Soc504: Basics

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Princeton

February 8, 2017

[1] These slides are a very lightly remixed version of slides for Gary King's course Gov2001 at Harvard.

Followup

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- Questions?

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- Questions?
- Replication Stories?

Simulation is used to:

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1. Solve probability problems
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3. Calculate features of probability densities
4. Transform statistical results into quantities of interest
5. Get the right answer: students get the right answer far more frequently by using simulation than math

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3. The estimate is arbitrarily precise for large n	3. The approximation is arbitrarily precise for large M

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2. Use the random sample to estimate a feature of the population	2. Use the random draws to approximate a feature of the distribution
3. The estimate is arbitrarily precise for large n	3. The approximation is arbitrarily precise for large M
4. Example: estimate the mean of the population	4. Example: Approximate the mean of the distribution

Monty Hall's Let's Make a Deal

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You have a choice among 3 doors. Behind a random door is a car; behind the other two are goats. You choose one at random. Monty peeks behind the other two doors and opens the one (or one of the two) with the goat and asks if you'd like to switch your door for the other door that hasn't been opened yet. Should you switch?

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doors <- c(1, 2, 3)
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sims <- 1000
WinNoSwitch <- 0
WinSwitch <- 0
doors <- c(1, 2, 3)
for (i in 1:sims) {
  WinDoor <- sample(doors, 1)
  choice <- sample(doors, 1)
  if (WinDoor == choice)                         # no switch
     WinNoSwitch <- WinNoSwitch + 1
  doorsLeft <- doors[doors != choice]            # switch
  if (any(doorsLeft == WinDoor))
     WinSwitch <- WinSwitch + 1
}
```

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       WinNoSwitch <- WinNoSwitch + 1
    doorsLeft <- doors[doors != choice]        # switch
    if (any(doorsLeft == WinDoor))
       WinSwitch <- WinSwitch + 1
  }
cat("Prob(Car | no switch)=", WinNoSwitch/sims, "\n")
cat("Prob(Car | switch)=", WinSwitch/sims, "\n")
```
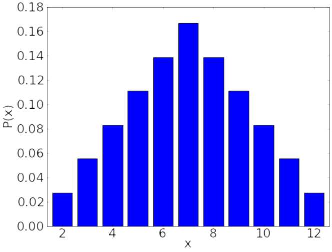
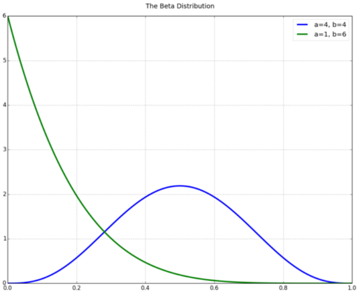

Let's Make a Deal

Pr(car|No Switch)	Pr(car|Switch)
.324	.676
.345	.655
.320	.680
.327	.673

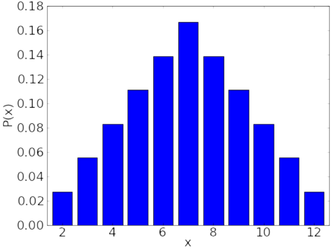
Computing Probabilities from PDFs

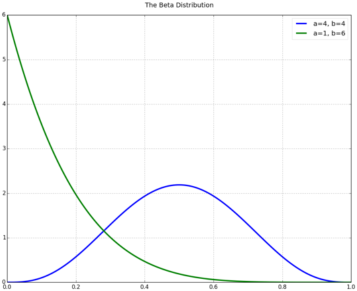
Computing Probabilities from PDFs





Computing Probabilities from PDFs

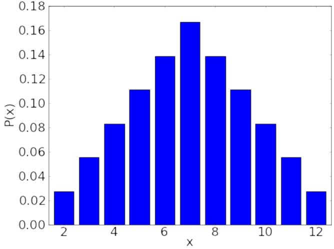


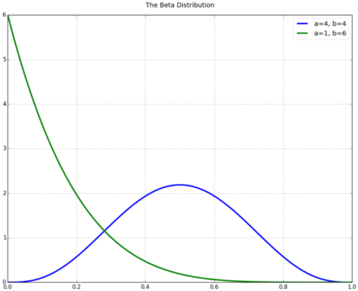


Required for a PDF:

Denoted: $\mathbb{P}(y)$

Computing Probabilities from PDFs





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1. $\mathbb{P}(y) \geq 0$ for every y

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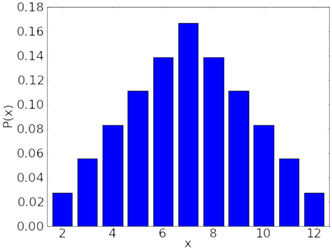

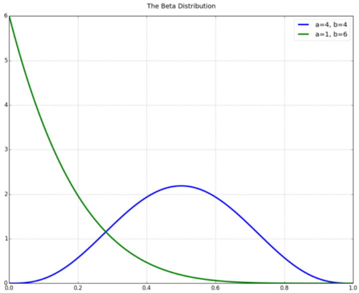

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2. $\sum_y \mathbb{P}(y) = 1$

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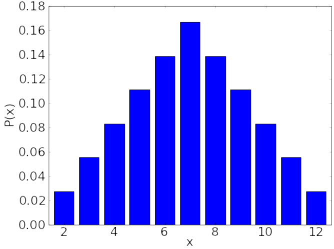

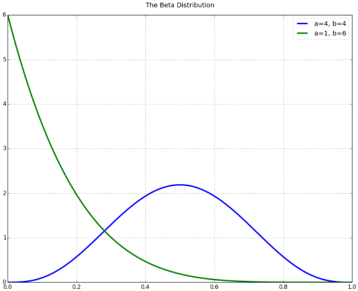

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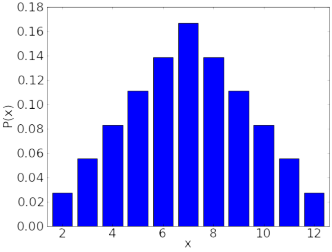

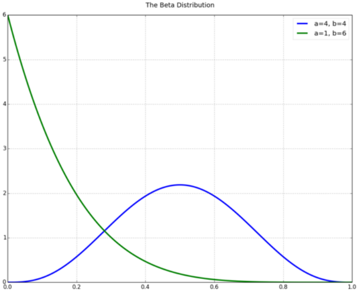

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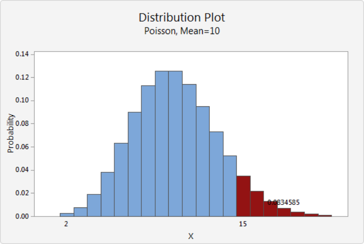

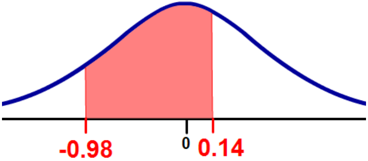

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- $\Pr(a \leq Y \leq b) = \int_a^b \mathbb{P}(y)dy$

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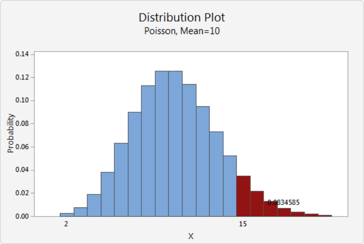

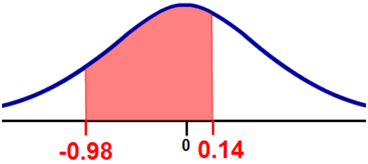

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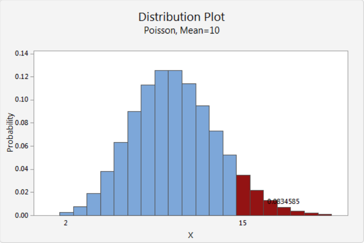

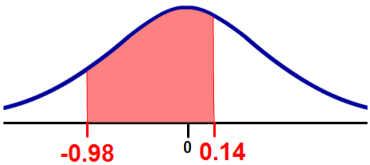

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Probability Calculations:

- $Pr(a \leq Y \leq b) = \int_a^b \mathbb{P}(y)dy$
- For discrete: $Pr(y) = \mathbb{P}(y)$
- For continuous: $Pr(y) = 0$

What you should know about every pdf

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- How to verify that the final expression is indeed a proper density

Uniform Density on the interval $[0, 1]$

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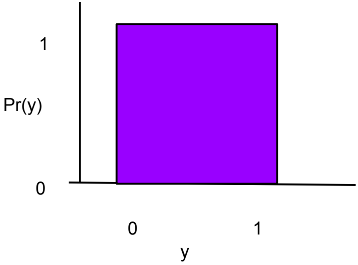

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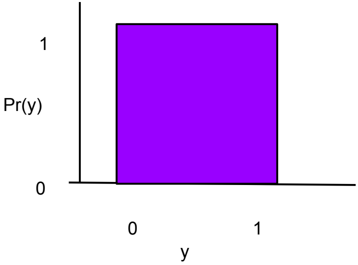

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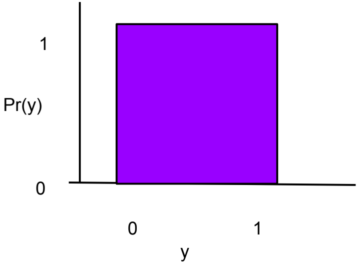

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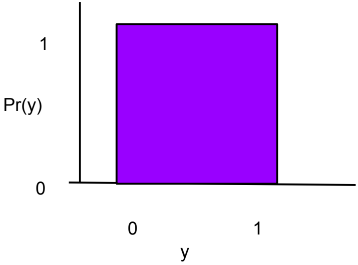

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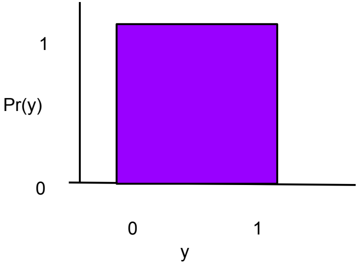

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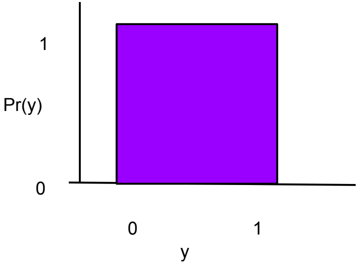

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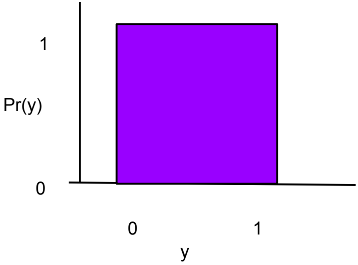

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Bernoulli pmf

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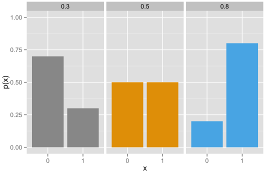
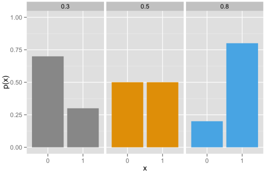


Bernoulli pmf



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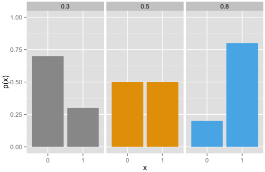

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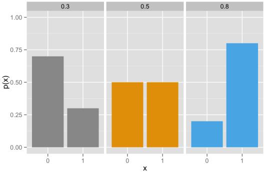

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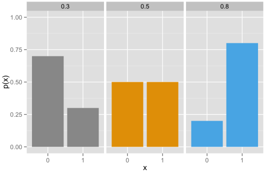

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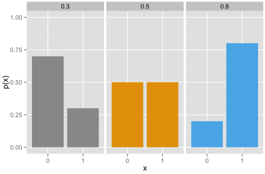

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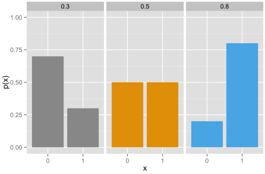

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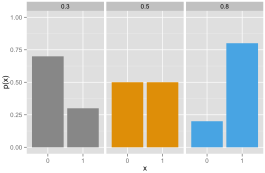

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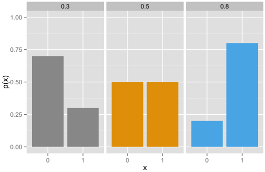

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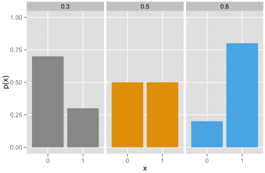

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  - The parameter π happens to be interpretable as a probability
  - $\implies \Pr(Y_i = y|\pi_i) = \pi_i^y(1 - \pi_i)^{1-y}$
  - Alternative notation: $\Pr(Y_i = y|\pi_i) = \text{Bernoulli}(y|\pi_i) = f_b(y|\pi_i)$

Features of the Bernoulli: analytically

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- Expected value:

$$
\begin{aligned}
E(Y) &= \sum_{\text{all } y} y\mathbb{P}(y) \\
&= 0\,\Pr(0) + 1\,\Pr(1) \\
&= \pi
\end{aligned}
$$

Features of the Bernoulli: analytically

- Expected value:

$$\begin{aligned} E(Y) &= \sum_{\text{all } y} y\mathbb{P}(y) \\ &= 0\,\Pr(0) + 1\,\Pr(1) \\ &= \pi \end{aligned}$$

- Variance:

Features of the Bernoulli: analytically

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- Variance:

$$V(Y) = E[(Y - E(Y))^2] \qquad \text{(The definition)}$$

Features of the Bernoulli: analytically

- Expected value:

$$\begin{aligned} E(Y) &= \sum_{\text{all } y} y\mathbb{P}(y) \\ &= 0\,\Pr(0) + 1\,\Pr(1) \\ &= \pi \end{aligned}$$

- Variance:

$$\begin{aligned} V(Y) &= E[(Y - E(Y))^2] && \text{(The definition)} \\ &= E(Y^2) - E(Y)^2 && \text{(An easier version)} \end{aligned}$$

Features of the Bernoulli: analytically

- Expected value:

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Think about where the maximum is. Does it accord with your intuition?

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- In practice, can use `rbinom(size=1)`

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- Mean $E(Y) = N\pi$
- Variance $V(Y) = \pi(1-\pi)/N$.

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- What can you do with the simulations?

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- In the midrange future we might be using quantum computers for this.

Discretization for random draws from discrete pmfs

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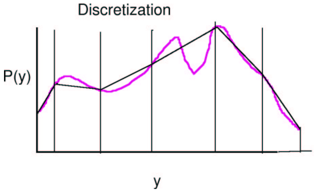

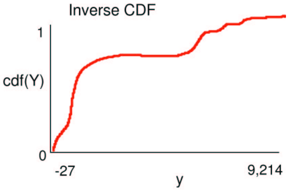

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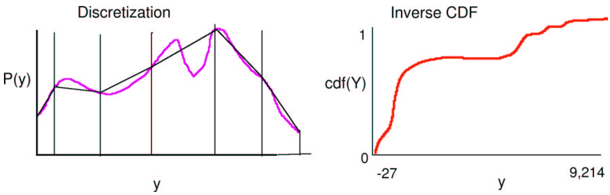

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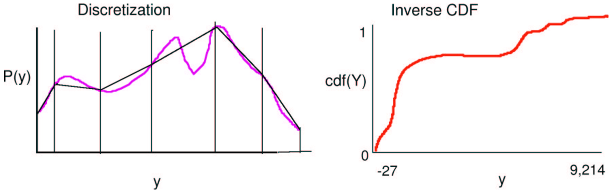

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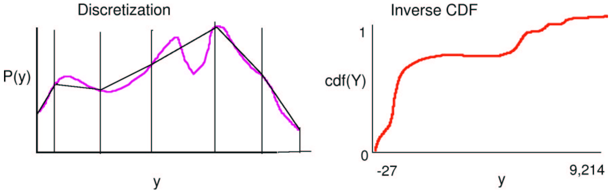

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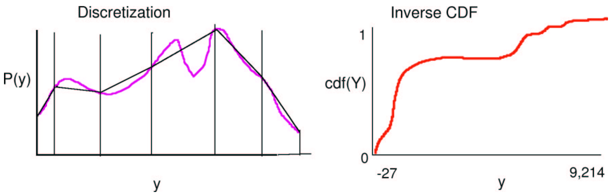

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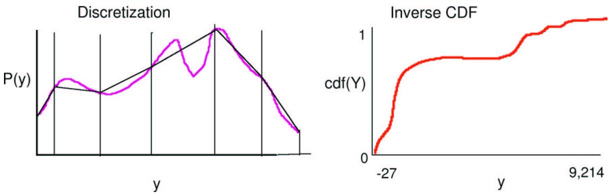

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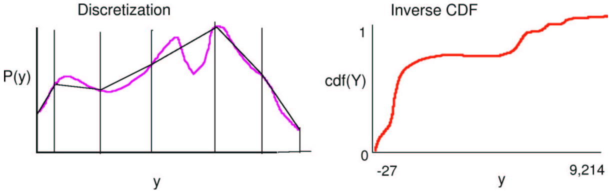

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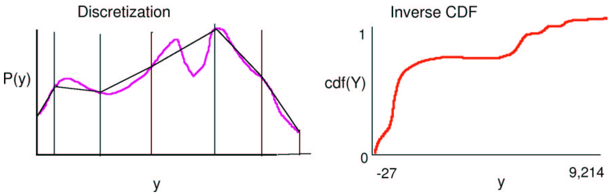
- (Works for a few dimensions, but infeasible for many)

Inverse CDF: drawing from arbitrary continuous pdfs



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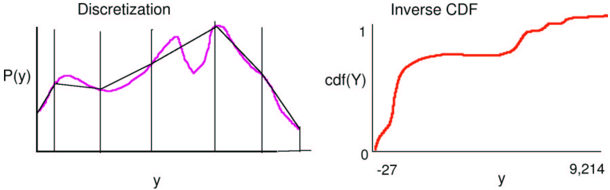

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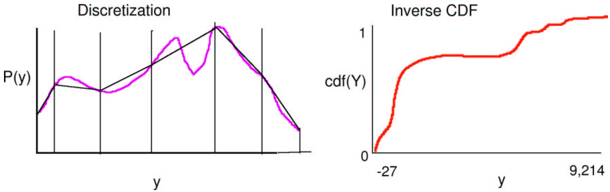

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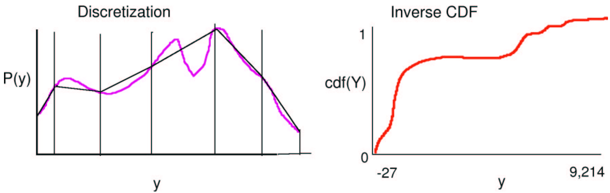

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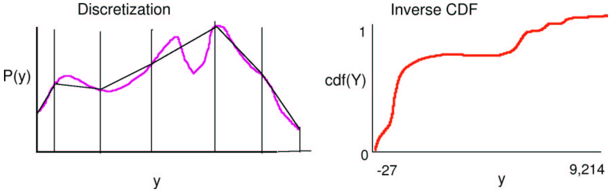

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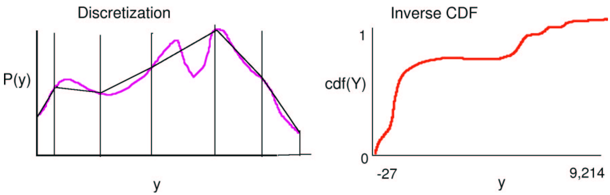

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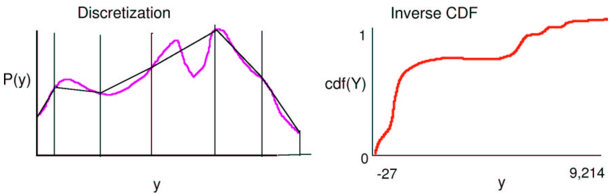

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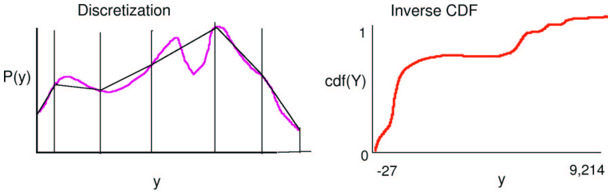

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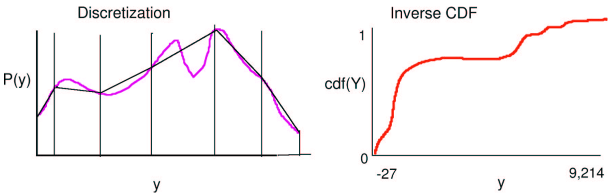

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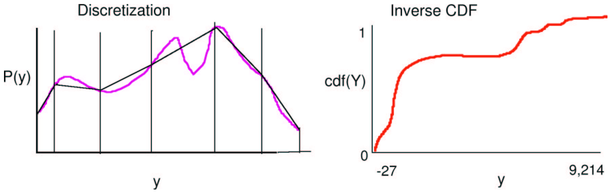

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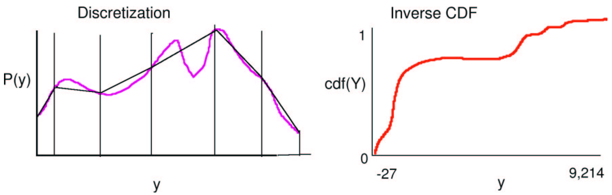

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- Also a decent literature on drawing samples with different speed/accuracy tradeoffs

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Multivariate Normal Distribution

- Let $Y_i \equiv \{Y_{1i}, \dots, Y_{ki}\}$ be a $k \times 1$ vector, jointly random:

$$Y_i \sim N(y_i | \mu_i, \Sigma)$$

where μ_i is $k \times 1$ and Σ is $k \times k$. For $k = 2$,

$$\mu_i = \begin{pmatrix} \mu_{1i} \\ \mu_{2i} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

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- Simulating <u>once</u> from this density produces k numbers. Special algorithms are used to generate normal random variates (in R, `mvrnorm()`, from the `MASS` library).

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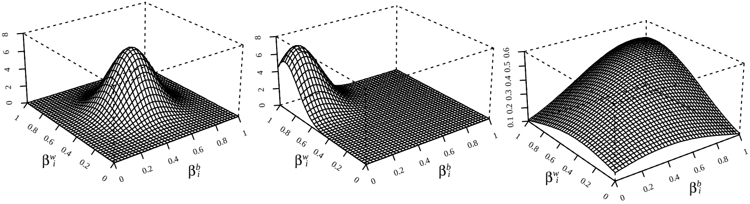
Truncated bivariate normal examples (for β^b and β^w)



(a) 0.5 0.5 0.15 0.15 0 (b) 0.1 0.9 0.15 0.15 0 (c) 0.8 0.8 0.6 0.6 0.5

Parameters are μ_1, μ_2, σ_1, σ_2, and ρ.

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By Next Wednesday: Read *UPM* Chapter 4

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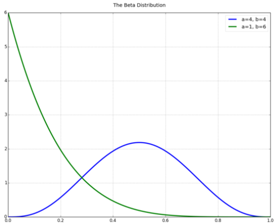

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Reparameterization like this will be key throughout the course.

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- Add up the $\tilde{z}$'s to get $y = \sum_j^N \tilde{z}_j$, which is a draw from the beta-binomial.

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&= \frac{N!}{y_i!(N-y_i)!} \prod_{j=0}^{y_i-1}(\mu+\gamma j) \prod_{j=0}^{N-y_i-1}(1-\mu+\gamma j) \prod_{j=0}^{N-1}(1+\gamma j)
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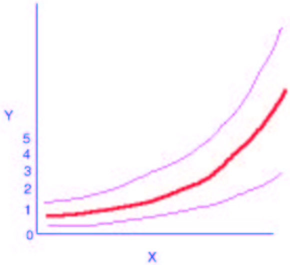

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- Draw Y from Poisson$(y|\tilde{\lambda})$, which gives one draw from the negative binomial.

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$$\text{NegBin}(y|\phi, \sigma^2) = \int_0^\infty \text{Poisson}(y|\lambda) \times \text{gamma}(\lambda|\phi, \sigma^2) d\lambda$$

Negative Binomial Derivation

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$$\begin{aligned} \text{NegBin}(y|\phi, \sigma^2) &= \int_0^\infty \text{Poisson}(y|\lambda) \times \text{gamma}(\lambda|\phi, \sigma^2) d\lambda \\ &= \int_0^\infty \mathbb{P}(y, \lambda|\phi, \sigma^2) d\lambda \end{aligned}$$

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\text{NegBin}(y|\phi, \sigma^2) &= \int_0^\infty \text{Poisson}(y|\lambda) \times \text{gamma}(\lambda|\phi, \sigma^2) d\lambda \\
&= \int_0^\infty \mathbb{P}(y, \lambda|\phi, \sigma^2) d\lambda \\
&= \frac{\Gamma\left(\frac{\phi}{\sigma^2-1} + y_i\right)}{y_i!\Gamma\left(\frac{\phi}{\sigma^2-1}\right)} \left(\frac{\sigma^2-1}{\sigma^2}\right)^{y_i} \left(\sigma^2\right)^{\frac{-\phi}{\sigma^2-1}}
\end{aligned}$$