Causal forests

A tutorial in high-dimensional causal inference

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General Exam
Frontiers of Causal Inference

12 October 2017

PC: Michael Schweppe via Wikimedia Commons

Note: These slides assume randomized treatment assignment until the section labeled "confounding."

Causal inference: A missing data problem

			Potential employment		
ID	Education X_i	Treated W_i	No job training $Y_i(0)$	Job training $Y_i(1)$	Treatment effect $\tau_i = Y_i(1) - Y_i(0)$
1	High school	0	0	1	1
2	High school	1	0	1	1
3	College	0	1	1	0
4	College	1	1	1	0

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$$
\begin{aligned}
\bar{\tau} &= \bar{Y}_{i:W_i=1}(1) - \bar{Y}_{i:W_i=0}(0) \\
&= 1 - 0.5 \\
&= 0.5
\end{aligned}
$$

Causal inference: A missing data problem

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If $W_i \perp\!\!\!\perp \{Y_i(0), Y_i(1)\}$, then

$$\begin{aligned}\hat{\hat{\tau}} &= \bar{Y}_{i:W_i=1} - \bar{Y}_{i:W_i=0} \\ &= 1 - 0.5 \\ &= 0.5\end{aligned}$$

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4	College	1	?	1	?

What if we want to study $\tau_i = f(X_i)$?

$$\hat{\hat{\tau}}_{\text{High school}} = \bar{Y}_{i:W_i=1,X_i=\text{High school}} - \bar{Y}_{i:W_i=0,X_i=\text{High school}} = 1 - 0.5 = 0.5$$

$$\hat{\hat{\tau}}_{\text{College}} = \bar{Y}_{i:W_i=1,X_i=\text{College}} - \bar{Y}_{i:W_i=0,X_i=\text{College}} = 1 - 1 = 0$$

Causal inference: A missing data problem

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What if there are dozens of X variables?

Causal inference: A missing data problem

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What if there are dozens of X variables?
What if X is continuous?

Causal inference: A missing data problem

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What if there are dozens of X variables?
What if X is continuous?

It's hard to know which subgroups of X
might show interesting effect heterogeneity

Start with a simpler prediction question.

Which subgroups of X have very different average outcomes?

Prediction: One tree

$\text{MSE}_0 = \frac{1}{n} \sum (Y_i - \bar{Y})^2$

All observations

Prediction: One tree

$$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

$$\text{MSE}_1 = \frac{1}{n}\sum(Y_i - \bar{Y}_{j:x_j \in \ell(x_i|\Pi_1)})^2$$

$$\begin{array}{ccc} & \text{All observations} & \\ \swarrow & & \searrow \\ X_i < k & & X_i \geq k \end{array}$$

Prediction: One tree

$$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

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$$\begin{array}{ccc} & \text{All observations} & \\ \swarrow & & \searrow \\ X_i < k & & X_i \geq k \end{array}$$

Choose k to minimize MSE_1

Prediction: One tree

$$\mathrm{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

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$$\begin{tikzcd}[column sep=small] & \text{All observations} \arrow[dl] \arrow[dr] & \\ X_i < 16 & & X_i \geq 16 \end{tikzcd}$$

Prediction: One tree

$$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

$$\text{MSE}_1 = \frac{1}{n}\sum(Y_i - \bar{Y}_{j:x_j \in \ell(x_i|\Pi_1)})^2$$

$$\text{MSE}_2 = \frac{1}{n}\sum(Y_i - \bar{Y}_{j:x_j \in \ell(x_i|\Pi_2)})^2$$

All observations

$X_i < 16$ $\quad$ $X_i \geq 16$

$X_i < k_1$ $\quad$ $X_i \geq k_1$ $\quad$ $X_i < k_2$ $\quad$ $X_i \geq k_2$

Prediction: One tree

$$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

All observations

$$\text{MSE}_1 = \frac{1}{n}\sum(Y_i - \bar{Y}_{j:x_j \in \ell(x_i|\Pi_1)})^2$$

$X_i < 16$ $\quad$ $X_i \geq 16$

$$\text{MSE}_2 = \frac{1}{n}\sum(Y_i - \bar{Y}_{j:x_j \in \ell(x_i|\Pi_2)})^2$$

$X_i < k_1$ $\quad$ $X_i \geq k_1$ $\quad$ $X_i < k_2$ $\quad$ $X_i \geq k_2$

Choose k_1 or k_2 to minimize MSE_2

Prediction: One tree

$$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$$

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All observations

$X_i < 16$ $\quad$ $X_i \geq 16$

$X_i < 12$ $\quad$ $X_i \geq 12$

Prediction: One tree

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$$\text{MSE}_3$$

All observations

$X_i < 16$ $X_i \geq 16$

$X_i < 12$ $X_i \geq 12$

$Z_i =$White $Z_i \neq$White

Prediction: One tree

$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$

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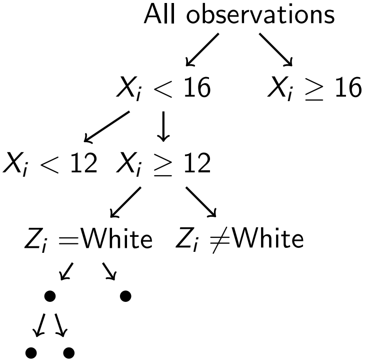
MSE_3



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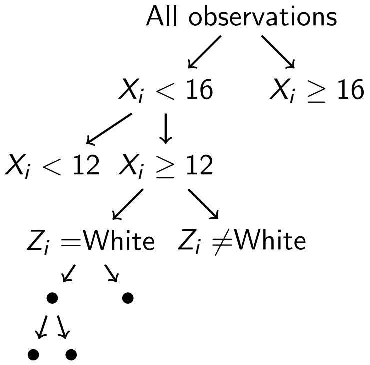

Could continue until all leaves had only one observation.

Unbiased but uselessly high variance!

Instead, regularize: keep only splits that improve MSE by more than c.

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$$\text{MSE}_3$$

All observations

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$X_i < 12$ $X_i \geq 12$

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$$\begin{tikzcd}
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 & X_i < 16 \arrow[dl] \arrow[d] & & X_i \geq 16 \\
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\end{tikzcd}$$

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All observations

$X_i < 16$ $\quad$ $X_i \geq 16$

Partition $\Pi \in \mathbb{P} \longrightarrow \left\{ \ell_1 = \{x_i : x_i < 16\},\ \ell_2 = \{x_i : x_i \geq 16\} \right\}$

Leaves

Prediction: One tree

$\text{MSE}_0 = \frac{1}{n}\sum(Y_i - \bar{Y})^2$

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Partition $\Pi \in \mathbb{P} \longrightarrow \left\{ \ell_1 = \{x_i : x_i < 16\},\ \ell_2 = \{x_i : x_i \geq 16\} \right\}$

Leaves

Prediction rule for new x:

$$\boxed{\hat{\mu}(x) = \bar{Y}_{j:x_j \in \ell(x_i|\Pi)}}$$

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Leaves

Prediction rule for new x:

$$\boxed{\hat{\mu}(x) = \bar{Y}_{j:x_j \in \ell(x_i|\Pi)}}$$

Could we use this method to find causal effects $\hat{\tau}(x)$ that are heterogeneous between leaves?

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   - One sample to choose partition
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Why is the split critical?

Fitting both on the training sample risks overfitting: Estimating many "heterogeneous effects" that are really just noise idiosyncratic to the sample.

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We want to search for true heterogeneity, not noise.

Sample splitting

$$\text{MSE}_\mu(S^{\text{te}}, S^{\text{est}}, \Pi) \equiv \frac{1}{\#(S^{\text{te}})} \sum_{i \in S^{\text{te}}} \left\{ \overbrace{(Y_i - \hat{\mu}(X_i; S^{\text{est}}, \Pi))^2}^{\text{MSE criterion}} - \overbrace{Y_i^2}^{\text{Authors add}} \right\}$$

Note: The authors include the final Y_i^2 term to simplify the math; it just shifts the estimator by a constant.

Sample splitting

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$$\mathrm{EMSE}_\mu(\Pi) \equiv \mathbb{E}_{S^{\mathrm{te}}, S^{\mathrm{est}}} \left[\mathrm{MSE}_\mu(S^{\mathrm{te}}, S^{\mathrm{est}}, \Pi) \right]$$

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Honest criterion: Maximize

This is S^{tr} in the classical approach

$$Q^H(\pi) \equiv -\mathbb{E}_{S^{\mathrm{te}}, S^{\mathrm{est}}, S^{\mathrm{tr}}} \left[\mathrm{MSE}_\mu(S^{\mathrm{te}}, S^{\mathrm{est}}, \pi(S^{\mathrm{tr}})) \right]$$

where $\pi : \mathbb{R}^{p+1} \to \mathbb{P}$ is a function that takes a training sample $S^{\mathrm{tr}} \in \mathbb{R}^{p+1}$ and outputs a partition $\Pi \in \mathbb{P}$.

Note: The authors include the final Y_i^2 term to simplify the math; it just shifts the estimator by a constant.

Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

Goal: Estimate expected MSE using only the training sample.

This will be used to place splits when training a tree.

Analytic estimator for $\mathrm{EMSE}_{\mu}(\Pi)$ (p. 7356)

$$
\begin{aligned}
-\mathrm{EMSE}_{\mu}(\Pi) &= -\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(Y_i - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2 - Y_i^2\right] \\
&= -\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi) + \mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2 - Y_i^2\right] \\
&= -\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi)\right)^2 - Y_i^2\right] \\
&\quad - \mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2\right] \\
&\quad - \mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[2\left(Y_i - \mu(X_i \mid \Pi)\right)\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)\right]
\end{aligned}
$$

Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

Expected mean squared error for a partition Π

$$
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&= -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi)\right)^2 - Y_i^2\right] \\
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Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

Expected mean squared error for a partition Π

$$-\text{EMSE}_\mu(\Pi) = -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \hat{\mu}(X_i \mid S^{\text{est}},\Pi)\right)^2 - Y_i^2\right]$$

Over estimation sets used to estimate the leaf-specific $\hat{\mu}$ and test sets to evaluate those

$$= -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi) + \mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\text{est}},\Pi)\right)^2 - Y_i^2\right]$$

$$= -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi)\right)^2 - Y_i^2\right]$$

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Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

Expected mean squared error for a partition Π

Prediction based on S^{est} from the leave $\ell(X_i)$ containing X_i

$$-\text{EMSE}_\mu(\Pi) = -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \hat{\mu}(X_i \mid S^{\text{est}}, \Pi)\right)^2 - Y_i^2\right]$$

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Analytic estimator for $\mathrm{EMSE}_\mu(\Pi)$ (p. 7356)

$$-\mathrm{EMSE}_\mu(\Pi) = -\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(Y_i - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2 - Y_i^2\right]$$

Add a zero

$$= -\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi) + \mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2 - Y_i^2\right]$$

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$$-\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)^2\right]$$

$$-\mathbb{E}_{S^{\mathrm{te}},S^{\mathrm{est}}}\left[2\left(Y_i - \mu(X_i \mid \Pi)\right)\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\mathrm{est}},\Pi)\right)\right]$$

Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

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First term^2

Second term^2

$$= -\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(Y_i - \mu(X_i \mid \Pi)\right)^2 - Y_i^2\right]$$

$$-\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\text{est}},\Pi)\right)^2\right]$$

2(First term)(Second term)

$$-\mathbb{E}_{S^{\text{te}},S^{\text{est}}}\left[2\left(Y_i - \mu(X_i \mid \Pi)\right)\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{\text{est}},\Pi)\right)\right]$$

Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

$$-\text{EMSE}_\mu(\Pi) = -\mathbb{E}_{S^{te},S^{est}}\left[\left(Y_i - \hat{\mu}(X_i \mid S^{est},\Pi)\right)^2 - Y_i^2\right]$$

$$= -\mathbb{E}_{S^{te},S^{est}}\left[\left(Y_i - \mu(X_i \mid \Pi) + \mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\right)^2 - Y_i^2\right]$$

$$= -\mathbb{E}_{S^{te},S^{est}}\left[\left(Y_i - \mu(X_i \mid \Pi)\right)^2 - Y_i^2\right]$$

$E(A) = 0$ by assumption

$$-\mathbb{E}_{S^{te},S^{est}}\left[\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\right)^2\right]$$

$Cov(A, B) = 0$ because Y_i is from a sample independent of S^{est}

$$Cov(AB) = E(AB) - E(A)E(B)$$
$$0 = E(AB) - 0$$

$$-\mathbb{E}_{S^{te},S^{est}}\left[2\underbrace{\left(Y_i - \mu(X_i \mid \Pi)\right)}_{A}\underbrace{\left(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\right)}_{B}\right]$$

Analytic estimator for $\text{EMSE}_\mu(\Pi)$ (p. 7356)

$$-\text{EMSE}_\mu(\Pi) = -\mathbb{E}_{S^{te},S^{est}}\left[\Big(Y_i - \hat{\mu}(X_i \mid S^{est},\Pi)\Big)^2 - Y_i^2\right]$$

$$= -\mathbb{E}_{S^{te},S^{est}}\left[\Big(Y_i - \mu(X_i \mid \Pi) + \mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\Big)^2 - Y_i^2\right]$$

$$= -\mathbb{E}_{S^{te},S^{est}}\left[\Big(Y_i - \mu(X_i \mid \Pi)\Big)^2 - Y_i^2\right]$$

$$-\mathbb{E}_{S^{te},S^{est}}\left[\Big(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\Big)^2\right]$$

$$-\mathbb{E}_{S^{te},S^{est}}\left[2\overset{A}{\Big(Y_i - \mu(X_i \mid \Pi)\Big)}\overset{B}{\Big(\mu(X_i \mid \Pi) - \hat{\mu}(X_i \mid S^{est},\Pi)\Big)}\right] = 0$$

$E(A) = 0$ by assumption

$Cov(A, B) = 0$ because Y_i is from a sample independent of S^{est}

$$Cov(AB) = E(AB) - E(A)E(B)$$
$$0 = E(AB) - 0$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[(Y_i - \mu(X_i \mid \Pi))^2 - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[Y_i^2 + \mu^2(X_i \mid \Pi) - 2Y_i\mu(X_i \mid \Pi) - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= \ -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[\mu^2(X_i \mid \Pi) - 2\mu(X_i \mid \Pi)\mu(X_i \mid \Pi)\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

Athey & Imbens 2016, p. 7356

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[(Y_i - \mu(X_i \mid \Pi))^2 - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

Y_i^2 terms cancel

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[Y_i^2 + \mu^2(X_i \mid \Pi) - 2Y_i\mu(X_i \mid \Pi) - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[\mu^2(X_i \mid \Pi) - 2\mu(X_i \mid \Pi)\mu(X_i \mid \Pi)\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

Athey & Imbens 2016, p. 7356

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[(Y_i - \mu(X_i \mid \Pi))^2 - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}(Y_i) = \mathbb{E}_{X_i,S^{\text{est}}}\mu(X_i \mid \Pi)$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[Y_i^2 + \mu^2(X_i \mid \Pi) - 2Y_i\mu(X_i \mid \Pi) - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[\mu^2(X_i \mid \Pi) - 2\mu(X_i \mid \Pi)\mu(X_i \mid \Pi)\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

Athey & Imbens 2016, p. 7356

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[(Y_i - \mu(X_i \mid \Pi))^2 - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[Y_i^2 + \mu^2(X_i \mid \Pi) - 2Y_i\mu(X_i \mid \Pi) - Y_i^2\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

$$= -\mathbb{E}_{(Y_i,X_i),S^{\text{est}}}\left[\mu^2(X_i \mid \Pi) - 2\mu(X_i \mid \Pi)\mu(X_i \mid \Pi)\right]$$
$$- \mathbb{E}_{X_i,S^{\text{est}}}\left[(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi) - \mu(X_i \mid \Pi))^2\right]$$

They have $\hat{\mu}^2$ here but I think they are wrong

I think

$$= \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{\cancel{S^{\text{est}}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

Athey & Imbens 2016, p. 7356

$$-\mathsf{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\mathrm{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\mathrm{est}}, \Pi))\right]$$

Estimate with $\hat{\mathbb{V}}\left(\hat{\mu}(x \mid S^{\text{est}}, \Pi)\right) \equiv \frac{S^2_{S^{\text{tr}}}(\ell(x|\Pi))}{N^{\text{est}}(\ell(x|\Pi))}$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\text{Estimate with } \hat{\mathbb{V}}\left(\hat{\mu}(x \mid S^{\text{est}}, \Pi)\right) \equiv \frac{S^2_{S^{\text{tr}}}(\ell(x|\Pi))}{N^{\text{est}}(\ell(x|\Pi))}$$

$$\hat{\mathbb{E}}_{X_i}\left[\hat{\mathbb{V}}_{S^{\text{est}}}\left(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi)\right) \mid i \in S^{\text{te}}\right] = \sum_{\ell} p_\ell \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{est}}(\ell)}$$

$$-\text{EMSE}_\mu(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\text{Estimate with } \hat{\mathbb{V}}\left(\hat{\mu}(x \mid S^{\text{est}}, \Pi)\right) \equiv \frac{S^2_{S^{\text{tr}}}(\ell(x|\Pi))}{N^{\text{est}}(\ell(x|\Pi))}$$

$$\hat{\mathbb{E}}_{X_i}\left[\hat{\mathbb{V}}_{S^{\text{est}}}\left(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi)\right) \mid i \in S^{\text{te}}\right] = \sum_{\ell} p_{\ell} \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{est}}(\ell)}$$

$$\text{(assuming } \approx \text{ equal leaf sizes)} \approx \sum_{\ell} \frac{1}{\#\ell} \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{est}}/\#\ell}$$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\text{Estimate with } \hat{\mathbb{V}}\left(\hat{\mu}(x \mid S^{\text{est}}, \Pi)\right) \equiv \frac{S^2_{S^{\text{tr}}}(\ell(x|\Pi))}{N^{\text{est}}(\ell(x|\Pi))}$$

$$\begin{aligned}
\hat{\mathbb{E}}_{X_i}\left[\hat{\mathbb{V}}_{S^{\text{est}}}\left(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi)\right) \mid i \in S^{\text{te}}\right] &= \sum_{\ell} p_\ell \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{est}}(\ell)} \\
\text{(assuming } \approx \text{ equal leaf sizes)} &\approx \sum_{\ell} \frac{1}{\#\ell} \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{est}}/\#\ell} \\
&= \frac{1}{N^{\text{est}}} \sum_{\ell \in \Pi} S^2_{S^{\text{tr}}}(\ell)
\end{aligned}$$

$$-\text{EMSE}_\mu(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$-\text{EMSE}_\mu(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}},X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\mathbb{V}(\hat{\mu} \mid x, \Pi) = \mathbb{E}(\hat{\mu}^2 \mid x, \Pi) - \left[\mathbb{E}(\hat{\mu} \mid x, \Pi)\right]^2$$

$$-\mathrm{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\mathrm{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\mathrm{est}}, \Pi))\right]$$

$$\mathbb{V}(\hat{\mu} \mid x, \Pi) = \mathbb{E}(\hat{\mu}^2 \mid x, \Pi) - \left[\mathbb{E}(\hat{\mu} \mid x, \Pi)\right]^2$$

$$\frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))} \approx \hat{\mu}^2(x \mid S^{\text{tr}}\Pi) - \mu^2(x \mid \Pi)$$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\mathbb{V}(\hat{\mu} \mid x, \Pi) = \mathbb{E}(\hat{\mu}^2 \mid x, \Pi) - \left[\mathbb{E}(\hat{\mu} \mid x, \Pi)\right]^2$$

$$\frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))} \approx \hat{\mu}^2(x \mid S^{\text{tr}}\Pi) - \mu^2(x \mid \Pi)$$

$$\mu^2(x \mid \Pi) \approx \hat{\mu}^2(x \mid S^{\text{tr}}, \Pi) - \frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))}$$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\mathbb{V}(\hat{\mu} \mid x, \Pi) = \mathbb{E}(\hat{\mu}^2 \mid x, \Pi) - \left[\mathbb{E}(\hat{\mu} \mid x, \Pi)\right]^2$$

$$\frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))} \approx \hat{\mu}^2(x \mid S^{\text{tr}}\Pi) - \mu^2(x \mid \Pi)$$

$$\mu^2(x \mid \Pi) \approx \hat{\mu}^2(x \mid S^{\text{tr}}, \Pi) - \frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))}$$

$$\hat{\mathbb{E}}_{X_i}(\mu^2(X_i \mid \Pi)) \approx \frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\mu}^2(x_i \mid S^{\text{tr}}, \Pi) - \sum_{\ell} \frac{1}{\#\ell} \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{tr}}/\#\ell}$$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$\mathbb{V}(\hat{\mu} \mid x, \Pi) = \mathbb{E}(\hat{\mu}^2 \mid x, \Pi) - \left[\mathbb{E}(\hat{\mu} \mid x, \Pi)\right]^2$$

$$\frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))} \approx \hat{\mu}^2(x \mid S^{\text{tr}}\Pi) - \mu^2(x \mid \Pi)$$

$$\mu^2(x \mid \Pi) \approx \hat{\mu}^2(x \mid S^{\text{tr}}, \Pi) - \frac{S^2_{S^{\text{tr}}}(\ell(x \mid \Pi))}{N^{\text{tr}}(\ell(x \mid \Pi))}$$

$$\hat{\mathbb{E}}_{X_i}(\mu^2(X_i \mid \Pi)) \approx \frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\mu}^2(x_i \mid S^{\text{tr}}, \Pi) - \sum_{\ell} \frac{1}{\#\ell} \frac{S^2_{S^{\text{tr}}}(\ell)}{N^{\text{tr}}/\#\ell}$$

$$= \frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\mu}^2(x_i \mid S^{\text{tr}}, \Pi) - \frac{1}{N^{\text{tr}}} \sum_{\ell} S^2_{S^{\text{tr}}}(\ell)$$

$$-\text{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\text{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\text{est}}, \Pi))\right]$$

$$-\widehat{\mathrm{EMSE}}_\mu(S^{\mathrm{tr}}, N^{\mathrm{est}}, \Pi) = \frac{1}{N^{\mathrm{tr}}} \sum_{i \in S^{\mathrm{tr}}} \hat{\mu}^2(X_i \mid S^{\mathrm{tr}}, \Pi) - \frac{1}{N^{\mathrm{tr}}} \sum_{\ell \in \Pi} S^2_{S^{\mathrm{tr}}}(\ell) - \frac{1}{N^{\mathrm{est}}} \sum_{\ell \in \Pi} S^2_{S^{\mathrm{tr}}}(\ell)$$

$$-\mathrm{EMSE}_\mu(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\mathrm{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\mathrm{est}}, \Pi))\right]$$

$$
\begin{aligned}
-\widehat{\mathrm{EMSE}}_{\mu}(S^{\mathrm{tr}}, N^{\mathrm{est}}, \Pi) &= \frac{1}{N^{\mathrm{tr}}} \sum_{i \in S^{\mathrm{tr}}} \hat{\mu}^2(X_i \mid S^{\mathrm{tr}}, \Pi) - \frac{1}{N^{\mathrm{tr}}} \sum_{\ell \in \Pi} S^2_{S^{\mathrm{tr}}}(\ell) \\
&\quad - \frac{1}{N^{\mathrm{est}}} \sum_{\ell \in \Pi} S^2_{S^{\mathrm{tr}}}(\ell) \\
&= \underbrace{\frac{1}{N^{\mathrm{tr}}} \sum_{i \in S^{\mathrm{tr}}} \hat{\mu}^2(X_i \mid S^{\mathrm{tr}}, \Pi)}_{\text{Conventional CART criterion}} - \underbrace{\left(\frac{1}{N^{\mathrm{tr}}} + \frac{1}{N^{\mathrm{est}}}\right) \sum_{\ell \in \Pi} S^2_{S^{\mathrm{tr}}}(\ell)}_{\text{Uncertainty about leaf means}}
\end{aligned}
$$

$$
-\mathrm{EMSE}_{\mu}(\Pi) = \mathbb{E}_{X_i}\left[\mu^2(X_i \mid \Pi)\right] - \mathbb{E}_{S^{\mathrm{est}}, X_i}\left[\mathbb{V}(\hat{\mu}(X_i \mid S^{\mathrm{est}}, \Pi))\right]
$$

Honest inference for treatment effects

Note: We still assume randomized treatment assignment

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\left[Y_i(w) \mid X_i \in \ell(x \mid \Pi) \right]$$

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Potential outcome for treatment w (heterogeneous by X_i)

Averaged over controls X_i in the leaf

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Average causal effect:

$$\tau(x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(1) - Y_i(0) \mid X_i \in \ell(x \mid \Pi)\Big] = \mu(1, x \mid \Pi) - \mu(0, x \mid \Pi)$$

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Average causal effect:

$$\tau(x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(1) - Y_i(0) \mid X_i \in \ell(x \mid \Pi)\Big] = \mu(1, x \mid \Pi) - \mu(0, x \mid \Pi)$$

Average effect evaluated at (potentially moderating) covariate value x

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Average causal effect:

$$\tau(x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(1) - Y_i(0) \mid X_i \in \ell(x \mid \Pi)\Big] = \mu(1, x \mid \Pi) - \mu(0, x \mid \Pi)$$

Difference in potential outcomes

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Average causal effect:

$$\tau(x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(1) - Y_i(0) \mid X_i \in \ell(x \mid \Pi)\Big] = \mu(1, x \mid \Pi) - \mu(0, x \mid \Pi)$$

Among observations in the leaf ℓ

Honest inference for treatment effects

Population-average potential outcomes within leaves:

$$\mu(w, x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(w) \mid X_i \in \ell(x \mid \Pi)\Big]$$

Average causal effect:

$$\tau(x \mid \Pi) \equiv \mathbb{E}\Big[Y_i(1) - Y_i(0) \mid X_i \in \ell(x \mid \Pi)\Big] = \mu(1, x \mid \Pi) - \mu(0, x \mid \Pi)$$

Compact notation

Estimate:

$$\hat{\mu}(w, x \mid S, \Pi) \equiv \frac{1}{\#(\{i \in S_w : X_i \in \ell(x|\Pi)\})} \sum_{i \in S_w : X_i \in \ell(x|\Pi)} Y_i^{\text{obs}}$$

Estimate:

$$\hat{\mu}(w, x \mid S, \Pi) \equiv \frac{1}{\#(\{i \in S_w : X_i \in \ell(x|\Pi)\})} \sum_{i \in S_w : X_i \in \ell(x|\Pi)} Y_i^{\text{obs}}$$

MSE for treatment effects:

$$\text{MSE}_\tau(S^{\text{te}}, S^{\text{est}}, \Pi) \equiv \frac{1}{\#(S^{\text{te}})} \sum_{i \in S^{\text{te}}} \left\{ \left(\tau_i - \hat{\tau}(X_i \mid S^{\text{est}}, \Pi) \right)^2 - \tau_i^2 \right\}$$

Estimate:

$$\hat{\mu}(w, x \mid S, \Pi) \equiv \frac{1}{\#(\{i \in S_w : X_i \in \ell(x|\Pi)\})} \sum_{i \in S_w : X_i \in \ell(x|\Pi)} Y_i^{\text{obs}}$$

MSE for treatment effects:

$$\text{MSE}_\tau(S^{\text{te}}, S^{\text{est}}, \Pi) \equiv \frac{1}{\#(S^{\text{te}})} \sum_{i \in S^{\text{te}}} \left\{ \left(\tau_i - \hat{\tau}(X_i \mid S^{\text{est}}, \Pi) \right)^2 - \tau_i^2 \right\}$$

Challenge! τ_i is *never* observed.

Adapt EMSE_μ to estimate EMSE_τ

$$-\widehat{\text{EMSE}}_\mu(S^{\text{tr}}, N^{\text{est}}, \Pi) = \underbrace{\frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\mu}^2(X_i \mid S^{\text{tr}}, \Pi)}_{\text{Conventional CART criterion}} - \underbrace{\left(\frac{1}{N^{\text{tr}}} + \frac{1}{N^{\text{est}}}\right) \sum_{\ell \in \Pi} S^2_{S^{\text{tr}}}(\ell)}_{\text{Uncertainty about leaf means}}$$

$$-\widehat{\text{EMSE}}_\tau(S^{\text{tr}}, N^{\text{est}}, \Pi) = \underbrace{\frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\tau}^2(X_i \mid S^{\text{tr}}, \Pi)}_{\substack{\text{Variance of treatment} \\ \text{effects across leaves}}}$$

$$- \underbrace{\left(\frac{1}{N^{\text{tr}}} + \frac{1}{N^{\text{est}}}\right) \sum_{\ell \in \Pi} \left(\frac{S^2_{S^{\text{tr}}_{\text{treat}}}(\ell)}{p} + \frac{S^2_{S^{\text{tr}}_{\text{control}}}(\ell)}{1-p}\right)}_{\text{Uncertainty about leaf treatment effects}}$$

Adapt EMSE_μ to estimate EMSE_τ

$$-\widehat{\text{EMSE}}_\mu(S^{\text{tr}}, N^{\text{est}}, \Pi) = \underbrace{\frac{1}{N^{\text{tr}}}\sum_{i\in S^{\text{tr}}} \hat{\mu}^2(X_i \mid S^{\text{tr}}, \Pi)}_{\text{Conventional CART criterion}} - \underbrace{\left(\frac{1}{N^{\text{tr}}} + \frac{1}{N^{\text{est}}}\right)\sum_{\ell\in\Pi} S^2_{S^{\text{tr}}}(\ell)}_{\text{Uncertainty about leaf means}}$$

$$-\widehat{\text{EMSE}}_\tau(S^{\text{tr}}, N^{\text{est}}, \Pi) = \underbrace{\frac{1}{N^{\text{tr}}}\sum_{i\in S^{\text{tr}}} \hat{\tau}^2(X_i \mid S^{\text{tr}}, \Pi)}_{\substack{\text{Variance of treatment}\\ \text{effects across leaves}}}$$

$$\underbrace{-\left(\frac{1}{N^{\text{tr}}} + \frac{1}{N^{\text{est}}}\right)\sum_{\ell\in\Pi}\left(\frac{S^2_{S^{\text{tr}}_{\text{treat}}}(\ell)}{p} + \frac{S^2_{S^{\text{tr}}_{\text{control}}}(\ell)}{1-p}\right)}_{\text{Uncertainty about leaf treatment effects}}$$

Prefers leaves with heterogeneous effects

Prefers leaves with good fit (leaf-specific effects estimated precisely)

Four partitioning estimators

1. Causal trees

Split by

$$-\widehat{\text{EMSE}}_\tau(S^{\text{tr}}, N^{\text{est}}, \Pi) = \underbrace{\frac{1}{N^{\text{tr}}} \sum_{i \in S^{\text{tr}}} \hat{\tau}^2(X_i \mid S^{\text{tr}}, \Pi)}_{\text{Variance of treatment effects across leaves}}$$

Prefers leaves with heterogeneous effects

$$\underbrace{- \left(\frac{1}{N^{\text{tr}}} + \frac{1}{N^{\text{est}}} \right) \sum_{\ell \in \Pi} \left(\frac{S^2_{S^{\text{tr}}_{\text{treat}}}(\ell)}{p} + \frac{S^2_{S^{\text{tr}}_{\text{control}}}(\ell)}{1-p} \right)}_{\text{Uncertainty about leaf treatment effects}}$$

Prefers leaves with good fit (leaf-specific effects estimated precisely)

- Benefit: Prioritizes heterogeneity ($\hat{\tau}$ varies a lot) and fit (within-leaf precision)
- Drawback: Cannot be done with off-the-shelf CART methods

2. Transformed outcome trees

Transform the outcome

$$Y_i^* = Y_i \frac{W_i - p}{p(1-p)} \to \mathbb{E}(Y_i^* \mid X_i = x) = \tau(x)$$

$$\begin{aligned}
\mathbb{E}(Y_i^*) &= \mathbb{E}\left[Y_i \frac{W_i - p}{p(1-p)}\right] \\
&= \mathbb{E}\left[Y_i \frac{W_i}{p(1-p)}\right] - \mathbb{E}\left[Y_i \frac{p}{p(1-p)}\right] \\
&= \mathbb{E}\left[Y_i(1) \frac{W_i}{p(1-p)}\right] - \mathbb{E}\left[\left(Y_i(1)W_i + Y_i(0)(1-W_i)\right)\frac{p}{p(1-p)}\right] \\
&= Y_i(1)\frac{1}{p(1-p)}\mathbb{E}[W_i] - Y_i(1)\frac{p}{p(1-p)}\mathbb{E}[W_i] - Y_i(0)\frac{p}{p(1-p)}\mathbb{E}[1-W_i] \\
&= Y_i(1)\frac{1-p}{p(1-p)}\mathbb{E}[W_i] - Y_i(0)\frac{p}{p(1-p)}\mathbb{E}[1-W_i] \\
&= Y_i(1)\frac{p(1-p)}{p(1-p)} - Y_i(0)\frac{p(1-p)}{p(1-p)} \\
&= Y_i(1) - Y_i(0) = \tau_i
\end{aligned}$$

2. Transformed outcome trees

- Benefit: Can use off-the-shelf CART methods for prediction
- Drawbacks: Inefficient. Treatment is ignored after transforming outcome.
  If within a leaf $\bar{W} \neq p$ (by chance), then sample average within leaf is a poor estimator of $\hat{\tau}$.

3. Fit-based trees

Replace

$$\mathrm{MSE}_{\mu}(S^{\mathrm{te}}, S^{\mathrm{est}}, \Pi) \equiv \frac{1}{\#(S^{\mathrm{te}})} \sum_{i \in S^{\mathrm{te}}} \left\{ (Y_i - \hat{\mu}(X_i; S^{\mathrm{est}}, \Pi))^2 - Y_i^2 \right\}$$

with the fit-based split rule

$$\mathrm{MSE}_{\mu,W}(S^{\mathrm{te}}, S^{\mathrm{est}}, \Pi) \equiv \sum_{i \in S^{\mathrm{te}}} \left\{ (Y_i - \hat{\mu}_w(W_i X_i; S^{\mathrm{est}}, \Pi))^2 - Y_i^2 \right\}$$

which loss by model fit within each leaf: the difference from the expected value for the treatment group of observation i.

Benefit: Prefers splits that lead to better fit.

Drawback: Does not prefer splits that lead to variation in treatment effects.

Zeileis et al. 2008

4. Squared T-statistic trees

Split based on:

$\hat{\tau}$ in left leaf in right leaf

$$T^2 \equiv N \frac{(\bar{Y}_L - \bar{Y}_R)^2}{S^2/N_L + S^2/N_R}$$

Benefit: Prefers splits that lead to variation in treatment effects.

Drawback: Missed opportunity to improve fit: ignores useful splits between leaves with similar treatment effects but very different average values.

Su et al. 2009

From trees to forests: Double-sample trees

An individual tree can be noisy. Instead, we might fit a forest.

1. Draw a sample of size s
2. Split into an $\mathcal{I}$ and $\mathcal{J}$ sample.
3. Grow a tree on the $\mathcal{J}$ sample
4. Estimate leaf-specific $\hat{\tau}_\ell$ using the $\mathcal{I}$ sample

Repeat many times.

Advantages of forests:

- Consistent for true $\tau(x)$
- Asymptotic normality
- Asymptotic variance is estimable

Why *double-sample* forests:

- Advantage: Trees search for heterogeneous effects
- Disadvantage: Requires sample splitting

Wager & Athey 2017

From trees to forests: Propensity trees

An individual tree can be noisy. Instead, we might fit a forest.

1. Draw a sample of size s
2. Grow a tree on the $\mathcal{J}$ sample to predict W
   - Each leaf must have at least k observations of each treatment class
3. Estimate $\hat{\tau}_\ell$ on each leaf

Repeat many times.

Advantages of forests:

- Consistent for true $\tau(x)$
- Asymptotic normality
- Asymptotic variance is estimable

Why *propensity* forests:

- Advantage: Can use full sample
- Disadvantage: Does not search for heterogeneous effects

Wager & Athey 2017

Summary of causal trees and forests

- There is no ground truth: We never observe τ_i

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- With selection on observables, the general recommendation is propensity forests
  - Maximizes the goal of addressing confounding by ignoring heterogeneous effects when choosing splits
  - Generalized random forests also perform well (Athey, Tibshirani, & Wager 2017)
  - But "the challenge in using adaptive methods. . . is that selection bias can be difficult to quantify" (Wager & Athey p. 24).

If treatment is not randomized

Causal trees find heterogeneous effects but cannot guarantee that confounding is addressed.

Next we focus on why high-dimensional confounding is hard

Why aren't causal trees guaranteed to address confounding?

Plan

1. What does address confounding? Standardization
2. Why is tree-based standardization biased? Regularization
3. Is there anything we can do? Chernozhukov et al.

What works: Nonparametric standardization

What if $\{Y_i(0), Y_i(1)\} \not\perp\!\!\!\perp W_i$ but $\{Y_i(0), Y_i(1)\} \perp\!\!\!\perp W_i \mid X_i$?

What works: Nonparametric standardization

What if $\{Y_i(0), Y_i(1)\} \not\perp\!\!\!\perp W_i$ but $\{Y_i(0), Y_i(1)\} \perp\!\!\!\perp W_i \mid X_i$?

			Potential employment		
	Education	Treated	No job training	Job training	Treatment effect
ID	X_i	W_i	$Y_i(0)$	$Y_i(1)$	$\tau_i = Y_i(1) - Y_i(0)$
1	High school	0	0	1	1
2	High school	0	0	1	1
3	High school	1	0	1	1
4	College	0	1	1	0
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We need to estimate $\hat{\tau}$ within each level of X_i.

			Potential employment		
	Education	Treated	No job training	Job training	Treatment effect
ID	X_i	W_i	$Y_i(0)$	$Y_i(1)$	$\tau_i = Y_i(1) - Y_i(0)$
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What works: Nonparametric standardization

$$
\begin{aligned}
\hat{\hat{\tau}} &= \sum_{x \in \text{Support of } X} \mathbb{P}(X = x)\left(\bar{Y}_{i:W_i=1,X_i=x} - \bar{Y}_{i:W_i=0,X_i=x}\right) \\
&= \mathbb{P}(X_i = \text{High school})\left(\bar{Y}_{i:W_i=1,X_i=\text{High school}} - \bar{Y}_{i:W_i=0,X_i=\text{High school}}\right) \\
&\quad + \mathbb{P}(X_i = \text{College})\left(\bar{Y}_{i:W_i=1,X_i=\text{College}} - \bar{Y}_{i:W_i=0,X_i=\text{College}}\right) \\
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\end{aligned}
$$

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What works: Nonparametric standardization

But when there are many cells of the covariates X_i,

nonparametric standardization is impossible!

Why is tree-based standardization biased? Regularization

With no regularization, a tree would grow until each leaf was completely homogenous in X_i.

But this tree would be very noisy! We prune our trees so that leaves contain more observations.

- Treatment effects are more precisely estimated
- But treatment effects are biased if there is confounding within leaves

Is there anything we can do? Chernozhukov et al.

$$\overbrace{Y = D\theta_0 + g_0(X) + U}^{\text{Outcome equation}} \qquad \overbrace{D = m_0(X) + V}^{\text{Treatment assignment}}$$

One might be tempted to estimate $\hat{g}_0(X)$ by machine learning and then state:

$$\hat{\theta}_0 = \frac{\frac{1}{n}\sum_{i\in\mathcal{I}} D_i(Y_i - \hat{g}_0(X_i))}{\frac{1}{n}\sum_{i\in\mathcal{I}} D_i^2}$$

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This will be biased because the estimator $\hat{g}_0$ is regularized.

$$b = \frac{1}{\mathbb{E}(D_i^2)}\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{I}} \overbrace{\Big(m_0(X_i)(g_0(X_i) - \hat{g}_0(X_i)\Big)}^{\text{Does not have mean 0}} + o_P(1)$$

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Key: D_i is centered at $m_0(X) \neq 0$. We should recenter D_i.

Is there anything we can do? Chernozhukov et al.

$$\overbrace{Y = D\theta_0 + g_0(X) + U}^{\text{Outcome equation}} \qquad \overbrace{D = m_0(X) + V}^{\text{Treatment assignment}}$$

1. Split the sample into $\mathcal{I}$ and $\mathcal{J}$
2. Estimate $\hat{g}_0(X)$ using sample $\mathcal{J}$
3. Estimate $\hat{m}_0(X)$ using sample $\mathcal{J}$
4. Orthogonalize D on X (approximately)

$$\hat{V} = D - \hat{m}_0(X)$$

5. Estimate the treatment effect

Biased

$$\hat{\theta}_0 = \frac{\frac{1}{n}\sum_{i\in\mathcal{I}} D_i(Y_i - \hat{g}_0(X_i))}{\frac{1}{n}\sum_{i\in\mathcal{I}} D_i^2}$$

De-biased

$$\hat{\theta}_0 = \frac{\frac{1}{n}\sum_{i\in\mathcal{I}} \hat{V}_i(Y_i - \hat{g}_0(X_i))}{\frac{1}{n}\sum_{i\in\mathcal{I}} \hat{V}_i D_i}$$

Chernozhukov et al. 2016

Bias remaining in de-biased estimator (Chernozhukov et al.)

$$\sqrt{n}(\hat{\theta}_0 - \theta_0) = a^* + b^* + c^*$$

Bias remaining in de-biased estimator (Chernozhukov et al.)

$$\sqrt{n}(\hat{\theta}_0 - \theta_0) = a^* + b^* + c^*$$

$$a^* = \frac{1}{\mathbb{E}(V^2)} \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} V_i U_i \to N(0, \Sigma)$$

Because a^* converges to mean 0, we don't worry about it.

Bias remaining in de-biased estimator (Chernozhukov et al.)

$$\sqrt{n}(\hat{\theta}_0 - \theta_0) = a^* + b^* + c^*$$

Regularization bias:

$$b^* = \frac{1}{\mathbb{E}(V^2)} \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} \Big(\hat{m}_0(X_i) - m_0(X_i)\Big)\Big(\hat{g}_0(X_i) - g_0(X_i)\Big)$$

Vanishes "under a broad range of data-generating processes."

Bounded above by

$$\sqrt{n}\, n^{-\psi_m}\, n^{-\psi_g}$$

Rate of convergence of $\hat{m}_0 \to m$

Rate of convergence of $\hat{g}_0 \to g$

Bias remaining in de-biased estimator (Chernozhukov et al.)

$$\sqrt{n}(\hat{\theta}_0 - \theta_0) = a^* + b^* + c^*$$

An example of the third term in the partially linear model:

$$c^* = \frac{1}{\sqrt{n}} \sum_{i \in \mathcal{I}} V_i \Big(\hat{g}_0(X_i) - g_0(X_i) \Big)$$

If $\hat{g}_0$ is estimated on an auxiliary sample $\mathcal{J}$, then V_i and $\hat{g}_0(X_i)$ will be uncorrelated and $\mathbb{E}(c^*) = 0$.

BART: Bayesian Additive Regression Trees

Differs from random forests:

- Fixed number of trees
- Backfits repeatedly over the fixed number of trees
- Strong prior encourages shallow trees
- Uncertainty comes automatically from posterior samples

Chipman, George, & McCulloch 2010

BART model

$$Y = \sum_{j=1}^{m} g_j(x \mid T_j, M_j) + \epsilon$$

$$\epsilon \sim N(0, \sigma^2)$$

T_j prior

$$P(\underbrace{D_j = d}_{\text{Tree depth}}) = \alpha(1+d)^{-\beta}$$

$$\text{Split variable} \sim \text{Uniform(Available variables)}$$

$$\text{Split value} \sim \text{Uniform(Available split values)}$$

$\mu_{ij} \mid T_j$ prior

$$\underbrace{\mu_{ij}}_{\text{Tree } i \text{ leaf } j} \sim N\Bigg(\underbrace{\mu_m, \sigma^2_\mu}_{\substack{\text{Chosen so that} \\ \text{high probability of} \\ E(Y|x)\in(y_{\min}, y_{\max})}}\Bigg)$$

σ prior

$$\sigma \sim \frac{\nu\lambda}{\chi^2_\nu} \text{ (inverse chi-square)}$$

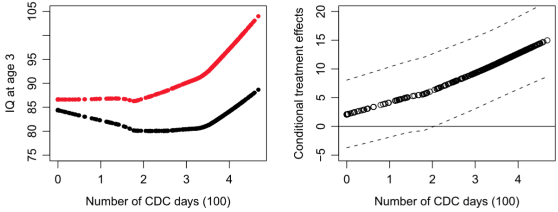
They recommend $\{\alpha = .95, \beta = 2\} \rightarrow$ 97% of prior probability is on 4 or fewer terminal nodes.

Chipman, George, & McCulloch 2010

BART for causal inference

Goal: Model the response surface as a function of treatment and pre-treatment covariates

1. Fit a flexible model for $Y = f(X, W)$
2. Set $W = 0$ to predict $\hat{Y}_i(0)$ for all i
3. Set $W = 1$ to predict $\hat{Y}_i(1)$ for all i
4. Difference to estimate $\hat{\tau}_i$
5. Plot effects



BART: Benefits and drawbacks

Benefits

- Less researcher discretion for tuning parameters
- Automatic posterior uncertainty estimates

Drawbacks

- Not guaranteed to address confounding due to regularization
- No theoretical guarantees of centering over truth
- Splitting is based on prediction and is not explicitly optimized for causal inference within leaves

Summary

- Causal trees can detect high-dimensional covariate-based treatment effect heterogeneity
- Work well with high-order interactions
- Causal forests give theoretically valid confidence intervals
- Bayesian approaches (BART) are less theoretically verified but give easy uncertainty
- With high-dimensional confounding, all methods are biased but can be designed to be consistent.