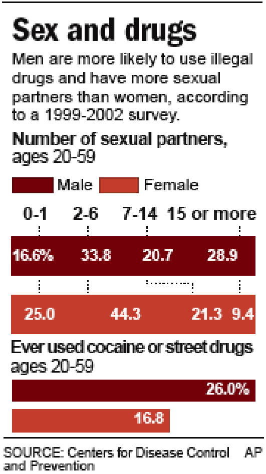
- Other variables, like gender, number of children, may be measured with less error

US Survey Data



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Let's assume that that the Gauss-Markov assumptions hold for the model with the true (but unobserved) variables so that OLS would be unbiased and consistent if we observed Y_i. Does the measurement error in Y_i^* cause any problems when fitting OLS to the observed data?

Measurement Error in the Dependent Variable

When we fit OLS to the observed data:

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- Since $V[U + E] = \sigma_U^2 + \sigma_E^2 > \sigma_U^2$ the estimates will be less precise

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- If $E[E_i] > 0$ the constant will be biased upwards and vice versa, but our slope estimates remain unbiased $E[\hat{\beta}] = \beta$
- Since $V[U + E] = \sigma_U^2 + \sigma_E^2 > \sigma_U^2$ the estimates will be less precise
- If E is correlated with X then OLS is inconsistent and biased.

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Does the measurement error cause any problems when fitting OLS to the observed data (i.e. using X^* instead of X)?

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- Note: This has nothing to do with assumptions about errors U , we always maintain that $Cov[X^*, U] = 0$ and $Cov[X, U] = 0$

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- Can we know the direction of the (asymptotic) bias?

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Bias is small if variance of observed measure σ_X^2 is large relative to variance of error term σ_E^2 (high signal to noise ratio).

Measurement Error in Multiple Independent Variables

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + u$$

- If only X_1 is measured with CEV type error (but X_2 and X_3 are correct), then β_1 exhibits attenuation bias, and β_2 and β_3 will be inconsistent and biased unless X_1 is uncorrelated with X_2 and X_3

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- the direction and magnitude of bias in X_2 and X_3 are not easy to derive and often unclear
- If we have CEV measurement error in multiple Xs then the size and direction of biases are unclear.

What can we do about measurement error?

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Note: This is true only under fairly strong assumptions including mean zero measurement error.

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- Don't let regression be a magic black box for you- understand why it is giving the answers it gives.

References

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Where We've Been and Where We're Going...

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Questions?

Residuals are still important. Look at them.

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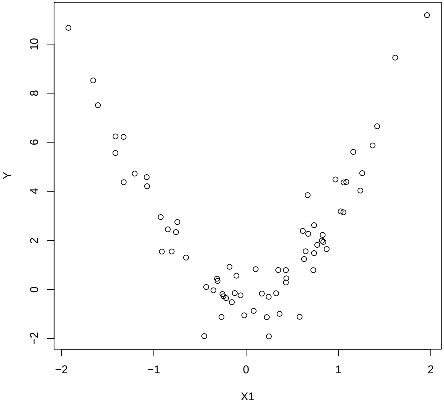

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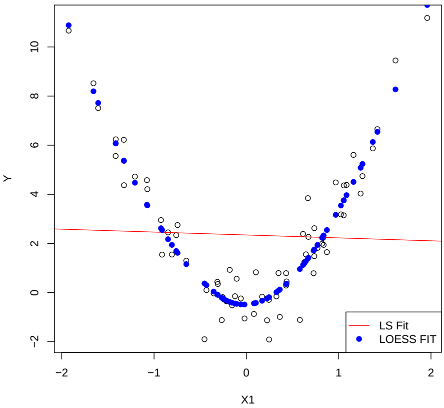

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- Usually we employ "linearity by default" but we should try to make sure this is appropriate: detect non-linearities and model them accurately

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  - Non-parametric multiple regression techniques (beyond the scope of this course)

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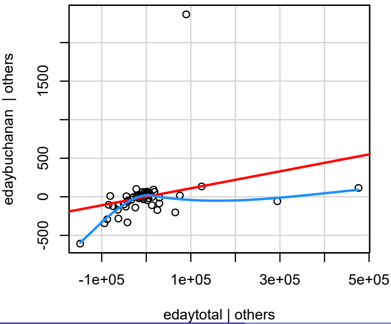
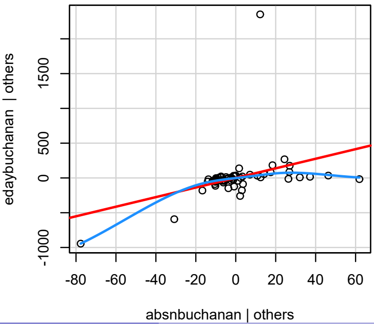
- Use local smoother (loess) to detect any non-linearity

Buchanan AV plot

R Code

```
par(mfrow = c(1,2))
out <- avPlots(mod3, "edaytotal")
lines(loess.smooth(x = out$edaytotal[,1],
 y= out$edaytotal[,2]), col = "dodgerblue", lwd = 2)
out2 <- avPlots(mod3, "absnbuchanan")
lines(loess.smooth(x = out2$absnbuchanan[,1],
y= out2$absnbuchanan[,2]), col = "dodgerblue", lwd = 2)
```





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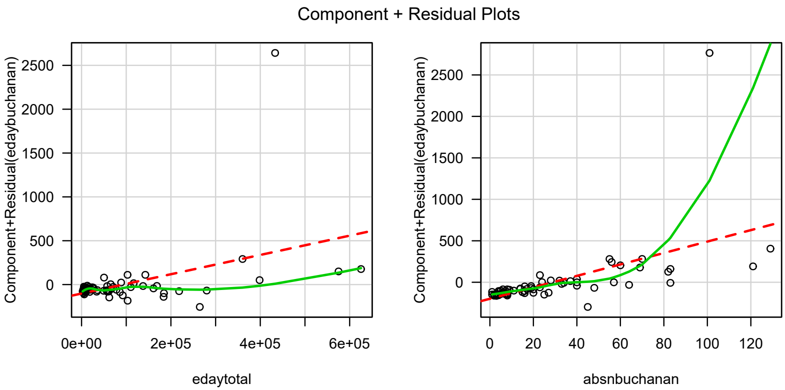
Buchanan CR plot

R Code

```
crPlots(mod3, las = 1)
```



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    - This suggests that linearizing the relationship between the Xs through transformations can be helpful
    - Experience suggests weak non-linearities among Xs do not invalidate CR plots

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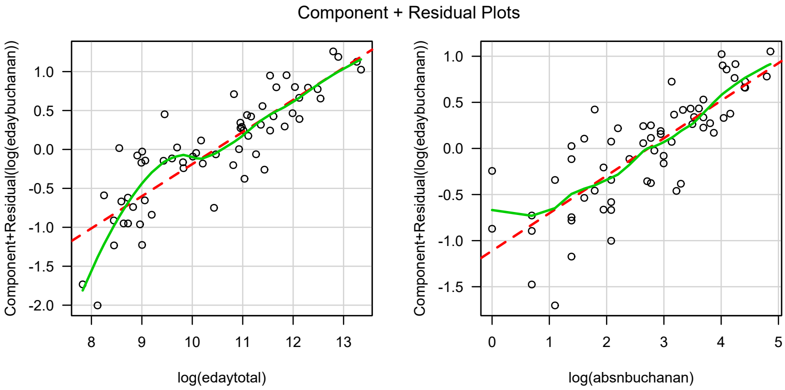
- Transformations such as logs
- Generalized Additive Models (GAM)
- Many more flexible, nonlinear regression models exist beyond the scope of this course.

Transformed Buchanan regression

R Code

```
mod.nopb2 <- lm(log(edaybuchanan) ~ log(edaytotal) + log(absnbuchanan),
data = flvote, subset = county != "Palm Beach")
crPlots(mod.nopb2, las = 1)
```



Bias-Variance Tradeoff

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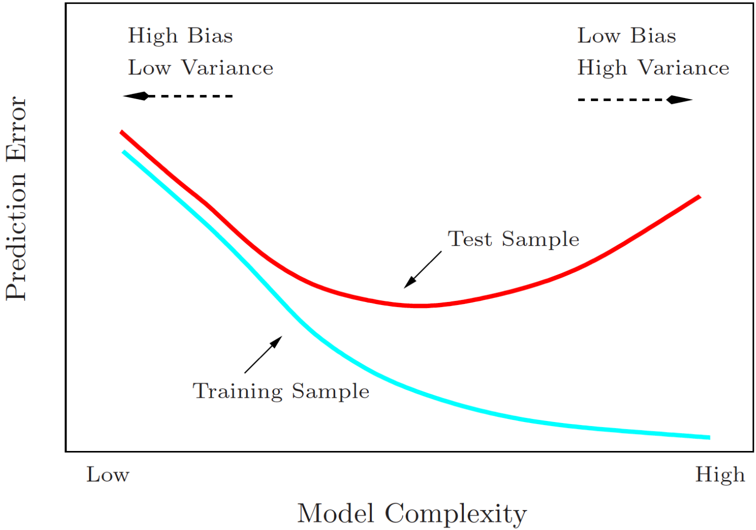
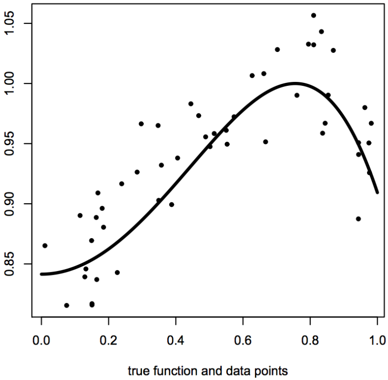


Example Synthetic Problem

$$y = \sin(1 + x^2) + \epsilon$$



This section adapted from slides by Radford Neal.

Linear Basis Function Models

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- In general the idea is to do a linear regression of y on $\phi_1(x), \phi_2(x), \dots, \phi_{m-1}(x)$ where ϕ_j are basis functions.
- The model is now:

$$y = f(x, \beta) + \epsilon$$

$$f(x, \beta) = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$

Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.

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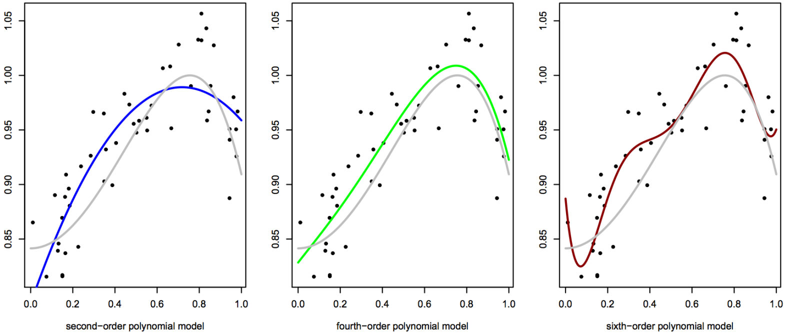

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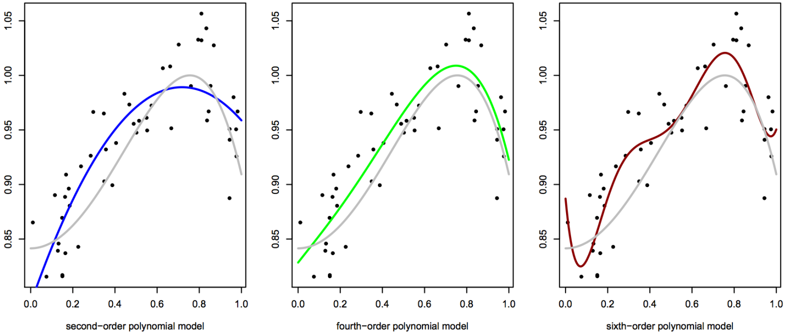

It appears that the last model is too complex and is overfitting a bit.

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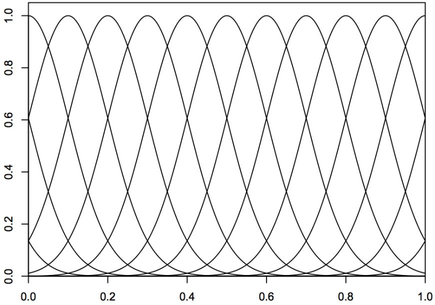
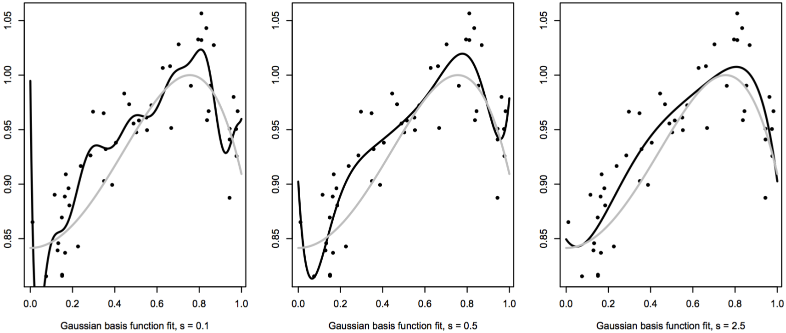


Gaussian basis functions, s = 0.1

Gaussian Basis Fits



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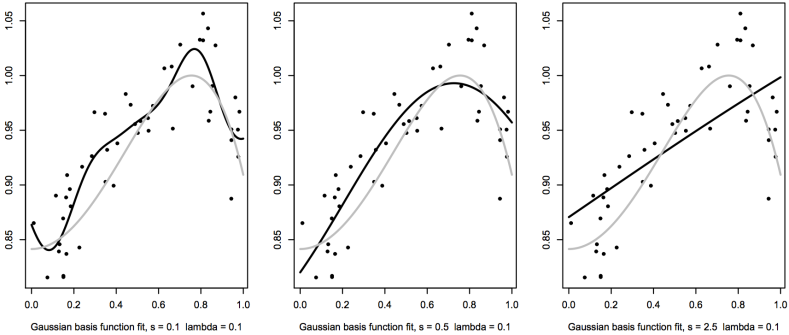
- The trick in general is how to set λ

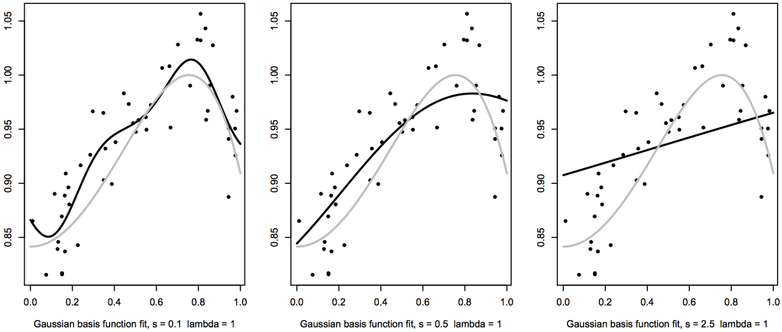
Results

Here are the results with $\lambda = 0.1$:



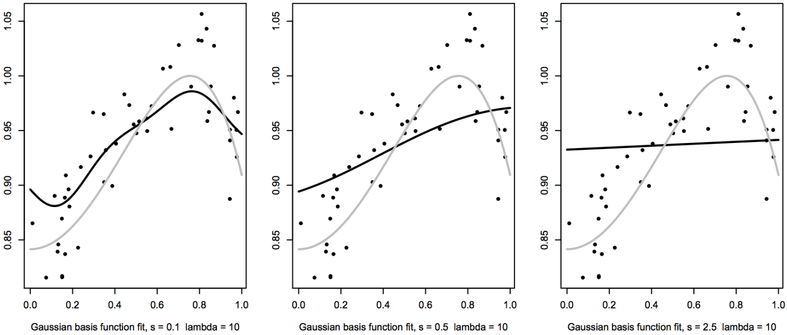
Results

Here are the results with $\lambda = 1$:



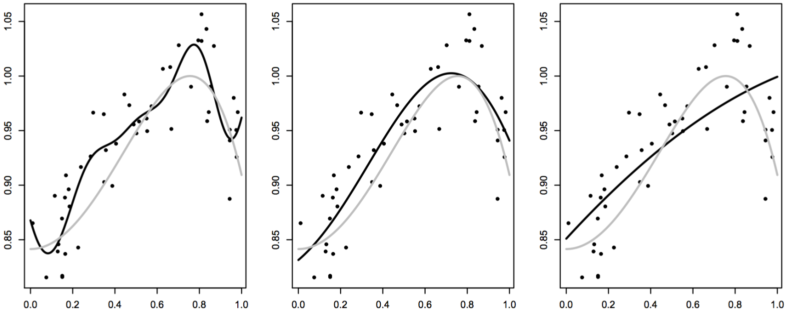
Results

Here are the results with $\lambda = 10$:



Results

Here are the results with $\lambda = 0.01$:



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- next up, Generalized Additive Models

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$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + x_{3i}\beta_3 + u_i$$

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- Theory and estimation are somewhat involved, but they are easy to use:
  - ```
    gam.out <- gam(y~s(x1)+s(x2)+x3)
    plot(gam.out)
    ```
  - Multiple functions but I recommend `mgcv` package
    ```

Generalized Additive Models (GAM)

The GAM approach can be extended to allow interactions ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

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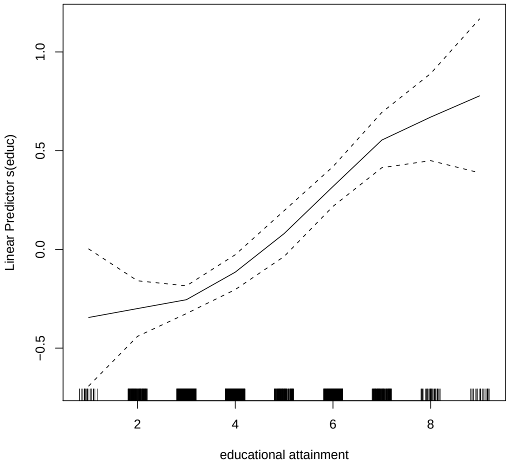
It can also be used for hybrid models where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

GAM Fit to Attitudes Toward Immigration



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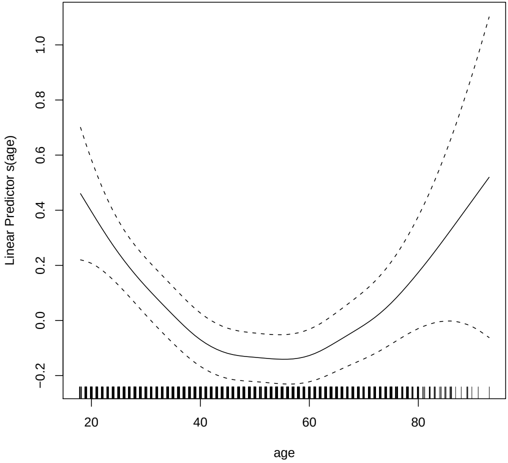

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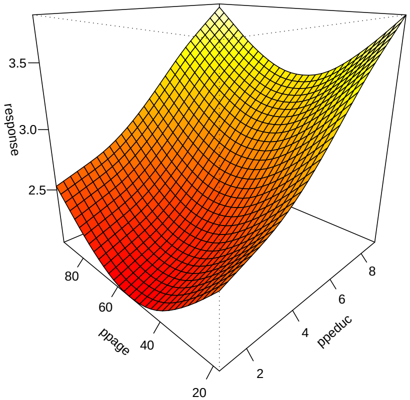

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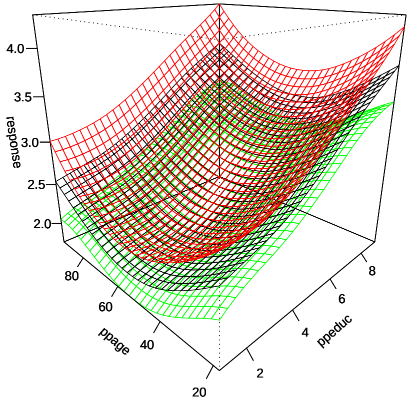
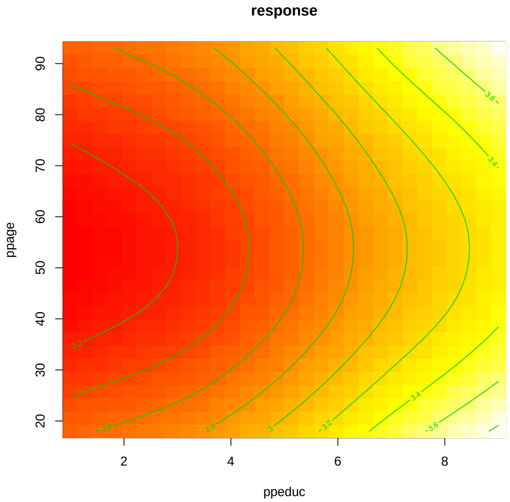
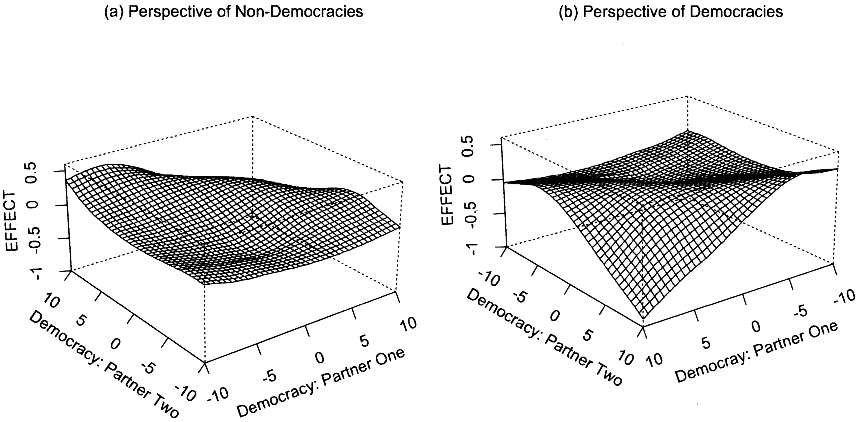


red/green are +/− 2 s.e.

GAM Fit to Attitudes Toward Immigration



GAM Fit to Dyadic Democracy and Militarized Disputes



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- Non-linearity is pretty easy to detect and can substantially change our inferences
- GAMs are a great way to model/detect non-linearity but transformations are often simpler
- However, be wary of the global properties of transformations and polynomials
- Non-linearity concerns are most relevant for continuous covariates with a large range (age)

Fun With Kernels

Hainmueller and Hazlett (2013). "Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach" *Political Analysis*.[2]

[2] I thank Chad Hazlett for sharing many of the slides that follow

Motivation: Misspecification Bias

Consider a data generating process such as:

```
> # Predictors
> GDP = runif(500)
> Polity  = .5*GDP^2 + .2*runif(200)
>
> # True Model
> Stability = log(GDP)+rnorm(500)
```

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> Stability = log(GDP)+rnorm(500)
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Regressing `Stability` on `polity` and `GDP`:

```
> # OLS
> lm(Stability ~ Polity + GDP)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.3000     0.1039 -22.145  < 2e-16 ***
Polity       -3.1983     0.7613  -4.201 3.15e-05 ***
GDP           4.3443     0.4237  10.252  < 2e-16 ***
```

Motivation: Misspecification Bias

Consider a data generating process such as:

```
> # Predictors
> GDP = runif(500)
> Polity  = .5*GDP^2 + .2*runif(200)
>
> # True Model
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Entirely wrong conclusions!

Misspecification Bias

Try more flexible method that still reports marginal effects:

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> krls(y=Stability,X=cbind(GDP,Polity))

Average Marginal Effects:
              Est Std. Error    t value     Pr(>|t|)
GDP     3.3855912  0.5217110  6.4893996 2.084441e-10
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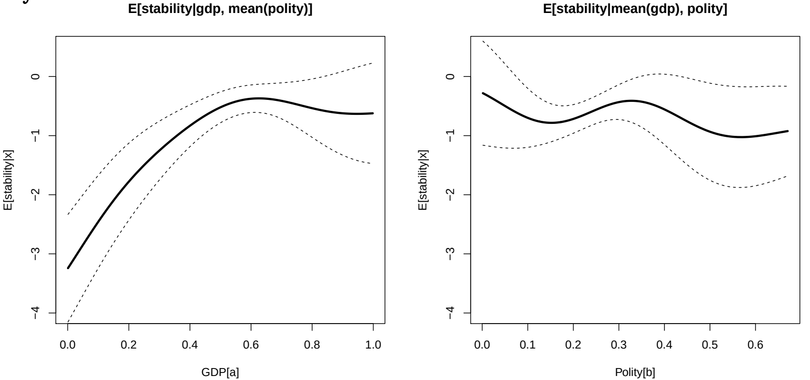

Kernel Basics

Kernel

For now, a kernel is a function $\mathbb{R}^{\mathbb{P}} \times \mathbb{R}^{\mathbb{P}} \to \mathbb{R}$

$$k(x_i, x_j) \to \mathbb{R}$$

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Some kernels are naturally interpretable as a distance metric, e.g. the Gaussian:

Gaussian Kernel

$$k(\cdot,\cdot) : \mathbb{R}^D \times \mathbb{R}^P \mapsto \mathbb{R}$$

$$k(x_j, x_i) = e^{-\frac{||x_j - x_i||^2}{\sigma^2}}$$

where $||X_j - X_i||$ is the Euclidean distance between X_j and X_i

Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$

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$$\underset{\theta \in \mathbb{R}^{P'}}{\text{argmin}} \sum_{i=1}^{N} (Y_i - \phi(X_i)^T \theta)^2 + \lambda \langle \theta, \theta \rangle$$

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- Solve the F.O.C.s:

$$R(\theta, \lambda) = \sum_{i=1}^{N} (Y_i - \phi(X_i)^\top \theta)^2 + \lambda \theta^\top \theta$$

$$\frac{\partial R(\theta, \lambda)}{\partial \theta} = -2 \sum_{i=1}^{N} \phi(X_i)(Y_i - \phi(X_i)^T \theta) + 2\lambda\theta = 0$$

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Similarity model:

$$E[alt|lat, long] = c_1(\text{similarity to obs1}) + \ldots + c_5(\text{similarity to obs5})$$

Intuition: Similarity

Think of this function space as built on similarity:

$$
\begin{aligned}
f(X^\star) &= \sum_{i=1}^{N} c_i k(X^\star, X_i) \\
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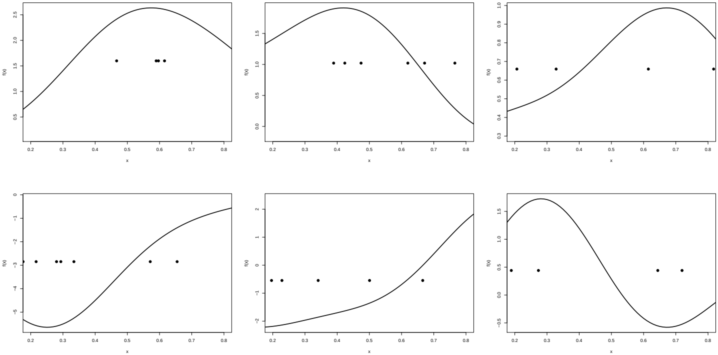
Some random functions from this space:



A real example: Harff 2003

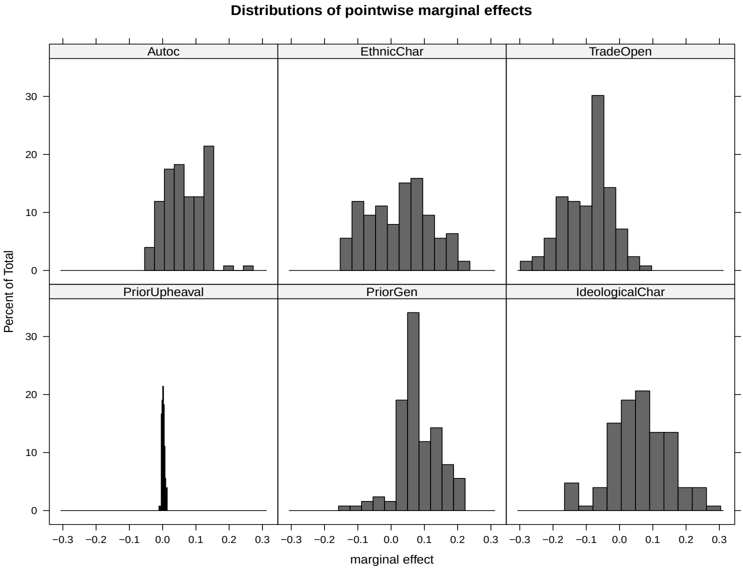
From `summary(krls(y,X))`

DV: Genocide onset

	β_{OLS}	$E[\frac{\hat{dy}}{dx_i}]$
Prior upheaval	0.009*	0.00
	(0.004)	0.00
Prior genocide	0.26*	0.19*
	(0.12)	(0.08)
Ideological char. elite	0.15*	0.13*
	(0.084)	(0.08)
Autocracy	0.16*	0.12*
	(0.077)	(0.07)
Ethnic char. elite	0.12	0.05
	(0.084)	(0.08)
log(trade openness)	-0.17*	-0.09*
	(0.057)	(0.03)

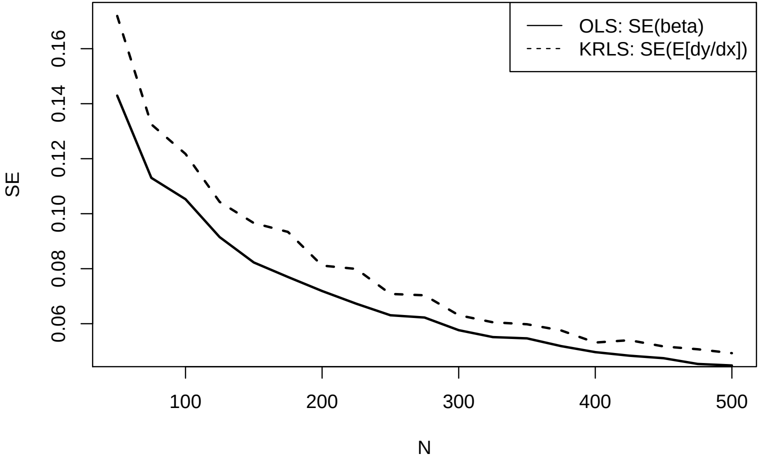
Behind the averages

`plot(krls(X,y))`

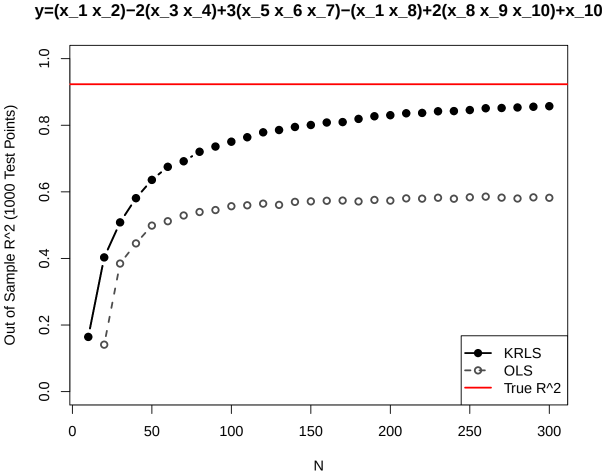


Efficiency Comparison

$$y = 2x + \epsilon,\ x \sim N(0,1), \epsilon \sim N(0,1)$$



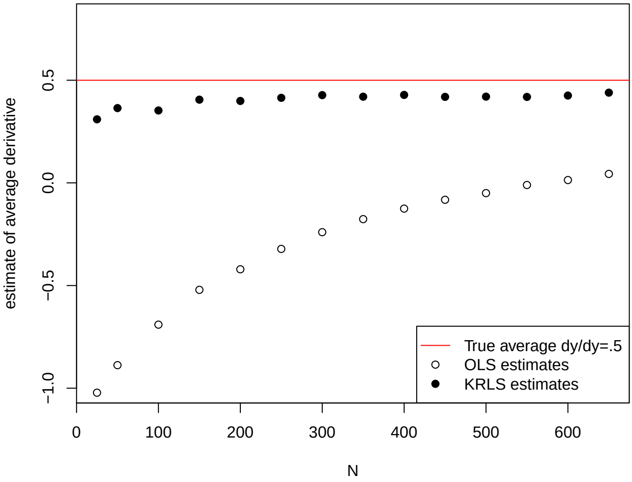
High-dimensional data with non-linearities



$y = (X_1X_2) - 2(X_3X_4) + 3(X_5X_6X_7) - (X_1X_8) + 2(X_8X_9X_{10}) + X_{10} + \epsilon$ where all X are i.i.d. *Bernoulli*(p) at varying p, $\varepsilon \sim N(0, .5)$. $1,000$ test points.

Linear model with bad leverage points

- $y = .5x + \varepsilon$ where $\varepsilon \sim N(0, .3)$
- One bad point, $(y_i = -5, x_i = 5)$.



Interaction or non-linearity?

Truth: $y = 5x_1^2 + \varepsilon, \quad \rho(x_1, x_2) = .72$
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Estimator	OLS	KRLS			
$\partial y/\partial x_{ij}$	Average	Average	1st Qu.	Median	3rd Qu.
const	-1.50				
	(0.34)				
x_1	7.51	9.22	5.22	9.38	14.03
	(0.40)	(0.52)	(0.82)	(0.85)	(0.79)
x_2	-1.28	0.02	-0.08	0.00	0.10
	(0.21)	(0.13)	(0.19)	(0.16)	(0.20)
$(x_1 \cdot x_2)$	1.24				
	(0.15)				
N	250				

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  - computation scales in number of datapoints ($O(N^3)$) which means it doesn't work for more than about 5000 datapoints
  - it may model deep interactions but it is still hard to summarize deep interactions

References

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Where We've Been and Where We're Going...

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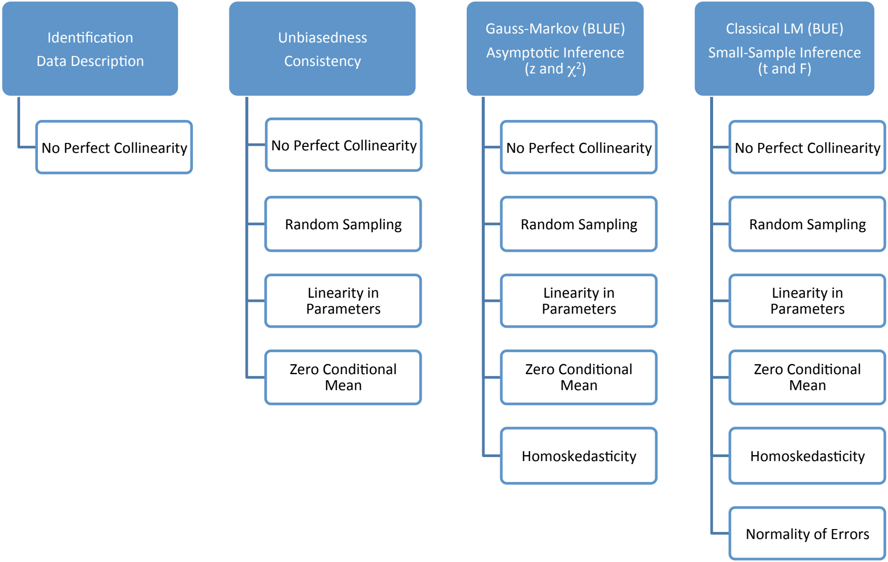
Questions?

A Quick Note of Thanks

Review of the OLS Assumptions



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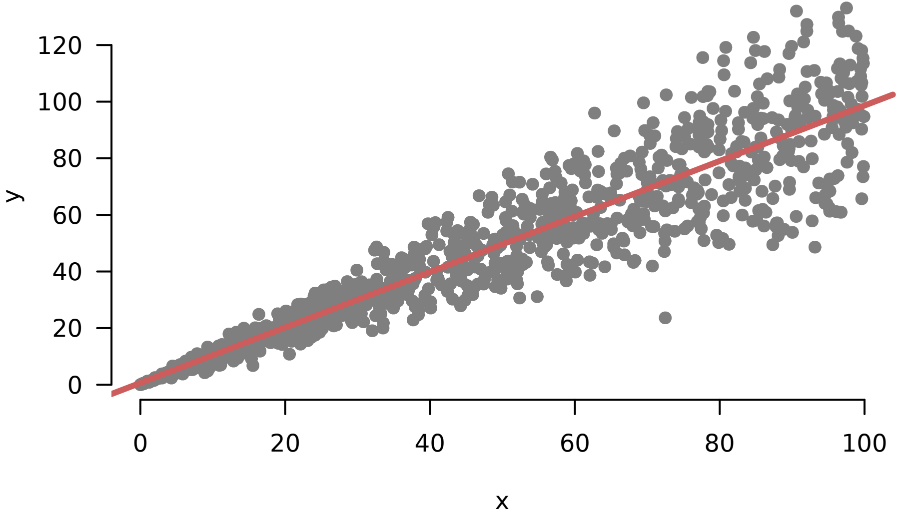
How Do We Deal With This?



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Then we will discuss a contrarian view

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- Using assumptions 1 and 4, we can show that we have the following (derivation in the appendix):

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- With homoskedasticity, $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$

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\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \\
&= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \text{ (by homoskedasticity)} \\
&= \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1} \\
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Review of Homoskedasticity

- Remember:

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\end{aligned}$$

- Replace σ^2 with estimate $\widehat{\sigma}^2$ will give us our estimate of the covariance matrix

Non-constant Error Variance

- Homoskedastic:

$$V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

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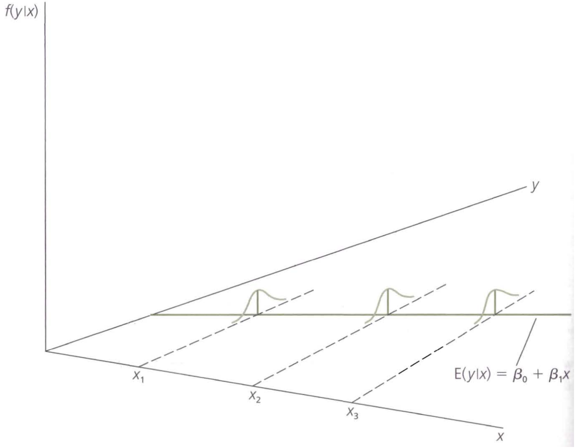
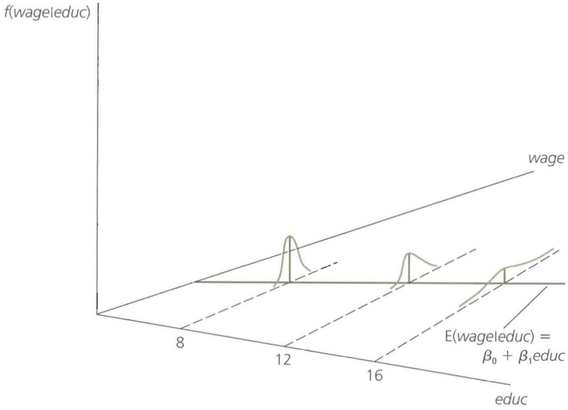
- Independent, not identical
- $\text{Cov}(u_i, u_j|\mathbf{X}) = 0$
- $\text{Var}(u_i|\mathbf{X}) = \sigma_i^2$

Example: $V[\mathbf{u}|\mathbf{X}] = \sigma^2$ Homoskedasticity



Example: $V[\mathbf{u}|\mathbf{X}] = \sigma_i^2$ Heteroskedasticity



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    - $\widehat{\boldsymbol{\beta}}$ still unbiased and consistent for $\boldsymbol{\beta}$
    - degree of the problem depends on how serious the heteroskedasticity is

Visual diagnostics

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   - Usually has loess trend curve to check if variance varies with fitted values
   - In R, `plot(mod, which = 3)`

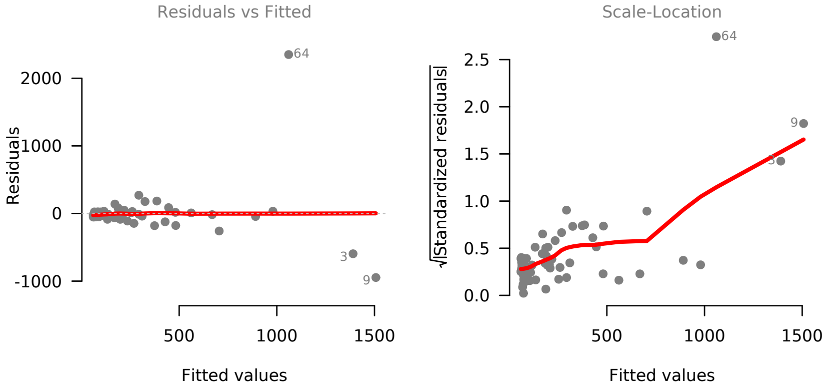
Example: Buchanan votes

```
flvote <- foreign::read.dta("flbuchan.dta")
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.423e+01  4.914e+01   1.104    0.274
## edaytotal   2.323e-03  3.104e-04   7.483 2.42e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 332.7 on 65 degrees of freedom
## Multiple R-squared:  0.4628, Adjusted R-squared:  0.4545
## F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10
```

Diagnostics

```
par(mfrow = c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 1, lwd = 3)
plot(mod, which = 3, lwd = 3)
```



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  - In R, `bptest` in the `lmtest` package

Breush-Pagan Example

```
library(lmtest)
bptest(mod)

##
##  studentized Breusch-Pagan test
##
## data:  mod
## BP = 12.59, df = 1, p-value = 0.0003878
```

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   (this will affect other model assumptions)

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4. Admit we have the wrong model and use a different approach

Variance Stabilizing Transformations

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Examples:

Transformation	Mean/Variance Relationship
$\sqrt{Y}$	$\sigma_i^2 \propto \mathbf{x}_i\boldsymbol{\beta}$
$\log Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^2$
$1/Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^4$

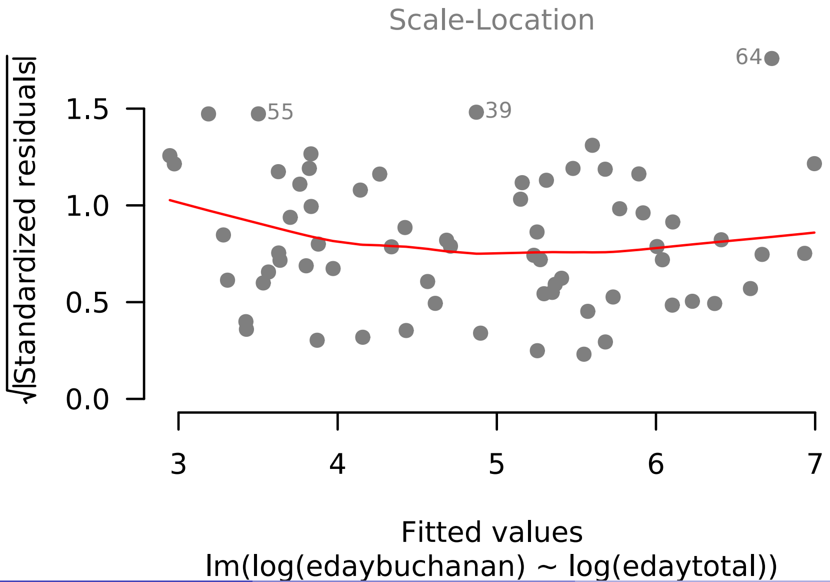
Example: Transforming Buchanan Votes

```
mod2 <- lm(log(edaybuchanan) ~ log(edaytotal), data = flvote)
summary(mod2)

##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -2.72789    0.39956  -6.827  3.5e-09 ***
## log(edaytotal)  0.72853    0.03803  19.154  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4688 on 65 degrees of freedom
## Multiple R-squared:  0.8495, Adjusted R-squared:  0.8472
## F-statistic: 366.9 on 1 and 65 DF,  p-value: < 2.2e-16
```

Example: Transformed Scale-Location Plot

```
plot(mod2, which=3)
```



Example: Transformed

```
bptest(mod, studentize=FALSE)

##
##  Breusch-Pagan test
##
## data:  mod
## BP = 250.07, df = 1, p-value < 2.2e-16

bptest(mod2, studentize=FALSE)

##
##  Breusch-Pagan test
##
## data:  mod2
## BP = 0.01105, df = 1, p-value = 0.9163
```

Appendix: Weighted Least Squares

- Suppose that the heteroskedasticity is known up to a multiplicative constant:

$$\text{Var}[u_i|\mathbf{X}] = a_i\sigma^2$$

  where $a_i = a_i(\mathbf{x}_i')$ is a positive and known function of $\mathbf{x}_i'$
- WLS: multiply y_i by $1/\sqrt{a_i}$:

$$y_i/\sqrt{a_i} = \beta_0/\sqrt{a_i} + \beta_1 x_{i1}/\sqrt{a_i} + \cdots + \beta_k x_{ik}/\sqrt{a_i} + u_i/\sqrt{a_i}$$

Appendix: Weighted Least Squares Intuition

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- Rescales errors to $u_i/\sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:

$$\text{Var}\left[\frac{1}{\sqrt{a_i}}u_i|\mathbf{X}\right] = \frac{1}{a_i}\text{Var}\left[u_i|\mathbf{X}\right]$$

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- But makes the error variance constant again:

$$\begin{aligned} \text{Var}\left[\frac{1}{\sqrt{a_i}}u_i|\mathbf{X}\right] &= \frac{1}{a_i}\text{Var}\left[u_i|\mathbf{X}\right] \\ &= \frac{1}{a_i}a_i\sigma^2 \\ &= \sigma^2 \end{aligned}$$

- If you know a_i, then you can use this approach to makes the model homoskedastic and, thus, BLUE again
- When do we know a_i?

Appendix: Weighted Least Squares procedure

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- Define the weighting matrix:

$$\mathbf{W} = \begin{bmatrix} 1/\sqrt{a_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{a_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1/\sqrt{a_n} \end{bmatrix}$$

- Run the following regression:

$$\begin{aligned} \mathbf{W}\mathbf{y} &= \mathbf{W}\mathbf{X}\beta + \mathbf{W}\mathbf{u} \\ \mathbf{y}^* &= \mathbf{X}^*\boldsymbol{\beta} + \mathbf{u}^* \end{aligned}$$

- Run regression of $\mathbf{y}^* = \mathbf{W}\mathbf{y}$ on $\mathbf{X}^* = \mathbf{W}\mathbf{X}$ and all Gauss-Markov assumptions are satisfied
- Plugging into the usual formula for $\widehat{\boldsymbol{\beta}}$:

$$\widehat{\boldsymbol{\beta}}_W = (\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{y}$$

Appendix: WLS Example

- In R, use `weights` = argument to `lm` and give the weights squared: $1/a_i$
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

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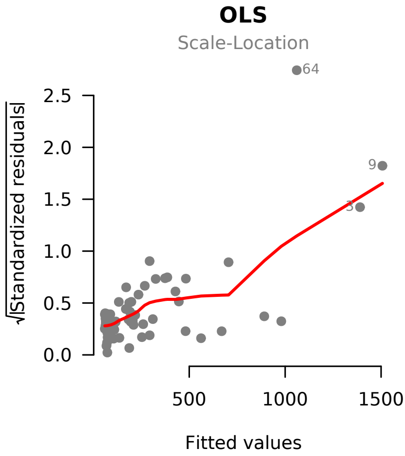
- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

```
mod.wls <- lm(edaybuchanan ~ edaytotal, weights = 1/edaytotal,
                     data = flvote)
summary(mod.wls)

##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept) 2.707e+01  8.507e+00   3.182  0.00225 **
## edaytotal   2.628e-03  2.502e-04  10.503 1.22e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5645 on 65 degrees of freedom
## Multiple R-squared:  0.6292, Adjusted R-squared:  0.6235
## F-statistic: 110.3 on 1 and 65 DF,  p-value: 1.22e-15
```

Appendix: Comparing WLS to OLS

```
par(mfrow=c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 3, main = "OLS", lwd = 2)
plot(mod.wls, which = 3, main = "WLS", lwd = 2)
```



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- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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- When $\boldsymbol{\Sigma} \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\boldsymbol{\Sigma}\mathbf{X} \left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

Heteroskedasticity Consistent Estimator

- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

- When $\boldsymbol{\Sigma} \neq \sigma^2 \mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \mathbf{X}'\Sigma\mathbf{X} \left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- Idea: If we can consistently estimate the components of $\boldsymbol{\Sigma}$, we could directly use this expression by replacing $\boldsymbol{\Sigma}$ with its estimate, $\hat{\boldsymbol{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form:

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$$\widehat{V[\hat{\beta}|\mathbf{X}]} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \begin{bmatrix} \hat{\mathbf{u}}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{\mathbf{u}}_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \hat{\mathbf{u}}_n^2 \end{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ under any form of heteroskedasticity consistent with $V[\mathbf{u}]$ above.

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is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ under any form of heteroskedasticity consistent with $V[\mathbf{u}]$ above.

The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors.

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3. Plug $\widehat{\mathbf{\Sigma}}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

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- There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] \;=\; \frac{n}{n-k-1}\cdot(\mathbf{X}'\mathbf{X})^{-1}\,\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}\,(\mathbf{X}'\mathbf{X})^{-1}$$

Regular & Robust Standard Errors in Florida Example

R Code

```
> library(sandwich)
> library(lmtest)
> coeftest(mod1) # homoskedasticity
t test of coefficients:
              Estimate Std. Error t value  Pr(>|t|)
(Intercept)  5.4231e+01 4.9141e+01  1.1036    0.2738
TotalVotes00 2.3229e-03 3.1041e-04  7.4831 2.417e-10 ***

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC0")) # classic White
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.0612e+01  1.3353  0.18642
TotalVotes00 2.3229e-03 8.7047e-04  2.6685  0.00961 **

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC1")) # small sample correction
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.1232e+01  1.3153  0.19304
TotalVotes00 2.3229e-03 8.8376e-04  2.6284  0.01069 *
```

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  - For small n, performance might be poor (correction factors exist but are often insufficient)

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- Called clustering or clustered dependence

Clustered Dependence: notation

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- Ignoring clustering is "cheating": units not independent

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- v_j and u_{ij} are assumed to be independent of each other
- $\rho \in (0, 1)$ is called the within-cluster correlation.
- What if we ignore this structure and just use ε_{ij} as the error?
- Variance of the composite error is σ^2:

$$\text{Var}[\varepsilon_{ij}] = \text{Var}[v_j + u_{ij}]$$
$$= \text{Var}[v_j] + \text{Var}[u_{ij}]$$
$$= \rho\sigma^2 + (1-\rho)\sigma^2 = \sigma^2$$

Lack of Independence

- Covariance between two units i and s in the same cluster is $\rho\sigma^2$:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sj}] = \rho\sigma^2$$

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- Zero covariance of two units i and s in different clusters j and k:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0$$

Example Covariance Matrix

$$\boldsymbol{\varepsilon} = \left[\begin{array}{cccccc} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{array}\right]'$$

$$\text{Var}[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \left[\begin{array}{cccccc} \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 \end{array}\right]$$

Appendix: Example 6 Units, 2 Clusters

$$\boldsymbol{\varepsilon} = [\ \varepsilon_{1,1} \quad \varepsilon_{2,1} \quad \varepsilon_{3,1} \quad \varepsilon_{4,2} \quad \varepsilon_{5,2} \quad \varepsilon_{6,2}\ ]'$$

$$V[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \begin{bmatrix} V[\varepsilon_{1,1}] & Cov[\varepsilon_{2,1}, \varepsilon_{1,1}] & Cov[\varepsilon_{3,1}, \varepsilon_{1,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{2,1}] & V[\varepsilon_{2,1}] & Cov[\varepsilon_{3,1}, \varepsilon_{2,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{3,1}] & Cov[\varepsilon_{2,1}, \varepsilon_{3,1}] & V[\varepsilon_{3,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{4,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{4,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{4,2}] & V[\varepsilon_{4,2}] & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{4,2}, \varepsilon_{5,2}] & V[\varepsilon_{5,2}] & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{4,2}, \varepsilon_{6,2}] & Cov[\varepsilon_{5,2}, \varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

which can be verified as follows:

- $V[\varepsilon_{ij}] = V[v_j + u_{ij}] = V[v_j] + V[u_{ij}] = \rho\sigma^2 + (1 - \rho)\sigma^2 = \sigma^2$
- $Cov[\varepsilon_{ij}, \varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] - E[\varepsilon_{ij}]E[\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] = E[(v_j + u_{ij})(v_j + u_{lj})]$
  $= E[{v_j}^2] + E[v_j u_{ij}] + E[v_j u_{lj}] + E[u_{ij} u_{lj}]$
  $= E[{v_j}^2] + E[v_j]E[u_{ij}] + E[v_j]E[u_{lj}] + E[u_{ij}]E[u_{lj}]$
  $= E[{v_j}^2] = V[v_j] + (E[v_j])^2 = V[v_j] = \rho\sigma^2$
- $Cov[\varepsilon_{ij}, \varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] - E[\varepsilon_{ij}]E[\varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] = E[(v_j + u_{ij})(v_k + u_{lk})]$
  $= E[v_j v_k] + E[v_j u_{lk}] + E[v_k u_{ij}] + E[u_{ij} u_{lk}]$
  $= E[v_j]E[v_k] + E[v_j]E[u_{lk}] + E[v_k]E[u_{ij}] + E[u_{ij}]E[u_{lk}] = 0$

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The variance-covariance matrix of the error, $\mathbf{\Sigma}$, is block diagonal:

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$$V[\varepsilon] = \mathbf{\Sigma} = \left[\begin{array}{c|c|c|c} \mathbf{\Sigma}_1 & 0 & \dots & 0 \\ \hline \mathbf{0} & \mathbf{\Sigma}_2 & \dots & \mathbf{0} \\ \hline & & \ddots & \\ \hline \mathbf{0} & \mathbf{0} & \dots & \mathbf{\Sigma}_M \end{array}\right]$$

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$$\mathbf{\Sigma}_j = \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \dots & \sigma^2 \cdot \rho \\ \sigma^2 \cdot \rho & \sigma^2 & \dots & \sigma^2 \cdot \rho \\ & & \ddots & \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \dots & \sigma^2 \end{bmatrix}$$

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- Under this clustered dependence, we can write this as:

$$\text{Var}[\hat{\boldsymbol{\beta}} | \mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1} \left(\sum_{j=1}^{m} \mathbf{X}_j' \boldsymbol{\Sigma}_j \mathbf{X}_j \right) \left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

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3. The within cluster correlation is then estimated as: $\hat{\rho} = \frac{\widehat{\sigma}^2 - \widehat{\tilde{\sigma}}^2}{\widehat{\sigma}^2}$

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- No canned function for CRSE in R; use our custom function posted on the course website

```
> source("vcovCluster.r")
> coeftest(model, vcov = vcovCluster(model, cluster = clusterID))
```

Example: Gerber, Green, Larimer

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

Social Pressure Model

```
load("gerber_green_larimer.RData")
social$voted <- 1 * (social$voted == "Yes")
social$treatment <- factor(social$treatment,
   levels = c("Control", "Hawthorne", "Civic Duty",
              "Neighbors", "Self"))
mod1 <- lm(voted ~ treatment, data = social)
coeftest(mod1)
```

```
##
## t test of coefficients:
##
##                       Estimate Std. Error  t value  Pr(>|t|)
## (Intercept)          0.2966383  0.0010612 279.5250 < 2.2e-16 ***
## treatmentHawthorne   0.0257363  0.0026007   9.8958 < 2.2e-16 ***
## treatmentCivic Duty 0.0178993  0.0026003   6.8835 5.849e-12 ***
## treatmentNeighbors   0.0813099  0.0026008  31.2634 < 2.2e-16 ***
## treatmentSelf        0.0485132  0.0026003  18.6566 < 2.2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Social Pressure Model, CRSEs

Again no canned CRSE in R, so we use our own.

```
source("vcovCluster.R")
coeftest(mod1, vcov = vcovCluster(mod1, "hh_id"))
```

```
##
## t test of coefficients:
##
##                          Estimate Std. Error  t value  Pr(>|t|)
## (Intercept)             0.2966383  0.0013096 226.5172 < 2.2e-16 ***
## treatmentHawthorne      0.0257363  0.0032579   7.8997 2.804e-15 ***
## treatmentCivic Duty     0.0178993  0.0032366   5.5302 3.200e-08 ***
## treatmentNeighbors      0.0813099  0.0033696  24.1308 < 2.2e-16 ***
## treatmentSelf           0.0485132  0.0033000  14.7009 < 2.2e-16 ***
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  - Block bootstrap can be a useful alternative (key idea: bootstrap by resampling the clusters)

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- Examples: a country over several years or a person over weeks/months
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- Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).

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- Generalizes to higher order serial correlation (e.g. an AR(2) model is given by $u_t = \rho u_{t-1} + \delta u_{t-2} + e_t$).

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ρ is usually positive, which implies that we underestimate the variance if we ignore serial correlation.

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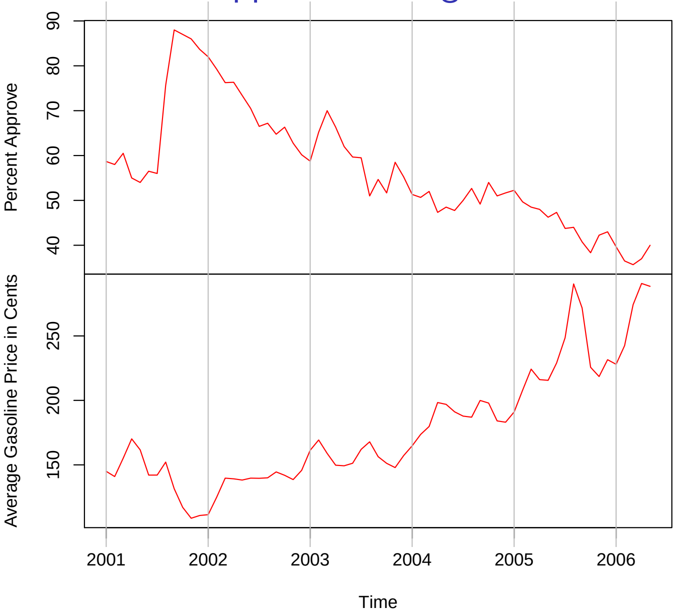
Monthly Presidential Approval Ratings and Gas Prices



Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(Zelig)
> data(approval)
> mod1 <- lm(approve ~ avg.price, data=approval)
> coeftest(mod1)

t test of coefficients:

              Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***
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Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

where $u_t = \rho u_{t-1} + e_t$, $e_t \sim N(0, \sigma^2)$, and ρ is our unknown autoregressive coefficient (with $|\rho| < 1$).

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$$DW = \frac{\sum_{t=2}^{n} \hat{u}_t - \hat{u}_{t-1}}{\sum_{t=1}^{n} \hat{u}_t^2} \quad \text{where} \quad DW \approx 2(1 - \widehat{\rho})$$

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- If $DW \approx 2$ then $\widehat{\rho} \approx 0$ (Note that $0 \leq DW \leq 4$)
- If $DW < 1$ we have serious positive serial correlation
- If $DW > 3$ we have serious negative serial correlation

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(lmtest)
> dwtest(approve ~ avg.price, data=approval)

        Durbin-Watson test

data:  approve ~ avg.price
DW = 0.4863, p-value = 1.326e-14
alternative hypothesis: true autocorrelation is greater than 0
```

The test suggests strong positive serial correlation. Standard errors are severely downward biased.

Corrections: HAC Standard Errors

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  - The `sandwich` package in R implements a variety of HAC estimators
  - A common option is `NeweyWest`

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> mod1 <- lm(approve~avg.price,data=approval)
> coeftest(mod1) # homoskedastic errors
t test of coefficients:

              Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***

> coeftest(mod1, vcov = NeweyWest) # HAC errors
t test of coefficients:

              Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076  14.499337  6.9294 2.652e-09 ***
avg.price    -0.243885   0.071733 -3.3999  0.001174 **
```

Once we correct for autocorrelation, standard errors increase dramatically.

Review of Standard Errors

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  - Newey-West HAC standard errors

Appendix: Derivation of Error Structure for the AR(1) Model

We have

$$V[u_t] = V[\rho u_{t-1} + e_t] = \rho^2 V[u_{t-1}] + \sigma^2$$

with stationarity, $V[u_t] = V[u_{t-1}]$, and so

$$V[u_t](1-\rho^2) = \sigma^2 \Rightarrow V[u_t] = \frac{\sigma^2}{(1-\rho^2)}$$

also

$$Cov[u_t, u_{t-1}] = E[u_t u_{t-1}] = E[(\rho u_{t-1} + e_t)e_{t-1}] = \rho V[e_{t-1}] = \rho \frac{\sigma^2}{(1-\rho^2)}$$

or generally

$$Cov[u_t, u_{t-h}] = \rho^h \frac{\sigma^2}{(1-\rho^2)}$$

A Contrarian View of Robust Standard Errors

King, Gary and Margaret E. Roberts. "How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It" *Political Analysis* (2015) 23: 159-179.[3]

[3] I thank Gary and Molly for the slides that follow.

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RSEs and SEs are the same

- Consistent with a correctly specified model
- RSEs are not useful, as a "fix"

Their Alternative Procedure

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- What they are not:

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1. Do RSEs and SEs differ?

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   - Evaluate misspecification
   - Respecify the model

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1. Do RSEs and SEs differ?
2. If they do:
    - Use model diagnostics (e.g. residual plots, qq-plots, misspecification tests)
    - Evaluate misspecification
    - Respecify the model
3. Keeping going, until they don't differ.

For RSEs to help: Everything has to be Juuuussttt Right

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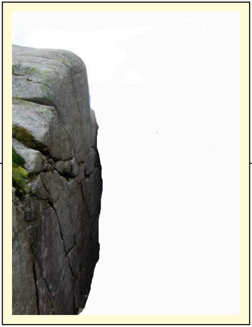

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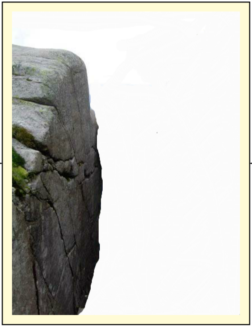

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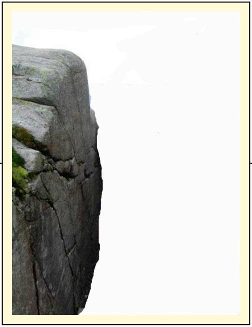
RSEs same as SEs



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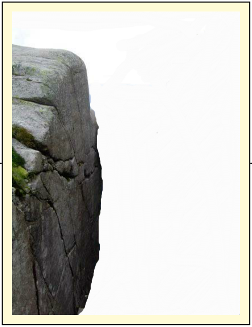
Model Correct

RSEs same as SEs

Point estimates correct



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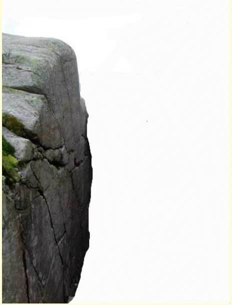

RSEs same as SEs

Point estimates correct

Awesome!

For RSEs to help: Everything has to be Juuuussttt Right

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Model Misspecified

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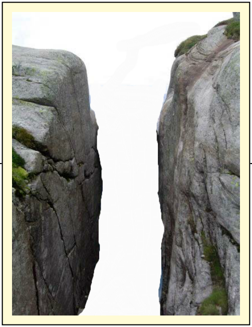

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RSEs differ from SEs

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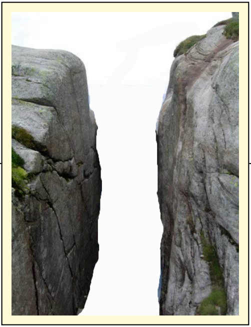

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RSEs same as SEs

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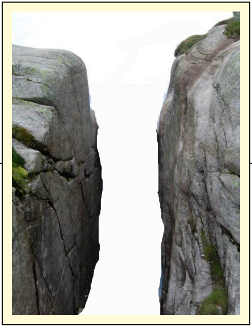

Model Misspecified

RSEs differ from SEs

Point estimates biased

Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Goldilocks Region

Model Correct

RSEs same as SEs

Point estimates correct

Awesome!

Model Misspecified

RSEs differ from SEs

Point estimates biased

Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Goldilocks Region

Model Correct	Model Misspecified
RSEs same as SEs	RSEs differ from SEs
Point estimates correct	Point estimates biased
Awesome!	Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Biased just enough to make RSEs useful,

Goldilocks Region

Model Correct

RSEs same as SEs

Point estimates correct

Awesome!

Model Misspecified

RSEs differ from SEs

Point estimates biased

Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Biased just enough to make RSEs useful,

Goldilocks Region

but not so much as to bias everything else

Model Correct

RSEs same as SEs

Point estimates correct

Awesome!

Model Misspecified

RSEs differ from SEs

Point estimates biased

Respecify!

The Goldilocks Region is not Idyllic

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In the Goldilocks region,

The Goldilocks Region is not Idyllic

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Difference Between SE and RSE Exposes Misspecification

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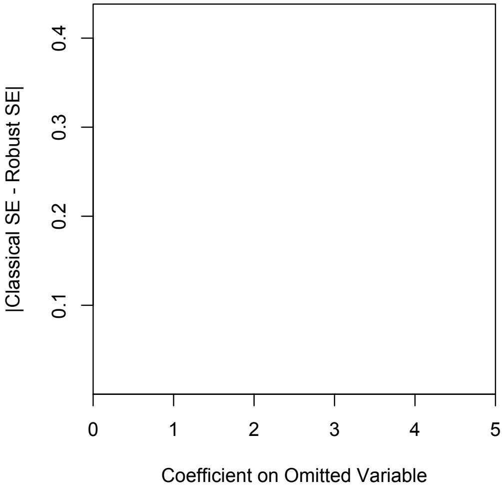

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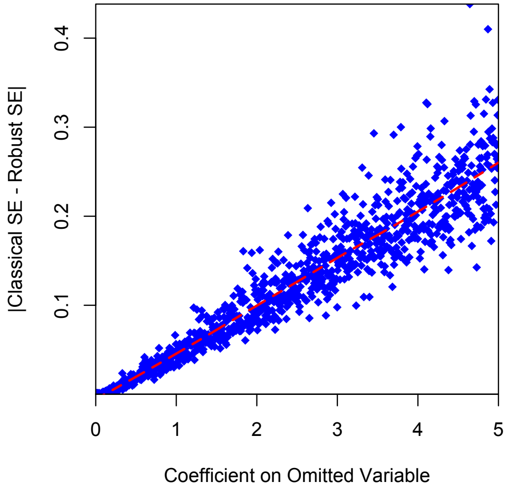

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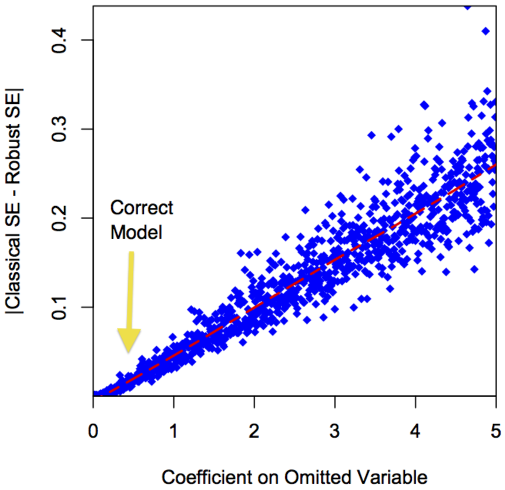

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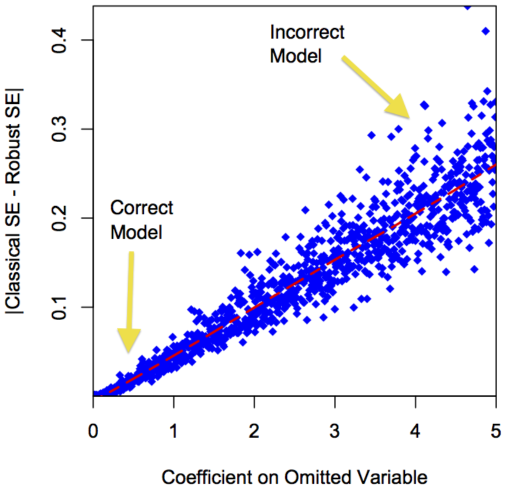

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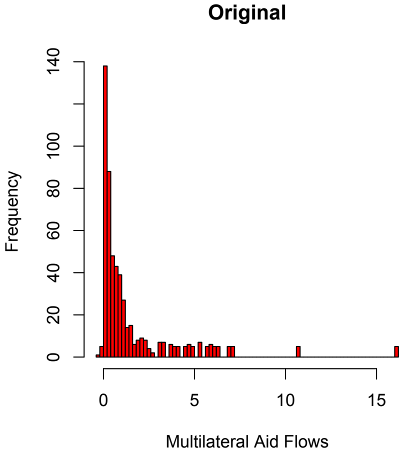

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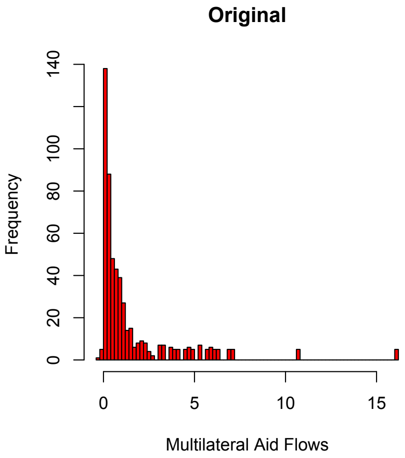

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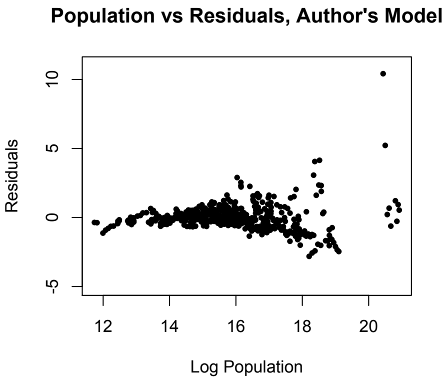

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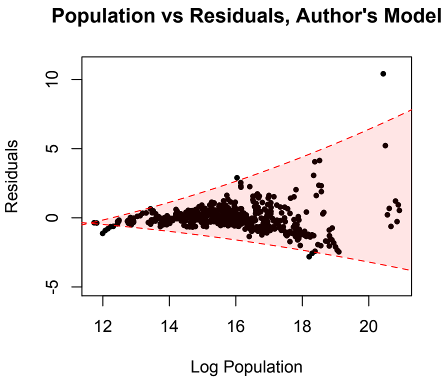

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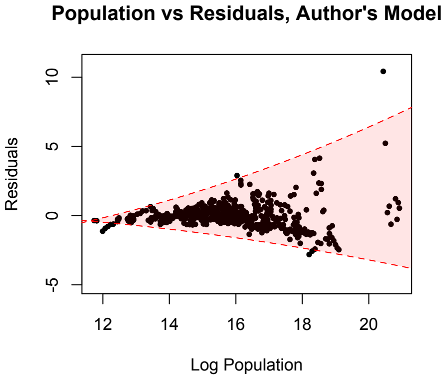

Textbook case of heteroskedasticity

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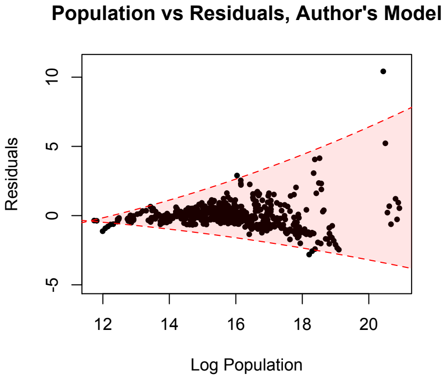

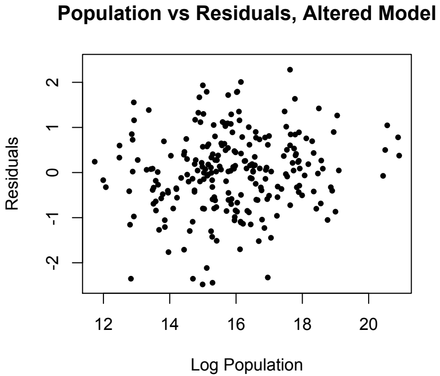

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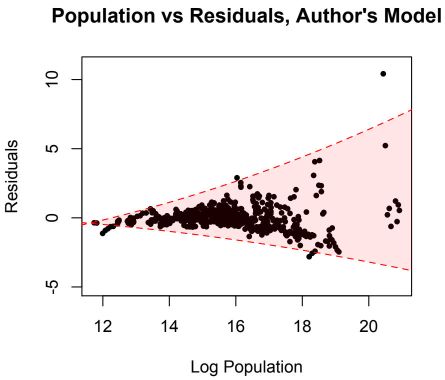

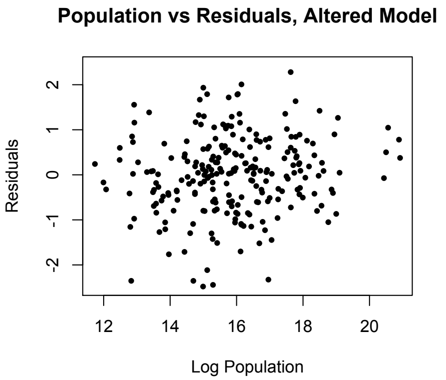

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After Fix: Different Conclusion

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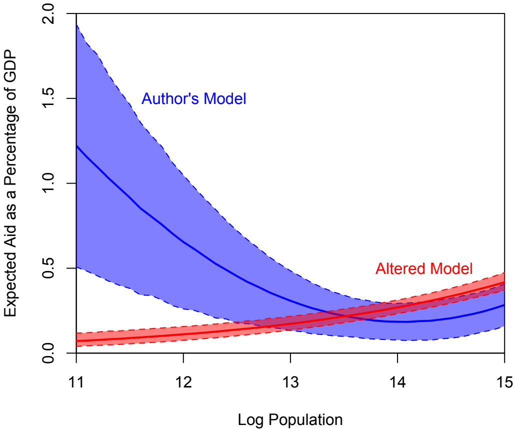

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- Easily identified with diagnostics
- Fixing these problems $\Rightarrow$ hugely different substantive conclusions

Concluding Thoughts on Diagnostics

Residuals are important. Look at them.

Next Week

- Causality with Measured Confounding
- Reading:
  - Angrist and Pishke Chapter 2 (The Experimental Ideal) Chapter 3.2 (Regression and Causality)
  - Morgan and Winship Chapters 3-4 (Causal Graphs and Conditioning Estimators)
  - Optional: Elwert and Winship (2014) "Endogenous selection bias: The problem of conditioning on a collider variable" *Annual Review of Sociology*
  - Optional: Morgan and Winship Chapter 11 Repeated Observations and the Estimation of Causal Effects
- As a side note: if you want to read the argument against the contrarian response: Aronow (2016) "A Note on 'How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It.'" It is an interesting piece- feel free to come talk to me about this debate!

Appendix: Derivation of Variance under Homoskedasticity

$$\begin{aligned}\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'(\mathbf{X}\boldsymbol{\beta} + \mathbf{u}) \\ &= \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}\end{aligned}$$

$$\begin{aligned}V[\hat{\boldsymbol{\beta}}|\mathbf{X}] &= V[\boldsymbol{\beta}|\mathbf{X}] + V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= V[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V[\mathbf{u}|\mathbf{X}]((\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}')' \text{ (note: } \mathbf{X} \text{ nonrandom } |\mathbf{X}) \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V[\mathbf{u}|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\ &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \text{ (by homoskedasticity)} \\ &= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}\end{aligned}$$

Replacing σ^2 with our estimator $\widehat{\sigma}^2$ gives us our estimator for the $(k+1)\times(k+1)$ variance-covariance matrix for the vector of regression coefficients:

$$\widehat{V[\hat{\boldsymbol{\beta}}|\mathbf{X}]} = \widehat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}$$

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Zhukov, Yuri M. and Brandon M. Stewart. "Choosing Your Neighbors: Networks of Diffusion in International Relations" *International Studies Quarterly* 2013; 57: 271-287.

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4. Different types of neighbors tell different stories

Visualization of Connections: Contiguity

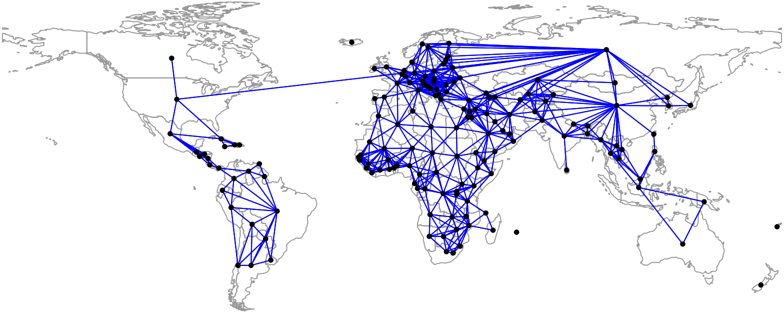
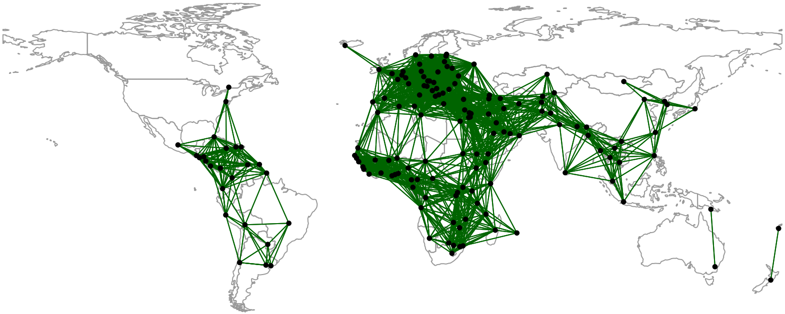


Figure: Contiguity neighbors with 500 km snap distance

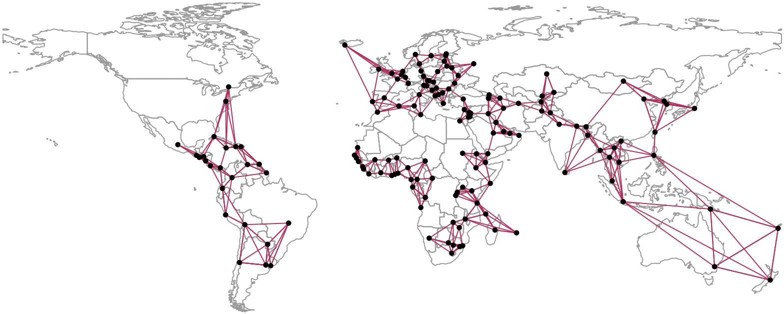
Visualization of Connections: Minimum Distance

Figure: Minimum distance neighbors (capital cities)



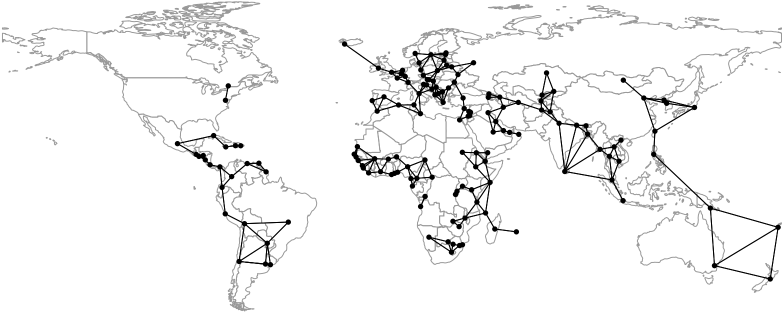
Visualization of Connections: K-Nearest Neighbors

Figure: $k = 4$ Nearest Neighbors (capital cities)



Visualization of Connections: Graph-based Neighbors

Figure: Sphere of Influence Neighbors (capital cities)



Application: Democratic Diffusion

Gleditsch and Ward (2006)

Changes of political regime modeled as a first-order Markov chain process with the transition matrix

$$\mathbf{K} = \begin{bmatrix} Pr(y_{i,t} = 0 | y_{i,t-1} = 0) & Pr(y_{i,t} = 1 | y_{i,t-1} = 0) \\ Pr(y_{i,t} = 0 | y_{i,t-1} = 1) & Pr(y_{i,t} = 1 | y_{i,t-1} = 1) \end{bmatrix}$$

where $y_{i,t} = 1$ if an (A)utocratic regime exists in country i at time t, and $y_{i,t} = 0$ if the regime is (D)emocratic.

. . . in other words:

$$\mathbf{K} = \begin{bmatrix} Pr(D \to D) & Pr(D \to A) \\ Pr(A \to D) & Pr(A \to A) \end{bmatrix}$$

Equilibrium Effects of Democratic Transition

If a regime transition takes place in country i, what is the change in predicted probability of a regime transition in country j (country i's neighbor)?

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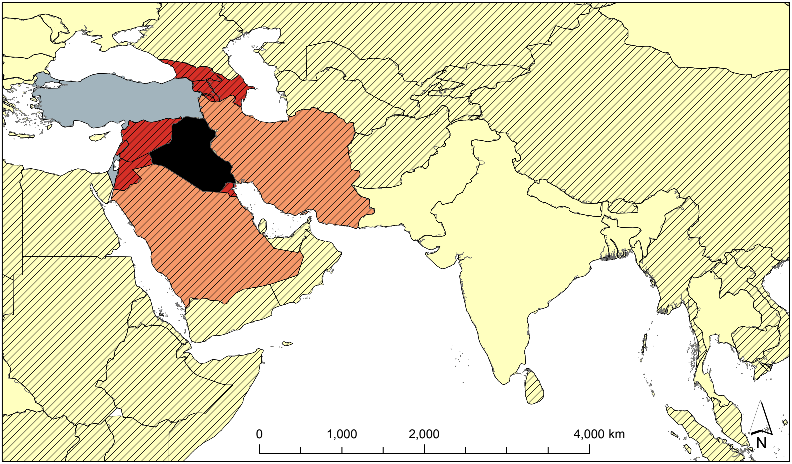
Illustrative cases

- Iraq transitions from autocracy to democracy.
- Russia transitions from democracy to autocracy.

Iraq's democratization and regional regime stability



Contiguity + 500 km

Iraq transitions from autocracy to democracy (1998 data)

Monte Carlo simulation (1,000 runs)

Iraq

Regime Type

- Democracy
- Autocracy

Change in Transition Probability

- -0.05- -0.025
- -0.025 - -0.001
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- 0.001 - 0.025
- 0.025 - 0.05

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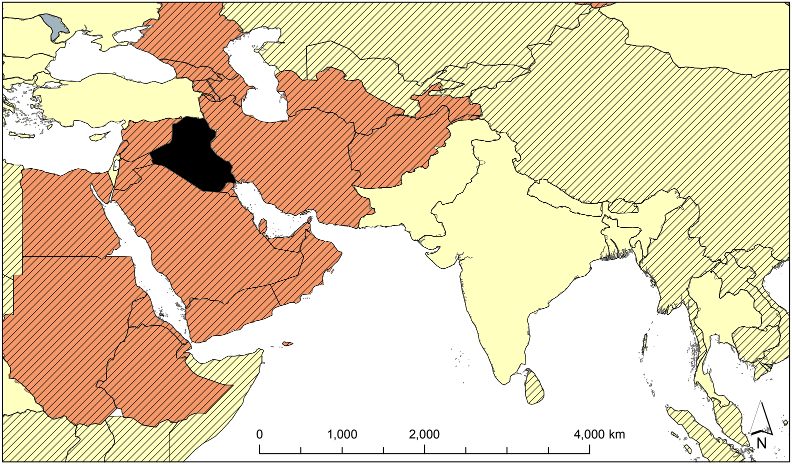

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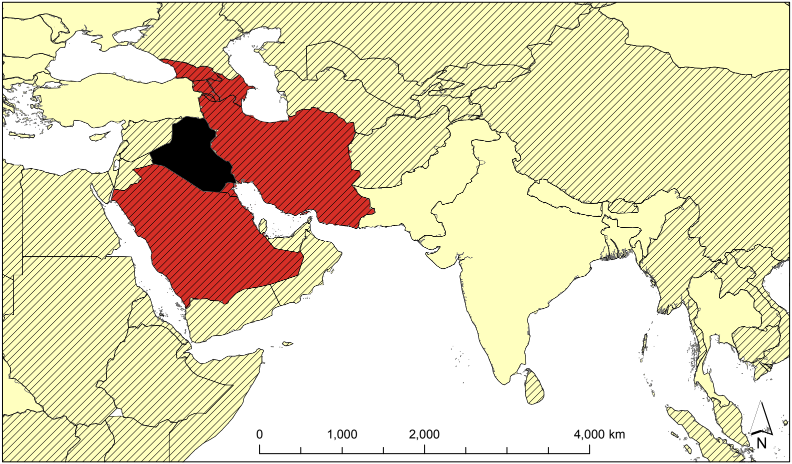

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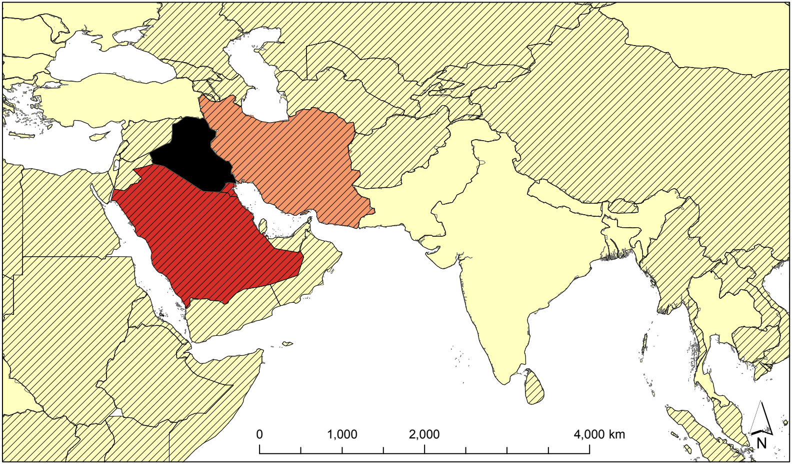

Sphere of Influence

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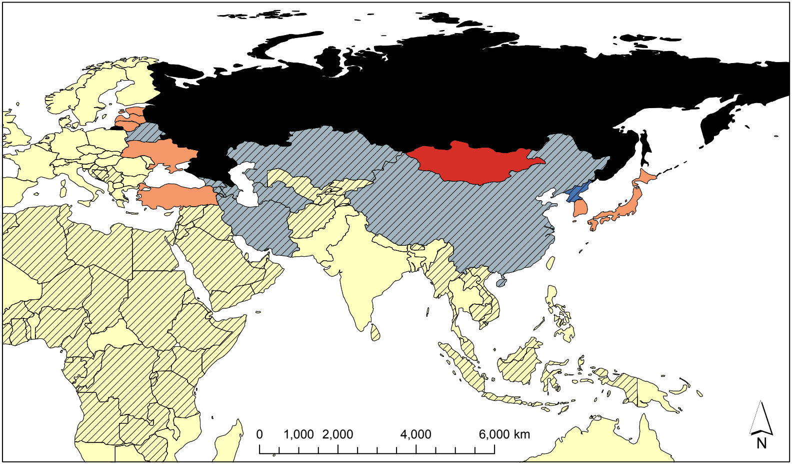

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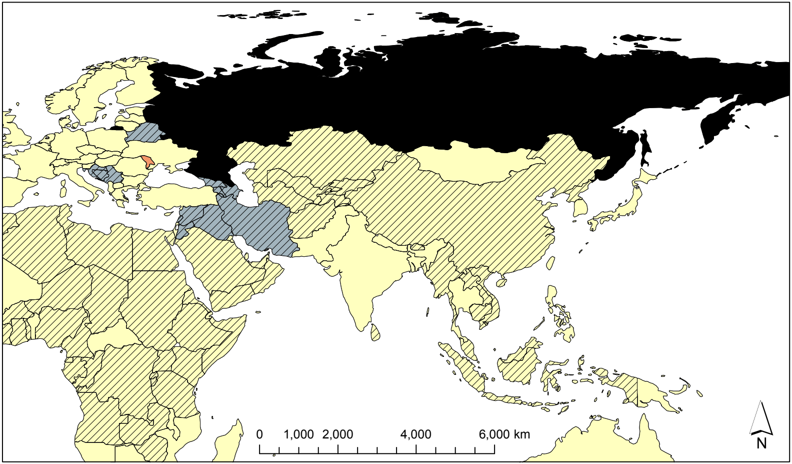

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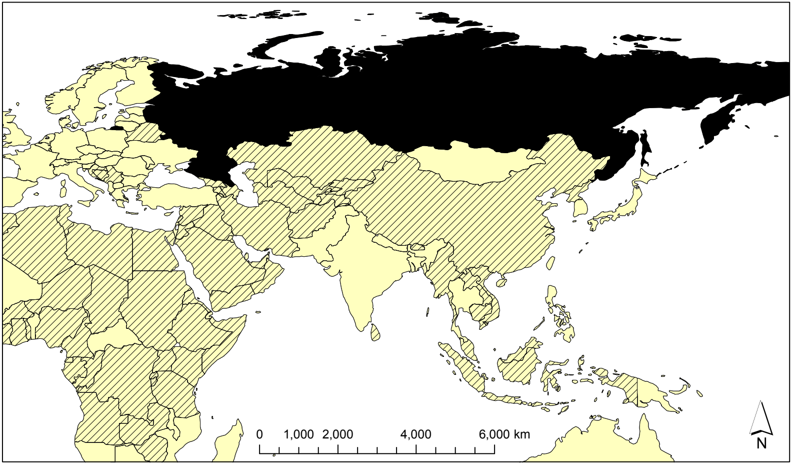

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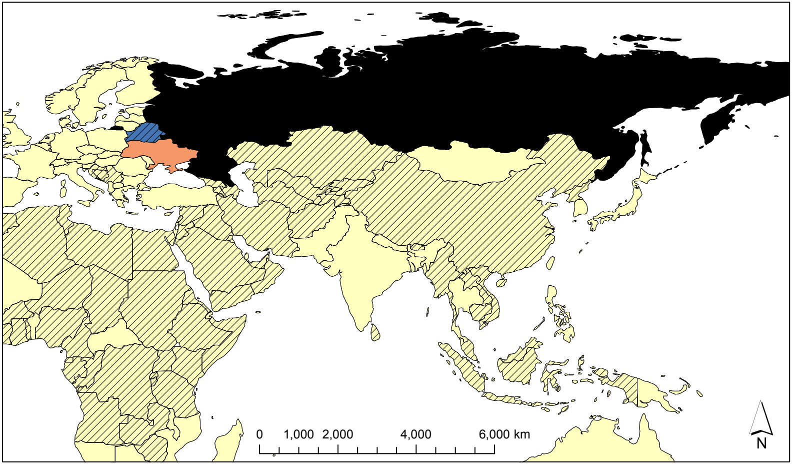
0.025 - 0.05

Russia's autocratization and regional regime stability



Sphere of Influence

Russia transitions from democracy to autocracy (1998 data)

Monte Carlo simulation (1,000 runs)

Russia

Regime Type

Democracy

Autocracy

Change in Transition Probability

-0.05- -0.025

-0.025 - -0.001

0

0.001 - 0.025

0.025 - 0.05

Kitten Wars

Kitten Wars


kittenwar
Warm, Flaky, Delicious
Crowd Pleaser.
Pillsbury™ Crescent Rolls
Save $1 Now
Alfredo vs. Amy
Home
Winningest Kittens
Losingest Kittens
Newest Kittens
Add your Kitten
The Daily Kitten
facebook group
faq
Privacy Policy
e-mail us
kitten search:
Go
Click on the cutest to decide the winner!!!
Can't decide? Refresh the page for a draw.

Kitten Wars

Winningest Kittens!

more

1. 

Bitsy
has won 76% of 8642 battles.

2. 

Freddie
has won 76% of 3768 battles.

Kitten Wars

Winningest Kittens!

more

1. 

Bitsy
has won 76% of 8642 battles.

2. 

Freddie
has won 76% of 3768 battles.

Losingest Kittens!

more

1. 

Scary Cat
has lost 79% of 11211 battles.

2. 

Beitsim
has lost 79% of 1919 battles.

Kitten Wars for Ideas (Salganik and Levy)

Kitten Wars for Ideas (Salganik and Levy)

RESEARCH ARTICLE

Wiki Surveys: Open and Quantifiable Social Data Collection

Matthew J. Salganik, Karen E. C. Levy

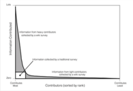
Published: May 20, 2015 • DOI: 10.1371/journal.pone.0123483

Article	Authors	Metrics	Comments	Related Content

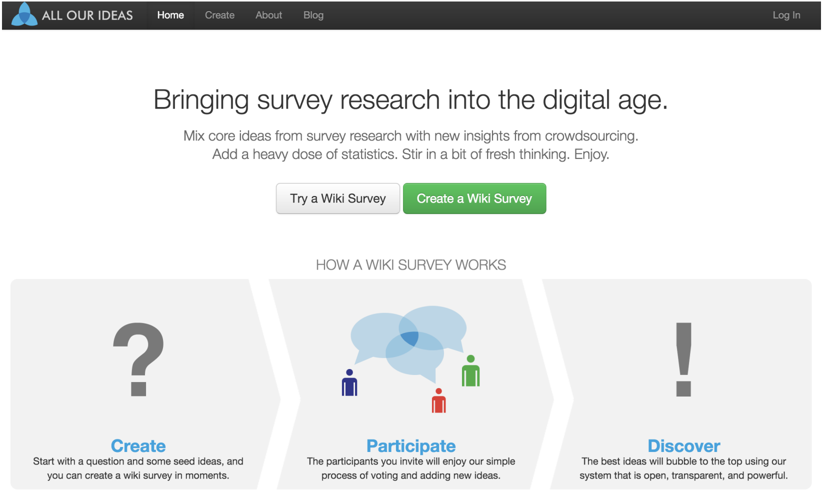
Abstract

In the social sciences, there is a longstanding tension between data collection methods that facilitate quantification and those that are open to unanticipated information. Advances in technology now enable new, hybrid methods that combine some of the benefits of both approaches. Drawing inspiration from online information aggregation systems like Wikipedia and from traditional survey research, we propose a new class of research instruments called *wiki surveys*. Just as Wikipedia evolves over time based on contributions from participants, we envision an evolving survey driven by contributions from respondents. We develop three general principles that underlie wiki surveys: they should be greedy, collaborative, and adaptive. Building on these principles, we develop methods for data collection and data analysis for one type of wiki survey, a pairwise wiki survey. Using two proof-of-concept case studies involving our free and open-source website www.allourideas.org, we show that pairwise wiki surveys can yield insights that would be difficult to obtain with other methods.

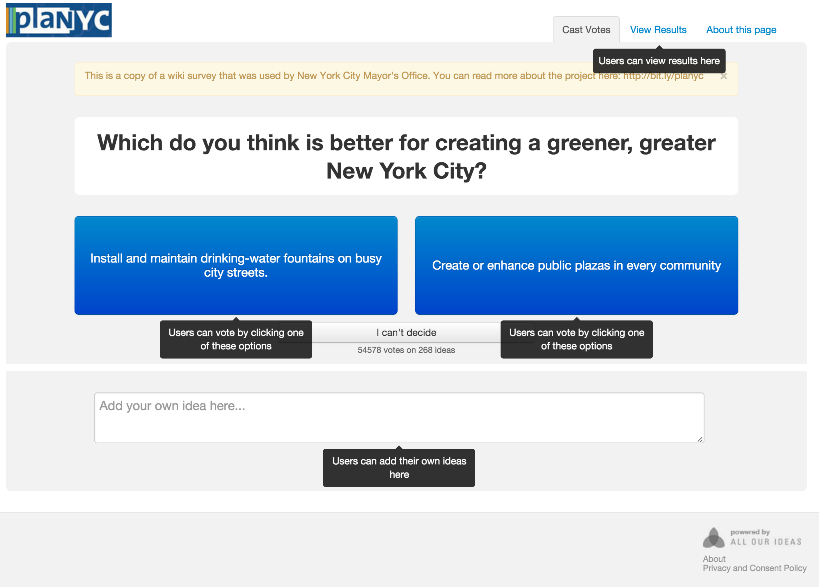
Figures




Kitten Wars for Ideas (Salganik and Levy)



Kitten Wars for Ideas (Salganik and Levy)



Kitten Wars for Ideas (Salganik and Levy)

Which do you think is better for creating a greener, greater New York City?

Ideas	Score (0 - 100)
Require all big buildings to make certain energy efficiency upgrades	67
Promote cycling by installing safe bike lanes	65
Promote the use of solar energy using the latest technology on all high-rise buildings.	65
Invest in multiple modes of transportation and provide both improved infrastructure and improved safety	65
Continue enhancing bike lane network, to finally connect separated bike lane systems to each other across all five boroughs.	65
Replace sodium vapor street lights with LED or other energy-saving lights.	64
Utilize NYC Rooftops to install Solar PV panels	63
Plant more trees	62
Create a network of protected bike paths throughout the entire city	62
Add improvements to the bike lanes in the inner city. This will encourage exercise and reduce city's carbon footprint.	62

The Power of Releasing Software

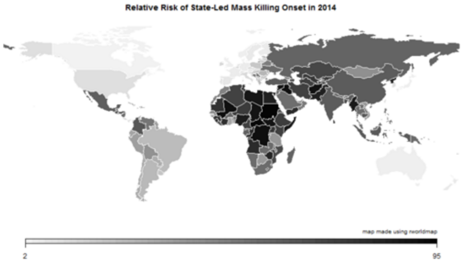
The Power of Releasing Software

The Governor asks ... again

Governor Tarso Genro of the state of Rio Grande do Sul, Brazil has done it again. The Governor and his team completed a second round of their amazing open government project called Governador Pergunta (The Governor Asks), which collects public feedback on important policy challenges using a customized version of allourideas.org.

The Power of Releasing Software

wiki surveys to assess risks of state-led mass killings



As part of their work with the Holocaust Museum's Center for the Prevention of Genocide, Jay Ulfelder and Ben Valentino launched a wiki survey to help assess the risks of state-led mass killing onsets in 2014. You can read about their results on this interesting blog post.

The Power of Releasing Software

UN Global Sustainability Report 2013

We are happy to announce that the United Nations Division for Sustainable Development is using allourideas.org to solicit ideas from scientists around the world for the 2013 UN Global Sustainability Report. The report will

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Powered By Research (You Can Do It Too!)

Backed by research

All Our Ideas is a research project based at Princeton University that is dedicated to creating new ways of collecting social data. You can learn more about the theory and methods behind our project by reading our paper or watching our talk. Thanks to Google, the National Science Foundation, and Princeton for supporting this research.

$$z_i \sim \begin{cases} N(\dot{\boldsymbol{x}}_i^T \boldsymbol{\theta}_{\boldsymbol{v}}, 1) I(z_i^* > 0) & \text{if } y_i = 1 \\ N(\dot{\boldsymbol{x}}_i^T \boldsymbol{\theta}_{\boldsymbol{v}}, 1) I(z_i^* < 0) & \text{if } y_i = 0 \end{cases}$$

Powered By Research (You Can Do It Too!)

Bradley–Terry model

From Wikipedia, the free encyclopedia

The **Bradley–Terry model** is a probability model that can predict the outcome of a comparison. Given a pair of individuals i and j drawn from some population, it estimates the probability that the pairwise comparison $i > j$ turns out true, as

$$P(i > j) = \frac{p_i}{p_i + p_j}$$

where p_i is a positive real-valued score assigned to individual i. The comparison $i > j$ can be read as "i is preferred to j", "i ranks higher than j", or "i beats j", depending on the application.

For example, p_i may represent the skill of a team in a sports tournament, estimated from the number of times i has won a match. $P(i > j)$ then represents the probability that i will win a match against j.[1][2] Another example used to explain the model's purpose is that of scoring products in a certain category by quality. While it's hard for a person to draft a direct ranking of (many) brands of wine, it may be feasible to compare a sample of pairs of wines and say, for each pair, which one is better. The Bradley–Terry model can then be used to derive a full ranking.[2]

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