Soc504: Generalized Linear Models

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Princeton

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[1] These slides are heavily influenced by Gary King with some material from Teppei Yamamoto, Patrick Lam and Yuri Zhukov. Some individual vignettes are built from the collective effort of generations of teaching fellows for Gov2001 at Harvard.

Followup

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- Questions?

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- Replication Stories?

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- Binary outcome variable:

$$Y_i \in \{0, 1\}$$

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- There are many possible binary outcome models, depending on the choice of $g(\cdot)$

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  - Always heteroskedastic

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- Probit: Standard normal CDF

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3. Y_i and Y_j are independent $\forall\ i \neq j$, conditional on X

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What do we do with this?

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   (d) Average effects: compute effects for every observation and average

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We may also want to include uncertainty (fundamental and estimation uncertainty)

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Variable	From		To	*FirstDifference*
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Income	\$35,000	$\rightarrow$	\$75,000	.14

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   (a) Max value of logit derivative: $\hat{\beta} \times 0.5(1-0.5) = \hat{\beta}/4$

   (b) Max value for probit $[\pi_i = \Phi(X_i\beta)]$ derivative: $\hat{\beta} \times 0.4$

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Y_i &= \begin{cases} 1 & \text{if } Y_i^* > 0 \\ 0 & \text{if } Y_i^* \leq 0 \end{cases} \\
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  - ϵ_i: Stochastic (random) component of utility

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  - Y_i^*: Utility from choosing $Y_i = 1$ instead of $Y_i = 0$
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- Make distributional assumptions about ϵ_i:

Latent Variable Interpretation

- Logit models can also be interpreted in terms of a latent variable Y_i^*
- Let

$$\begin{aligned} Y_i &= \begin{cases} 1 & \text{if } Y_i^* > 0 \\ 0 & \text{if } Y_i^* \leq 0 \end{cases} \\ Y_i^* &= X_i^\top \beta + \epsilon_i \quad \text{where} \quad \mathbb{E}(\epsilon_i) = 0 \end{aligned}$$

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Since Y^* is unobserved anyway, define the threshold as $\tau = 0$. (Plus the same independence assumption, which from now on is assumed implicit.)

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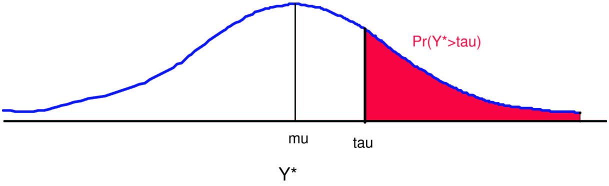

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5. $\implies$ interpret β as regression coefficients of Y^* on X: $\hat{\beta}_1$ is what happens to Y^* on average (or μ_i) when X_1 goes up by one unit, holding constant the other explanatory variables (and conditional on the model). In probit, one unit of Y^* is one standard deviation.

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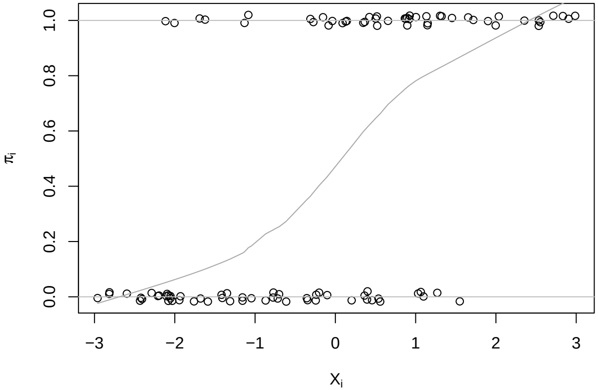
- If $\mathbb{P}(\cdot)$ is generalized extreme value, we get logit.

Comparing LPM, Logit, and Probit



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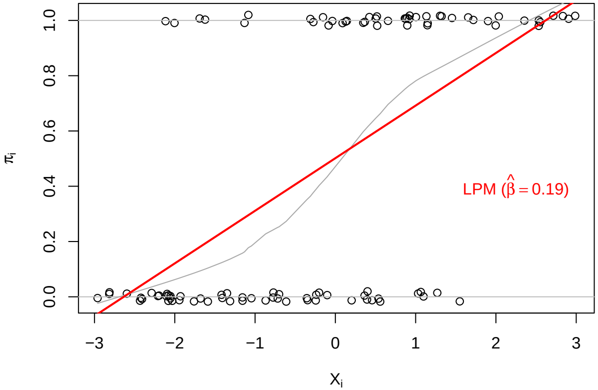

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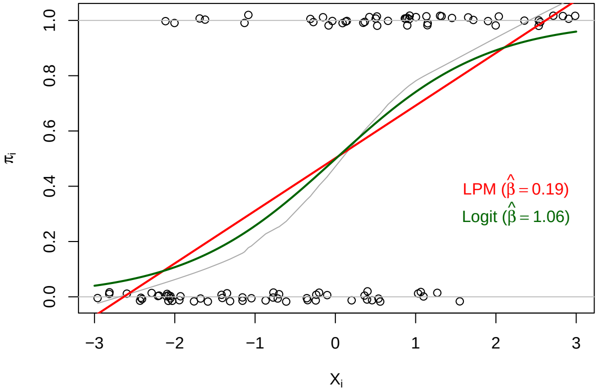

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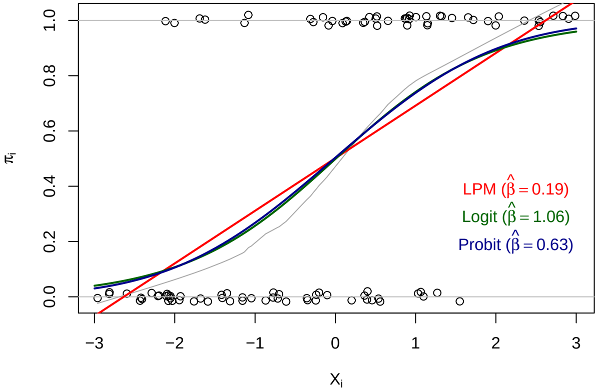

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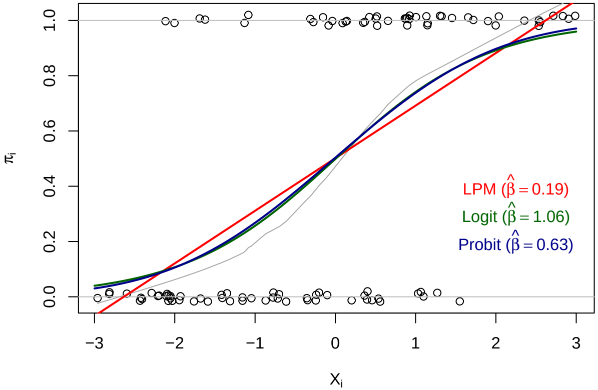

- LPM goes outside of $[0, 1]$ for extreme values of X_i
- LPM underestimates the marginal effect near center and overpredicts near extremes
- Logit has *slightly* fatter tails than probit, but no practical difference
- Note that $\hat{\beta}$ are completely different between the models

How Not to Present Statistical Results

TABLE 1
Predicting Which Ethnic Group Conquered Most of Bosnia

Attention to Bosnia crisis	.609**
Age	.007**
Education	.289**
Family income	.151**
Race (non-White/White)	.695**
Gender (female/male)	.789**
Region (South/non-South)	.076
Network coverage	.000
Education × Time	–.003*
Time in months	.078**
Constant	–9.257**
Number	7,021
–2 log-likelihood	7,215.231
Goodness of fit	6,789.45
Cox & Snell R^2	.212
Nagelkerke R^2	.295
Overall correct classification (%)	73.96

SOURCE: *Times Mirror* polls from September 1992, January 1993, September 1993, January 1994, and June 1995.
NOTE: Unstandardized coefficients for logistic regression. Dependent variable is knowledge of which group conquered most of Bosnia.
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3. Your work should satisfy a reader who hasn't taken this course

Reading

- King, Tomz, Wittenberg, "Making the Most of Statistical Analyses: Improving Interpretation and Presentation" *American Journal of Political Science*, Vol. 44, No. 2 (March, 2000): 341-355.
- Hamner and Kalkan (2013). Behind the Curve: Clarifying the Best Approach to Calculating Predicted Probabilities and Marginal Effects from Limited Dependent Variable Models. *American Journal of Political Science*.
- Greenhill, Ward, and Sacks (2011). The Separation Plot: A new visual method for evaluating the fit of binary models. *American Journal of Political Science*.

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  - Instead, always try to present your results in terms of an easy-to-interpret quantity

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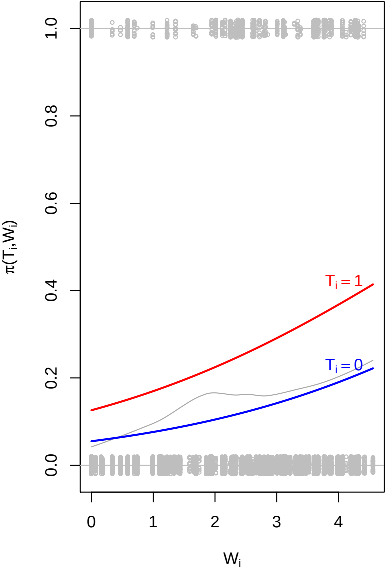

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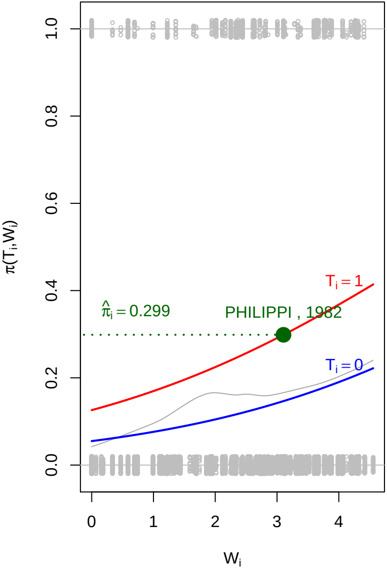

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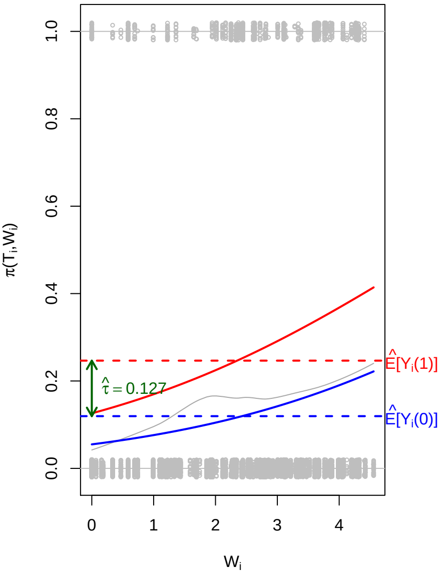

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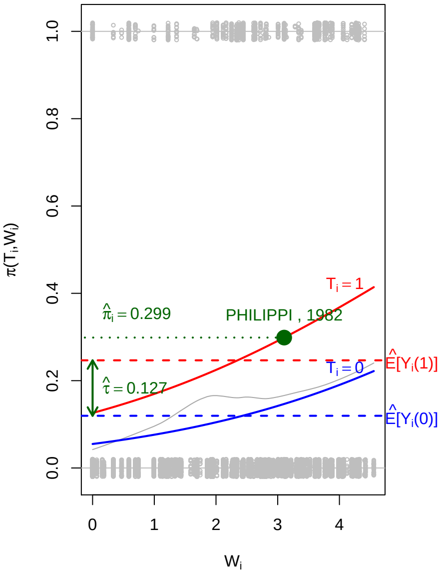

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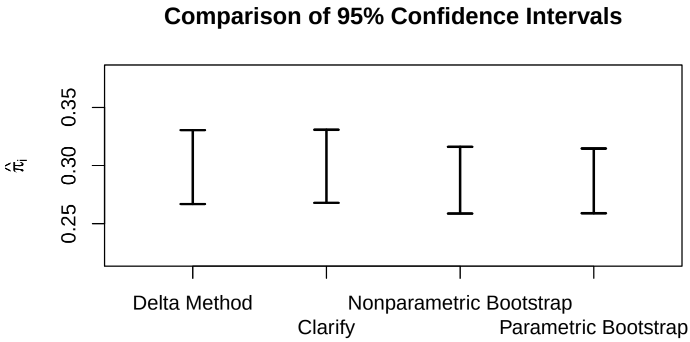
Example: Civil Conflict and Political Instability

Confidence Intervals for $\hat{\pi}(T_i = 1, W_i = 3.10)$:



	Delta Method	Clarify	Nonpara. B.	Para. B.
Normal Approximation	Yes	Yes	No	No
Simulations	No	Yes	Yes	Yes
Derivation-Free	No	Yes	Yes	No
Computation Speed	Instant	Fast	Very Slow	Slow

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2. Fundamental uncertainty: Represented by the stochastic component. (Due to the nature of nature!)

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(Note: the language is slightly different for the latent variable with observation mechanism but the result is the same)

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Let's do this together for a particular example.

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Let's try to predict assassination attempts with some of our covariates.

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What fits our application?

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We are interested in allowing some parameter of the distribution θ to vary as a (linear) function of covariates. So we specify a linear predictor.

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We are interested in allowing some parameter of the distribution θ to vary as a (linear) function of covariates. So we specify a linear predictor.

$$X\beta = \beta_0 + x_1\beta_1 + x_2\beta_2 + \cdots + x_k\beta_k$$

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- `tenure`: number of days in office
- `age`: age of leader, in years
- `dem_score`: polity score, -10 to 10
- `civil_war`: is there currently a civil war?
- `war`: is country in an international conflict?
- `pop`: the country's population, in thousands
- `energy`: energy usage

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Note that we usually use the **inverse link function** $g^{-1}(X\beta)$ rather than the link function.

Together with the linear predictor this forms the systematic component that we've been talking about all along.

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Log:

- Link: $\ln(\lambda) = X\beta$
- Inverse Link: $\lambda = \exp(X\beta)$

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Let's do probit. Why? Mostly to avoid giving away the problem set.

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c. Obtain an estimate of the variance by inverting the negative Hessian

Step 4a: Write Down the Likelihood

The model:

1. $Y_i \sim f_{\text{bern}}(y_i|\pi_i)$.
2. $\pi_i = \Phi(X_i\beta)$ where Φ is the CDF of the standard normal distribution.
3. Y_i and Y_j are independent for all $i \neq j$.

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3. Y_i and Y_j are independent for all $i \neq j$.

Like all CDF's, Φ has range 0 to 1, so it bounds our π_i to the correct space:

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp(\frac{z^2}{2}) dz.$$

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    phi <- pnorm(X%*%beta, log = TRUE)
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    logl <- sum(y*phi + (1-y)*opp.phi)
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1. the STN CDF is evaluated with pnorm. R's pre-programmed log of the CDF has greater range than `log(pnorm(Z))` (try Z=-50).
2. if `lower.tail = FALSE` gives $Pr(Z \geq z)$.
3. uses a logical test to check that an intercept column has been added

Step 4b: Maximize the Likelihood

```
y <- as$attempt
X <- as[,c("tenure","age","dem_score","civil_war",
          "war","pop","energy")]

yX <- na.omit(cbind(y,X))
y <- yX[,1]; X <- cbind(1,yX[,-1])

opt <- optim(par = rep(0,ncol(X)), fn = ll.probit, y=y, X=X,
      method = "BFGS", control = list(fnscale = -1,
      maxit = 1000), hessian = TRUE)
opt$par
[1] -1.836166916  0.029141779 -0.005751652 -0.012433873  0.066287021  0.317451778
[8]  0.026859441
```

Step 4c: Estimate the Variance-Covariance Matrix

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vcov <- solve(-opt$hessian)
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Now we can draw approximate the sampling distribution of beta.

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MASS::mvrnorm(n=1, mu=opt$par, Sigma=vcov)
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```

This is stochastic so we do it again and get a different answer:

```
MASS::mvrnorm(n=1, mu=opt$par, Sigma=vcov)
[1] -1.792636772  0.081477117 -0.006457063 -0.013436530  0.019081307  0.255634394  
[8]  0.073840550
```

5. Quantities of Interest

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What not to do...

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	est	SE
Intercept	-1.8362	1.4012
tenure	0.0291	0.2937
age	-0.0058	0.0251
dem_score	-0.0124	0.0445
civil_war	0.0663	0.8918
war	0.3175	1.0141
pop	0.0409	0.2368
energy	0.0269	0.2432

```
ses <- sqrt(diag(solve(-opt$hessian)))
table.dat <- cbind(opt$par, ses)
rownames(table.dat) <- colnames(X)
xtable::xtable(table.dat, digits = 4)
```

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3. Draw from distribution of Y for predicted values.

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a. Incorporating estimation uncertainty.
b. Incorporating fundamental uncertainty when making predictions.
c. Establishing appropriate baseline values for QOI, and considering plausible changes in those values.

Expected Values

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Let's consider a potentially high risk situations (we'll call them "highrisk","X_{HR}") then we can manipulate the risk factors:

Var.	Value
tenure	-0.30
age	54.00
dem_score	-3.00
civil_war	0.00
war	0.00
pop	-0.18
energy	-0.23

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What's the estimated probability of an assassination at X_{HR}?

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Draw $\tilde{\beta}$

```
beta.draws <- MASS::mvrnorm(10000, mu = opt$par, Sigma = vcov)
dim(beta.draws)
[1] 10000     8
```

Now we simulate the outcome (warning: inefficient code!)

```
nsims <- 10000
p.ests <- vector(length=nrow(beta.draws))
for(i in 1:nsims){
  p.ass.att <- pnorm(highrisk%*%beta.draws[i,])
  outcomes <- rbinom(nsims2, 1, p.ass.att)
  p.ests[i] <- mean(outcomes)
}
> mean(p.ests)
[1] 0.0166266
> quantile(p.ests, .025); quantile(p.ests, .975)
 2.5%    97.5%
0.0134  0.0201
```

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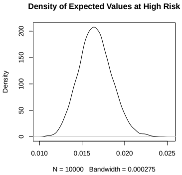
5. average over those draws to get one simulated $E[y|X_{HR}]$.
6. return to step 3.



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> mean(p.ests2)
[1] 0.01659705
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      2.5%        97.5%
0.01395935   0.01955867
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This shortcut works because $E[y|X_{HR}] = \pi_{HR}$; i.e. the parameter is the expected value of the outcome.

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Ex.: suppose that $y_i \sim Expo(\lambda_i)$ where $\lambda_i = exp(X_i\beta)$. We could find our likelihood, insert our parameterization of λ_i for each i, and then maximize to find $\hat{\beta}$ as usual.

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Its not too hard to show that if $y \sim Expo(\lambda)$, then $E[y] = \frac{1}{\lambda}$. The temptation is then to declare that because $E[\hat{\lambda}_{BL}] = E[exp(X_{BL}\hat{\beta})]$ then $\widehat{E[y]} = 1/E[\exp(X_{BL}\hat{\beta})]$.

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It turns out this is not the case because $E[1/\hat{\lambda}] \neq 1/E[\hat{\lambda}]$. The first averages over the sampling distribution of the means of y. The second averages over the sampling distribution of $\hat{\lambda}$ then plugs into the formula for the mean of y.

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Why this annoying wrinkle?

When wouldn't this work?

Why this annoying wrinkle? Jensen's inequality: given a random variable X, $E[g(X)] \neq g(E[X])$ (it's $\geq$ if $g(\cdot)$ is concave; $\leq$ if $g(\cdot)$ is convex).

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$\boxed{\text{\underline{Rule of thumb}: if } E[Y] = \theta\text{, you are safe taking the shortcut.}}$

More Expected Values

What if I want a bunch of these to see how expected values change with some variable?

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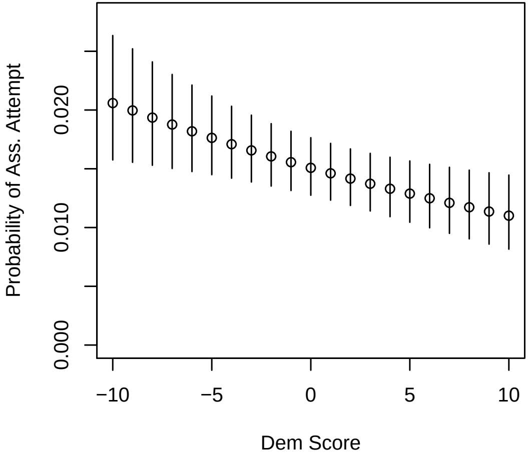
```
dem.rng <- -10:10
p.ests <- matrix(data = NA, ncol = length(dem.rng),
                 nrow=10000)

for(j in 1:length(dem.rng)){
  highrisk.dem <- highrisk
  highrisk.dem["dem_score"] <- dem.rng[j]
  p.ests[,j] <- pnorm(highrisk.dem%*%t(beta.draws))
}

plot(dem.rng, apply(p.ests,2,mean), ylim = c(0,.028))
segments(x0 = dem.rng, x1 = dem.rng,
         y0 = apply(p.ests, 2, quantile, .025),
         y1 = apply(p.ests, 2, quantile, .975))
```


Probability of Ass. Attempt
0.000
0.010
0.020
−10
−5
0
5
10
Dem Score

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Q: What will it look like?

There is no need to actually conduct the simulation, though. The simulated outcomes will be $Bern(\widehat{E[y|X]}) = Bern(.166)$. How is this different than the linear regression case?

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Compare expected values of the outcome for two different scenarios, usually all predictors held constant but one. Recall from our regression results that war seemed to have a big positive effect on probability of an assassination attempt.

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Each of these are just fitted values for the probability parameter, with all covariates at the highrisk values except war, which we control.

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```
highrisk.war <- highrisk
highrisk.war["war"] <- 1
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```

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highrisk.war <- highrisk
highrisk.war["war"] <- 1
highrisk.nowar <- highrisk
highrisk.nowar["war"] <- 0

fd.ests <- pnorm(highrisk.war%*%t(beta.draws)) -
           pnorm(highrisk.nowar%*%t(beta.draws))

> mean(fd.ests)
[1] 0.01891
> quantile(fd.ests, .025); quantile(fd.ests, .975)
   2.5%      97.5%
0.00578    0.03609
```

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   Denote the draw $\tilde{\gamma} = \text{vec}(\tilde{\beta}, \tilde{\alpha})$, which has *k* elements.

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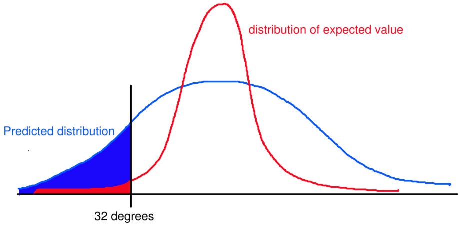
Repeat algorithm say $M = 1000$ times, to produce 1000 predicted values. Use these to compute a histogram for the full posterior, the average, variance, percentile values, or others.

The Distribution of Expected v. Predicted Values



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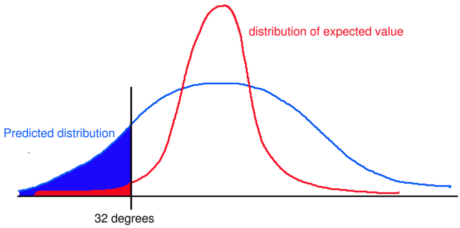

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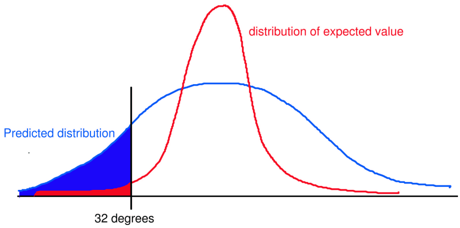

1. Predicted values: draws of Y that are or could be observed
2. Expected values: draws of fixed features of the distribution of Y, such as $E(Y)$.

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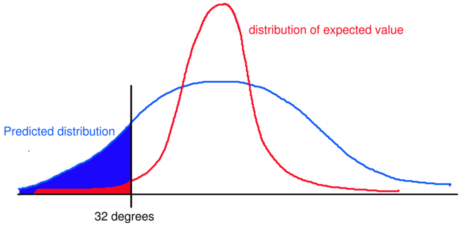
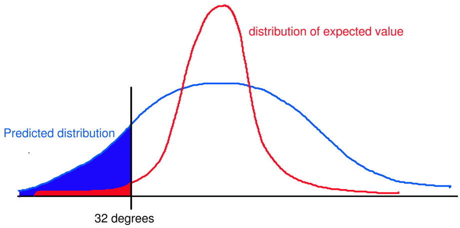
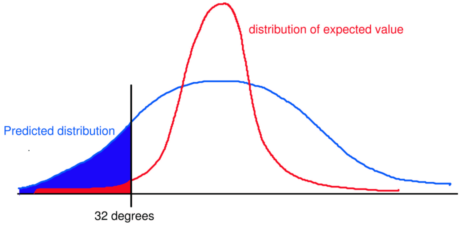


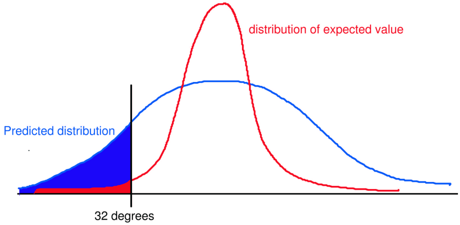
1. Predicted values: draws of Y that are or could be observed
2. Expected values: draws of fixed features of the distribution of Y, such as $E(Y)$.
3. Predicted values: include estimation and fundamental uncertainty.

The Distribution of Expected v. Predicted Values



1. Predicted values: draws of Y that are or could be observed
2. Expected values: draws of fixed features of the distribution of Y, such as $E(Y)$.
3. Predicted values: include estimation and fundamental uncertainty.
4. Expected values: average away fundamental uncertainty


distribution of expected value
Predicted distribution
32 degrees



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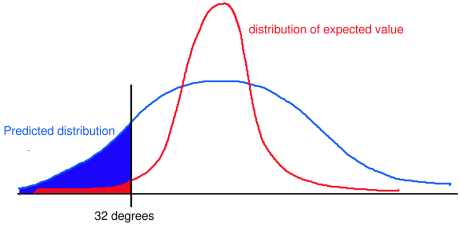

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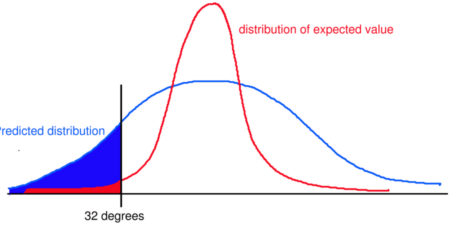

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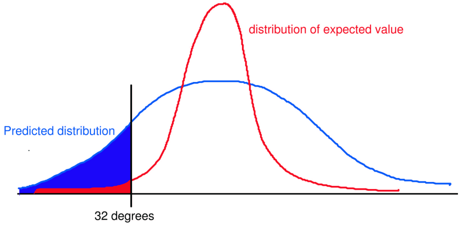

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8. Which to use for causal effects & first differences?

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5. Average over the fundamental uncertainty by calculating the mean of the m simulations to yield one simulated expected value $\tilde{E}(Y_c) = \sum_{k=1}^{m} \tilde{Y}_c^{(k)}/m$.

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5. When $E(Y_c) = \theta_c$, we can skip the last two steps. E.g., in the logit model, once we simulate π_i, we don't need to draw Y and then average to get back to π_i. (If you're unsure, do it anyway!)

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4. (To save computation time, and improve approximation, use the same simulated β in each.)

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   In all 3 cases, η is unbounded: estimate it, simulate from it, and reparameterize back to the scale you care about.

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   (c) Its wrong because the regression estimates $E[\ln(Y)]$, but $E[\ln(Y)] \neq \ln[E(Y)]$, so $\exp(E[\ln(Y)]) \neq Y$

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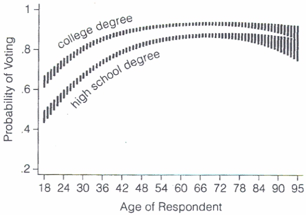
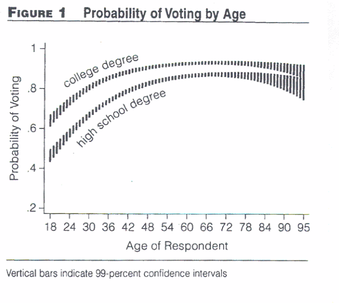
3. Use $M = 1000$ and compute 99% CI:

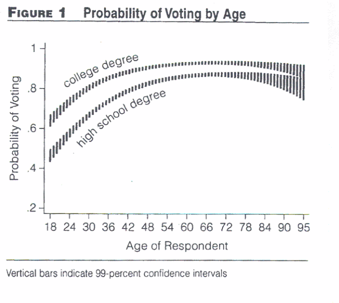
FIGURE 1 Probability of Voting by Age



Vertical bars indicate 99-percent confidence intervals



To create this graph, simulate:



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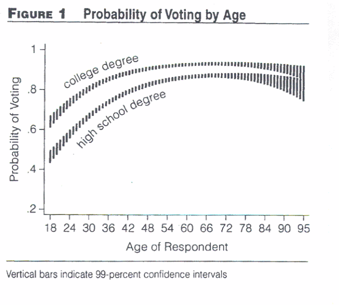
1. Set age=24, education=high school, income=average, Race=white



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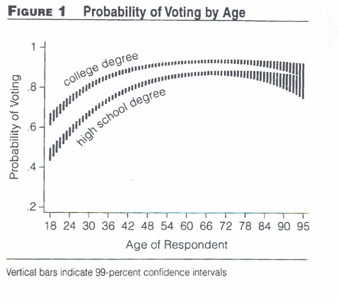

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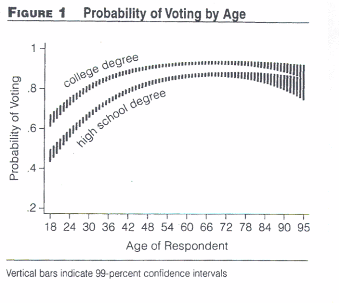

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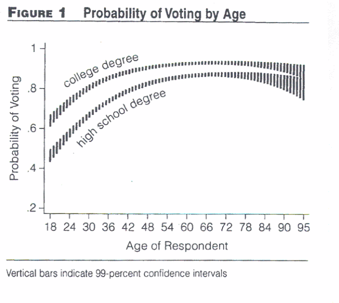

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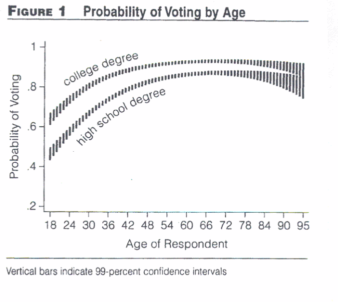

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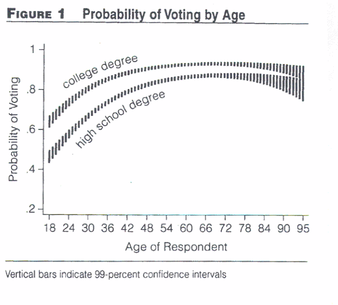

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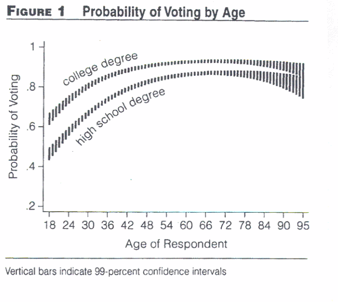

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8. Repeat for other ages and for college degree.

Replication of Garrett (King, Tomz and Wittenberg 2000)

- Dependent variable: Government Spending as % of GDP
- Key explanatory variable: left-labor power (high = solid line; low = dashed)
- Garrett used only point estimates to distinguish the eight quantities represented above. What new information do we learn with this approach?
- Left-labor power only has a clear effect when exposure to trade or capital mobility is high.

See Last Semester's Slides

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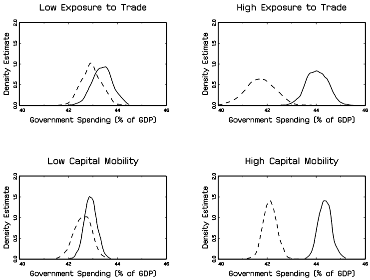

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- This one is due to McFadden (1974); many other variants exist

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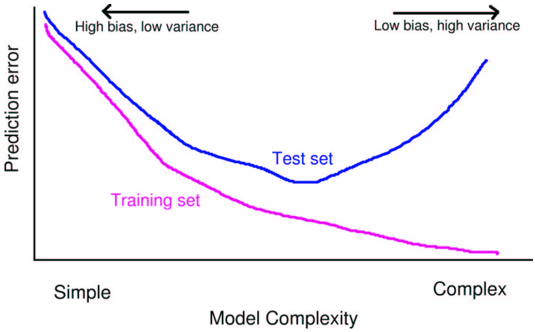

(See Trevor Hastie et al. 2001. The Elements of Statistical Learning, Springer, Chapter 7: Fig 7.1.)

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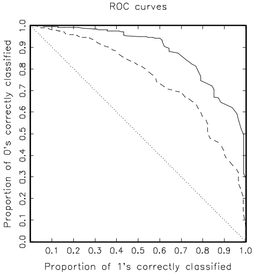

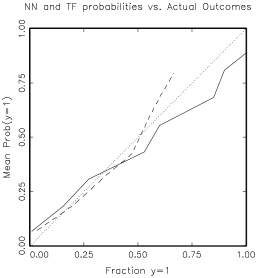

In sample ROC, on left (from Gary King and Langche Zeng. "Improving Forecasts of State Failure," World Politics, Vol. 53, No. 4 (July, 2001): 623-58)

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      - ★ transform heteroskedastic results
      - ★ highlight key results; label everything

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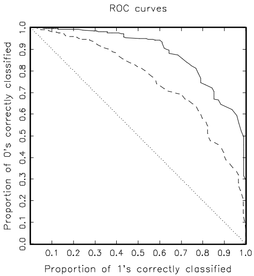
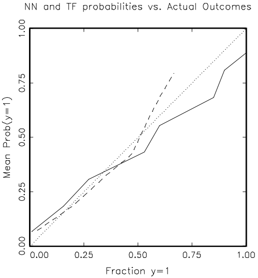
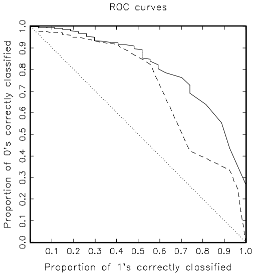
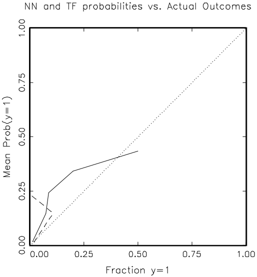




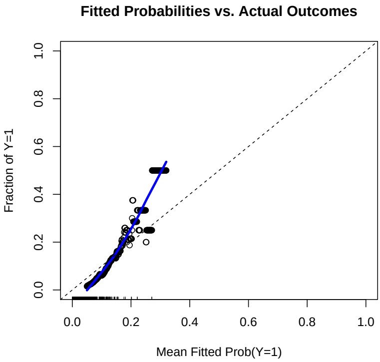
In sample calibration graph on right (from Gary King and Langche Zeng. "Improving Forecasts of State Failure," World Politics, Vol. 53, No. 4 (July, 2001): 623-58)

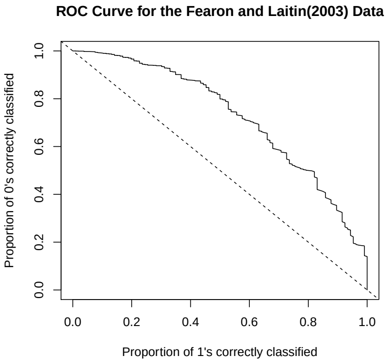




Out of sample calibration graph on right.

Out-of-Sample with Cross-Validation





New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

Table 1 Sample Data

Country	Actual Outcome (y)	Fitted Value ($\hat{p}$)
A	0	0.774
B	0	0.364
C	1	0.997
D	0	0.728
E	1	0.961
F	1	0.422

New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

Table 4 Rearrangement (and Coloring) of the Data Presented in Table 1 for Use in the Separation Plot

Country	Fitted Value ($\hat{p}$)	Actual Outcome (y)
B	0.364	0
F	0.422	1
D	0.728	0
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New Developments: Separation plots

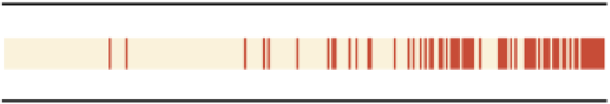
Greenhill, Ward and Sacks (2011)

FIGURE 2 Separation Plot Representing the Data Presented in Table 1

New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

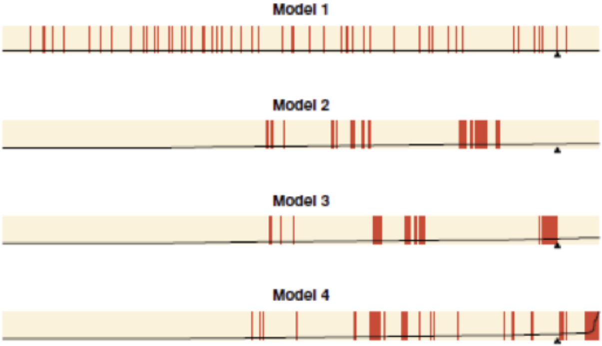
FIGURE 3 Separation Plot for a Larger Data Set



New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

FIGURE 7 Separation Plots Used in the Development of a Model of Insurgency in the Asia-Pacific Region, 1998–2004

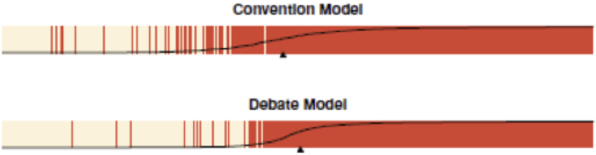


Note: For comparison, Models 1–4 have AUC scores of 0.500, 0.714, 0.744, and 0.816; Brier scores of 0.065, 0.063, 0.062, and 0.057; and ePCP scores of 0.869, 0.875, 0.876, and 0.887.

New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

FIGURE 8 Separation Plots for the Hillygus and Jackman (2003) Models of Voting Intentions in the 2000 Presidential Election



Note: The upper plot shows the results of the survey conducted in the period following the party conventions, while the lower plot shows the results of the survey conducted after the presidential debates. Both models make an excellent fit to the data. (For comparison, the convention and debate models have AUC scores of 0.964 and 0.982; Brier scores of 0.071 and 0.045; and ePCP scores of 0.859 and 0.909.)

New Developments: Separation plots

Greenhill, Ward and Sacks (2011)

FIGURE 9 **Comparison of the Separation Plots Produced by Replicating Model 1 of Fearon and Laitin (2003) and by Reestimating the Model with Logged GDP per Capita as the Only Covariate**

Note: For comparison, Model 1 and the GDP-only model have AUC scores of 0.760 and 0.671; Brier scores of 0.016 and 0.016; and ePCP scores of 0.968 and 0.967.

New Developments

Hanmer and Kalkan distinguish between:

New Developments

Hanmer and Kalkan distinguish between:

- average case (most common)

New Developments

Hanmer and Kalkan distinguish between:

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The Case for Observed Case

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Consider the case of voting for Bush in 2004.

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Consider the case of voting for Bush in 2004. Average case is:

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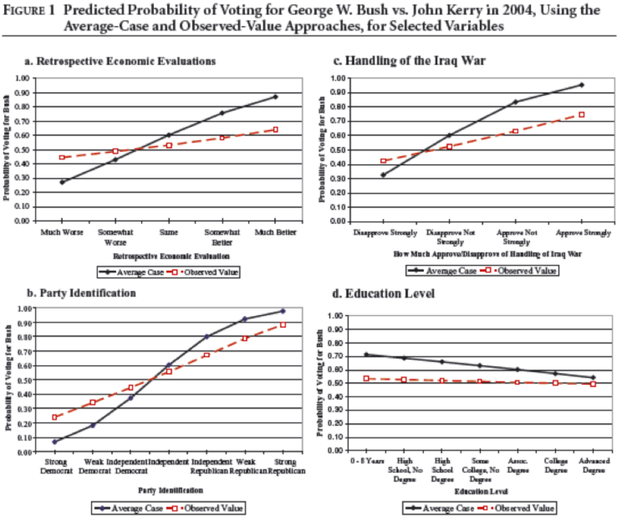
- 48 year old woman
- identifies as independent and a political moderate
- with an associates degree
- believes economic performance has been constant
- disapproves of the Iraq war but not strongly
- with an income between $45K and $50K

The Case for Observed Case



Notes: Data are from the 2004 ANES, using respondents who first answered the standard turnout question. Results are based on estimates from the model reported in SI Section B Table 1.

The Case for Observed Case

Try to come up with an argument for why the average-case method will tend to produce bigger changes than the observed-case method.

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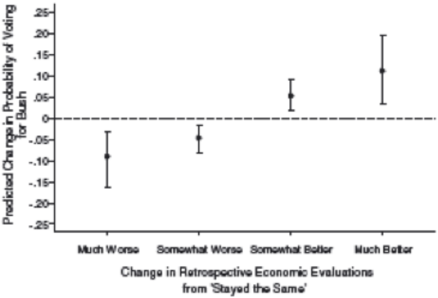
Average cases are likely to be in the "middle" of the data where the predicted probabilities are changing the fastest. Think about Bill O'Reilly and Rachel Maddow. They are going to show up in the observed-case method but not the average-case method.

The Case for Observed Case

FIGURE 3 Predicted Effects (First Differences) of Changing Retrospective Economic Evaluations on the Probability of Voting for George W. Bush vs. John Kerry in 2004, Using the Observed-Value Approach, with 95% Confidence Intervals



Notes: Data are from the 2004 ANES, using respondents who first answered the standard turnout question. Results are from statistical simulation.

UPM Remainder of Chapter 5.

Optionally: Greenhill et al. 2011, Hamner and Kalkan 2013

Also Helpful: Mood 2010, Berry, DeMeritt and Esarey 2010

Modeling Ordered Outcomes

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Ordered Dependent Variable Models

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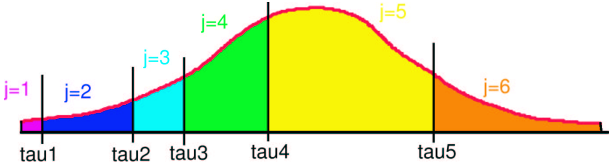

Ordered Logit and Probit Models

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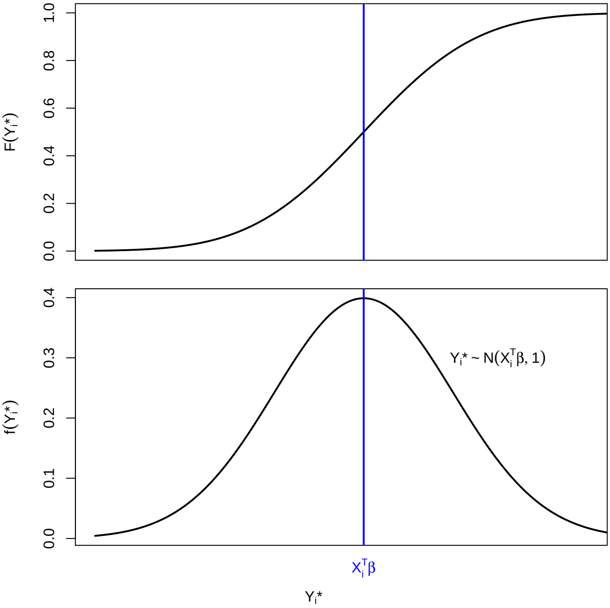
- $\epsilon_j \overset{\text{i.i.d.}}{\sim} \mathcal{N}(0,1) \Rightarrow$ the ordered probit model:

$$\Pr(Y_i \leq j \mid X_i) = \Phi\left(\psi_j - X_i^\top \beta\right)$$

Ordered Logit and Probit Models



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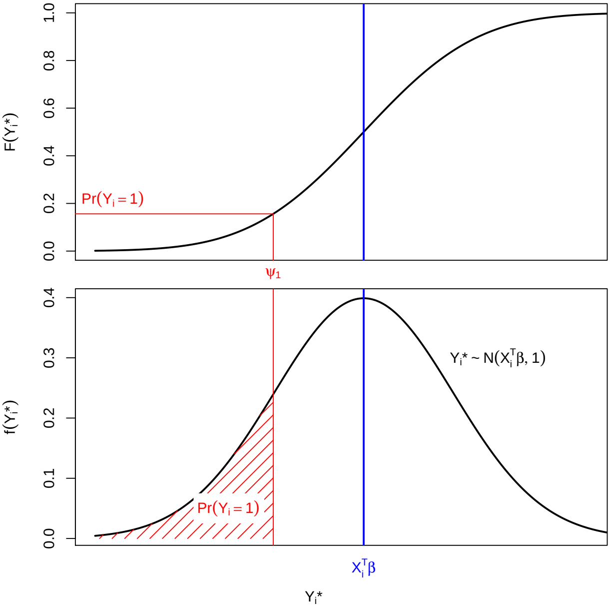

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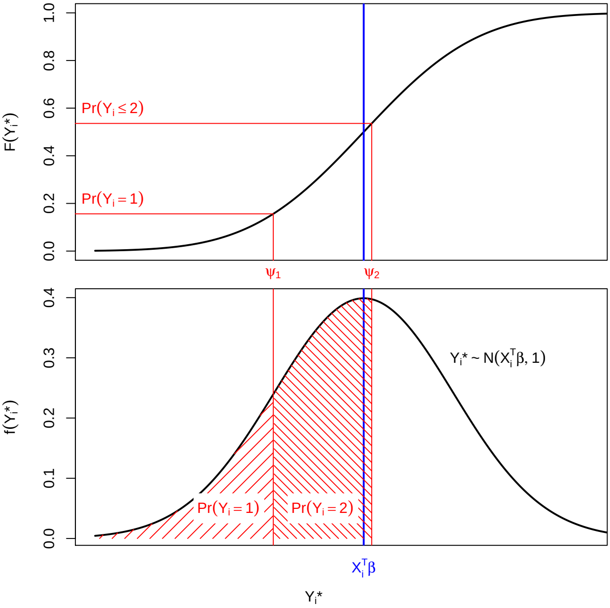

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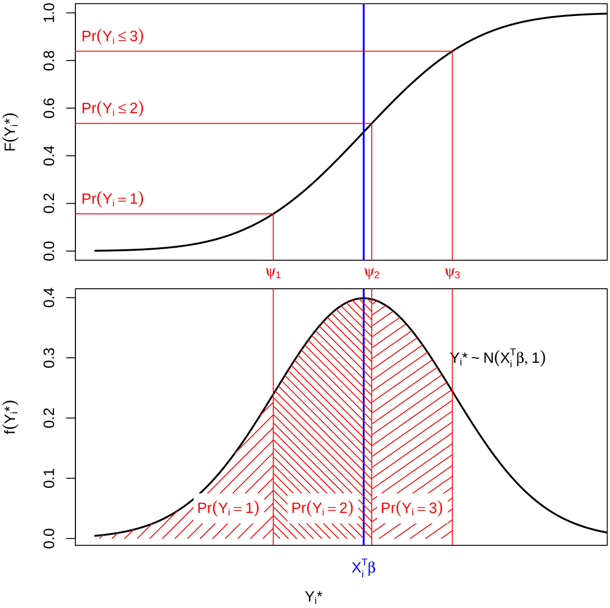
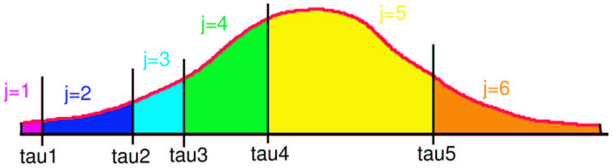


Ordered Dependent Variable Models: Connections



Ordered Dependent Variable Models: Connections

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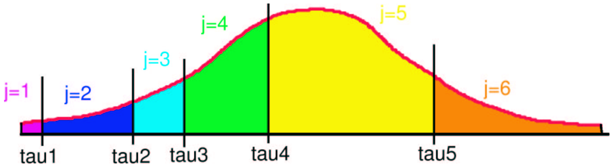

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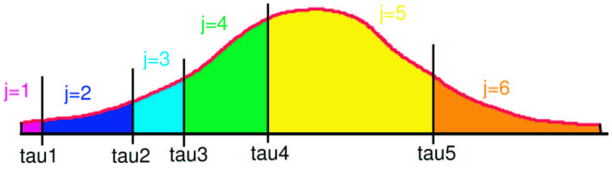

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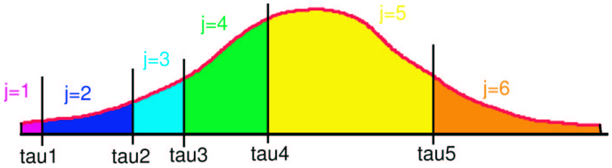

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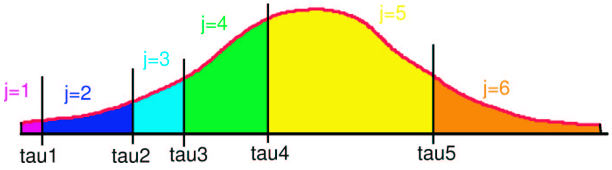

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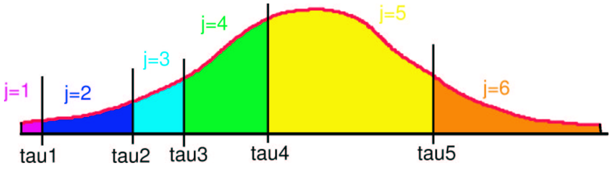
5. This is the same model, and the same parameters are being estimated; only the observation mechanism differs.



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(Constraints during optimization make this more complicated to do from scratch: $\tau_{j-1} < \tau_j$, $\forall j$)

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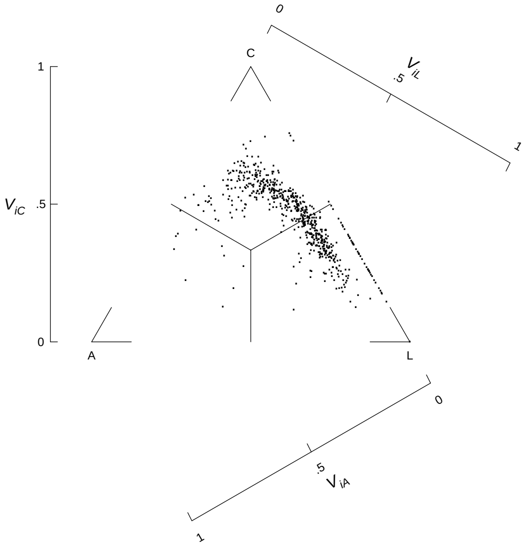
5. Can use ternary diagrams if $J = 3$

Representing 3 variables, with $Y_j \in [0, 1]$ and $\sum_{j=1}^{3} Y_j = 1$



Calculating Quantities of Interest

- Predicted probability:

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  $\longrightarrow$ Again, calculate quantities of interest, not just coefficients

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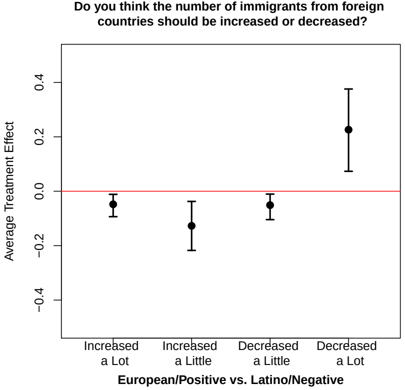

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The Risk Ratio measures the relative probability of being in the outcome category based on different values of the independent variable. Thus the RR for the Strong Reaction category for Female can be understood as

$$RR_{\text{Strong}} = \frac{\Pr(\text{Strong Reaction}|\text{Female})}{\Pr(\text{Strong Reaction}|\text{Male})}$$

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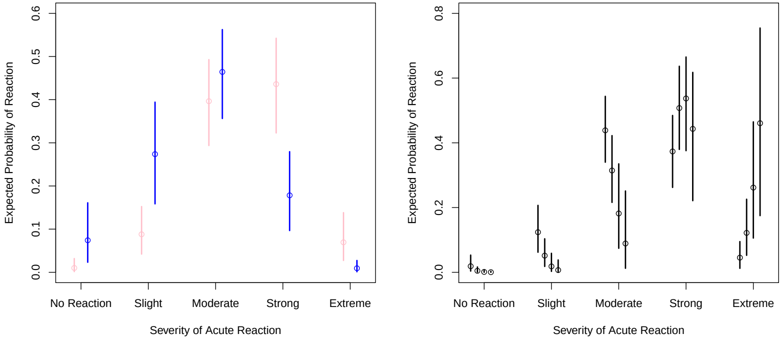

Figure: This plot shows the expected probabilities of being in each category of reaction given gender (left) and knowing 1 to 4 people (right) with 95% confidence intervals.

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- Visualization and appropriate quantities of interest can be tricky. Let the substance guide you.

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- Coefficients are relative to a baseline category- so again we want to compute quantities of interest for interpretation.

Independence of Irrelevant Alternatives

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- Computation is difficult because integral is intractable
- Moreover, # of parameters in Σ_J increases as J gets large, but data contain little information about Σ_J:

J	3	4	5	6	7
# of elements in Σ_J	6	10	15	21	28
# of parameters identified	2	5	9	14	20

The Poisson Distribution

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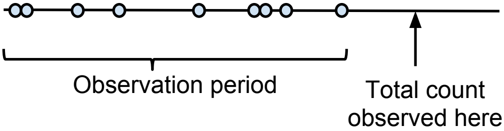

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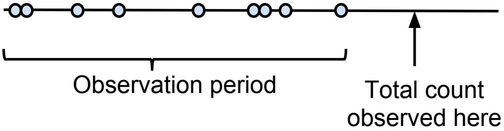

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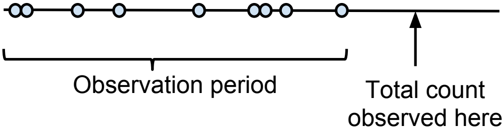

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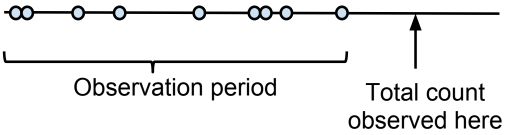

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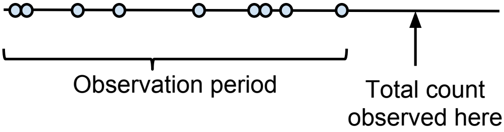

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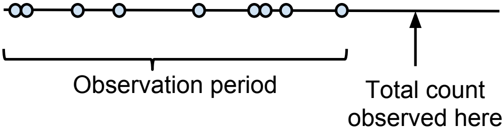

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6. $\Pr(\text{event at time } t \mid \text{all events up to time } t-1)$ is constant for all t.

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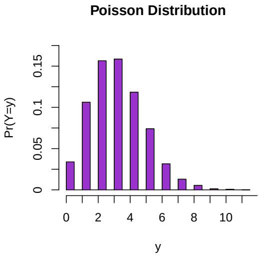

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It also turns out that $\mathrm{Var}(Y) = \lambda$, a feature of the model we will discuss later on.

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Derivation of the distribution has some other technical first principles, but the above is the most important.

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- For $Y_j | (X = k) \sim$Multinom(k, p_j) then each $Y_j \sim$Pois(λp_j).
- If $X_i \sim$Pois(λ_i) for $i = 1 \dots n$ independent then $Y = \sum_{i=1}^{n} X_i \sim \text{Pois}(\sum_{i=1}^{n} \lambda_i)$

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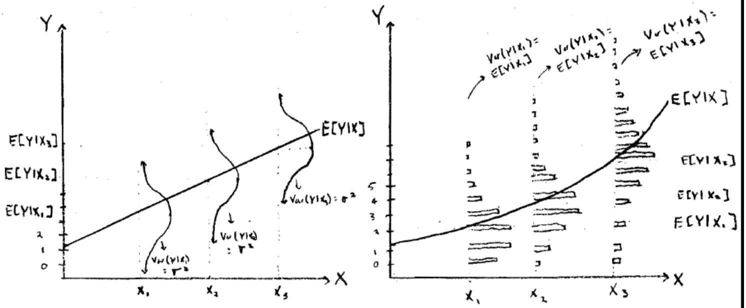
&\doteq \sum_{i=1}^{n} \{(x_i\beta)y_i - \exp(x_i\beta)\}
\end{aligned}$$

Comparing with the Linear Model



Example: Civil Conflict in Northern Ireland

Background: a conflict largely along religious lines about the status of Northern Ireland within the United Kingdom, and the division of resources and political power between Northern Ireland's Protestant (mainly Unionist) and Catholic (mainly Republican) communities.

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The data: the number of Republican deaths for every month from 1969, the beginning of sustained violence, to 2001 (at which point, most organized violence had subsided). Also, the unemployment rates in the two main religious communities.

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Our model is then:

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and

$$\lambda_i = E[Y_i | U_i^C] = exp(\beta_0 + \beta_1 * U_i^C).$$

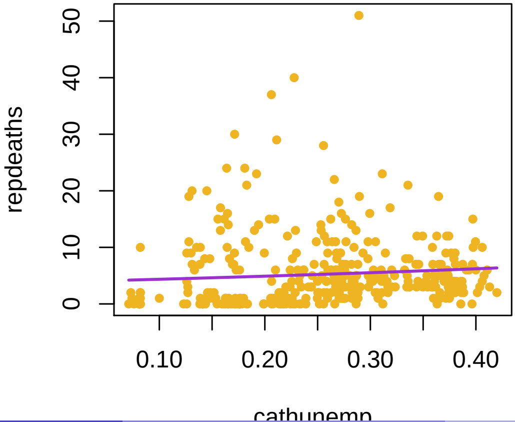
Estimate (just as we have all along!)

```
mod <- glm(repdeaths ~ cathunemp,
         data = troubles, family = poisson(link="log"))

> summary(mod)$coefficients
            Estimate Std. Error  z value     Pr(>|z|)
(Intercept) 1.295875  0.1805327 7.178064 7.070547e-13
cathunemp   1.406498  0.6689819 2.102445 3.551432e-02
```

Our fitted model

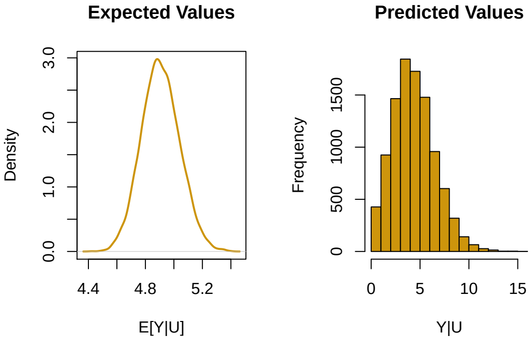
$$\lambda_i = E[Y_i | U_i^C] = exp(1.296 + 1.407 * U_i^C).$$



Some fitted and predicted values

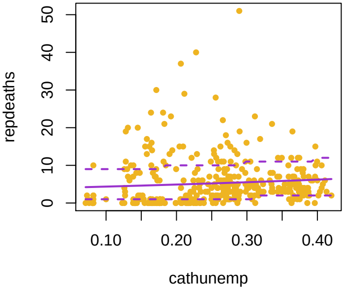
Suppose U_C is equal to .2.

```
mod.coef <- coef(mod); mod.vcov <- vcov(mod)
beta.draws <- mvrnorm(10000, mod.coef, mod.vcov)
lambda.draws <- exp(beta.draws[,1] + .2*beta.draws[,2])
outcome.draws <- rpois(10000, lambda.draws)
```



Overdispersion

36% of observations lie outside the 2.5% or 97.5% quantile of the Poisson distribution that we are alleging generated them.



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- When $\phi > 1$, this corresponds to a type of the negative binomial regression model (more on this later)

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Using a similar approach to that described in UPM pgs. 51-52 we can derive the marginal distribution of Y as

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Using a similar approach to that described in UPM pgs. 51-52 we can derive the marginal distribution of Y as

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2. we still have the same old systematic component: $\lambda_i = \exp(X_i\beta)$.

Estimates

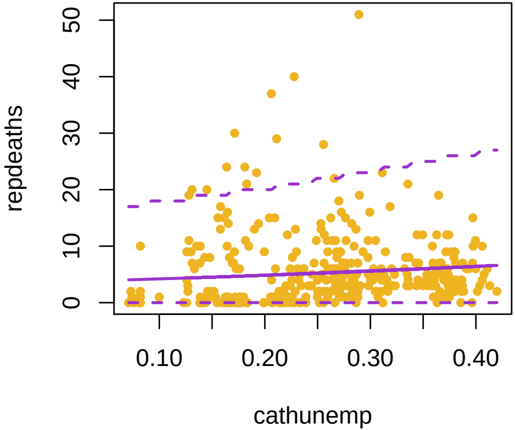
```
mod <- zelig(repdeaths ~ cathunemp, data = troubles,
             model = "negbin")
summary(mod)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)
(Intercept)   1.2959     0.1805   7.178 7.07e-13 ***
cathunemp     1.4065     0.6690   2.102   0.0355 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

...
              Theta:  0.8551
          Std. Err.:  0.0754
```

Overdispersion Handled!

5.68% of observations lie at or above the 95% quantile of the Negative Binomial distribution that we are alleging generated them.



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- Note that if $M_i = 1$ for all i, this reduces to a binary outcome model

Estimation and Overdispersion

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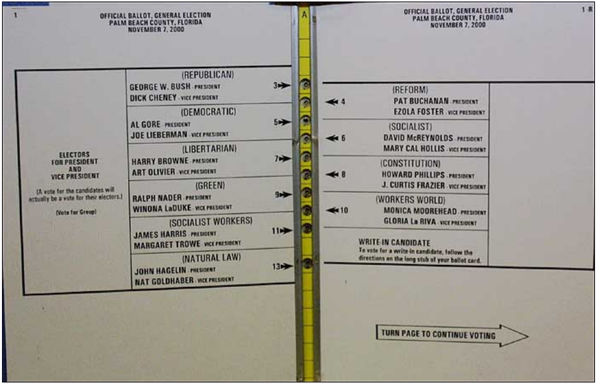
- Estimate β and ϕ via QMLE
- This is a GLM, so we have the same robustness properties as the Poisson case

Example: Butterfly Ballot in 2000 Presidential Election

Wand et al. (2001): Did the butterfly ballot give the election to Bush?



- Y_i: Number of votes cast for Buchanan in county i
- X_i: Past Republican & third-party vote shares, demographic covariates
- Wand et al. examine residuals to see how abberant the vote share was in Palm Beach

Fitting GLMs in R

Family	Canonical Link (Default)	Variance	Model
`gaussian`	`identity`	$\phi\ (= \sigma^2)$	normal linear
`binomial`	`logit`	$\mu(1-\mu)$	logit, probit, binomial
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- Observations should be measured in the same (temporal) units, i.e. don't have some units' duration measured in days and others in months

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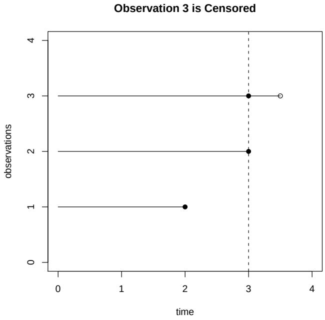

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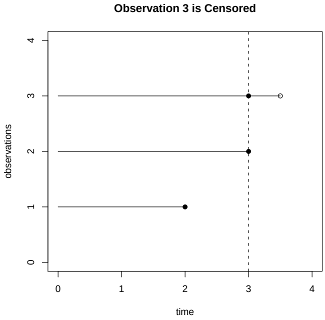

Observation 3 is censored in that it has not experienced the event at the time we stop collecting data, so we don't know its true duration

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   - OLS cannot handle multiple values of GDP per observation
   - You can set up data in a special way with duration models such that you can accommodate time-varying covariates

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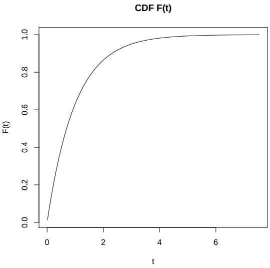

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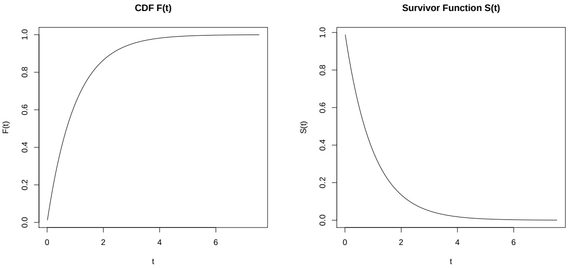

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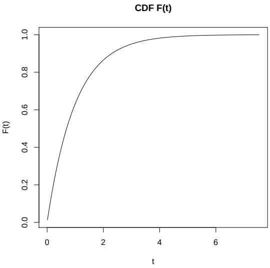
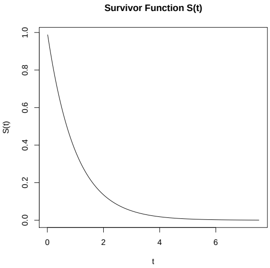
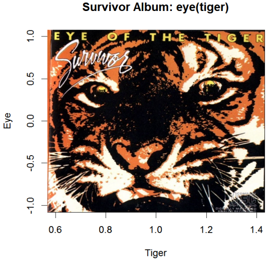
Eye of the Tiger: 1982 album by the band Survivor, which reached number 2 on the US Billboard 200 chart.







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Relating the Density, Survival, and Hazard Functions

$$h(t) = \frac{f(t)}{S(t)}$$

implies

$$\underbrace{f(t)}_{\text{density function}} = \underbrace{h(t)}_{\text{hazard function}} \cdot \underbrace{S(t)}_{\text{survival function}}$$

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and estimate β via maximum likelihood.

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4. Interpret quantities of interest (hazard ratios, expected survival times).

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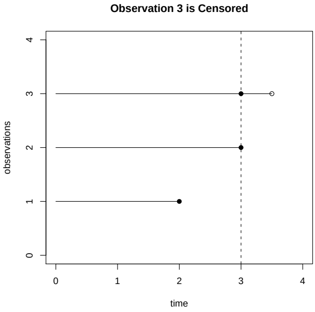

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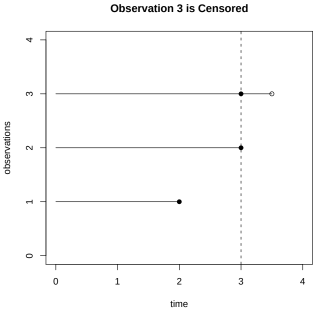

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Observation 3 is censored because it had not experienced the event when we collected the data, so we don't know its true duration.

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For each observation, let's create a censoring indicator variable, c_i, such that

$$c_i = \begin{cases} 1 & \text{if not censored} \\ 0 & \text{if censored} \end{cases}$$

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So uncensored observations contribute to the density function and censored observations contribute to the survivor function in the likelihood.

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- $N_\tau =$ number of arrivals in time period of length τ
  - $N_\tau \sim \text{Poisson}(\lambda\tau)$

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- Memorylessness property: how much you have waited already is irrelevant

$$P(T > t + k | T > t) = P(T > k)$$

$$P(T > 3 + 5 | T > 3) = P(T > 5)$$

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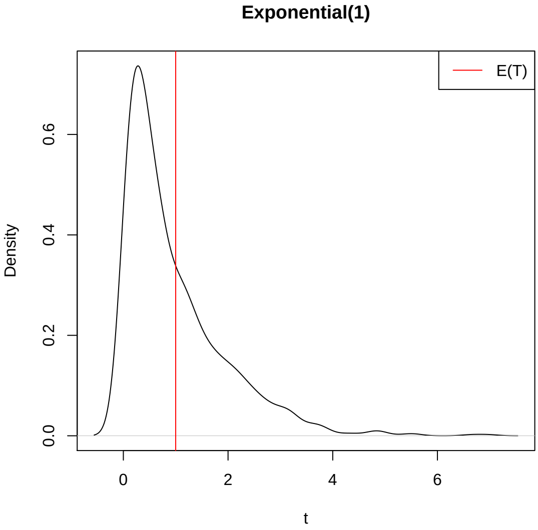

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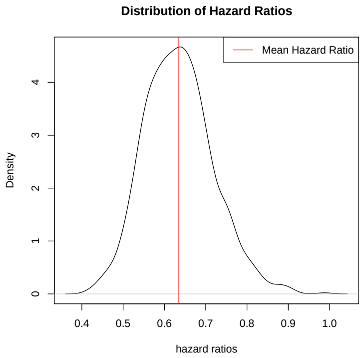
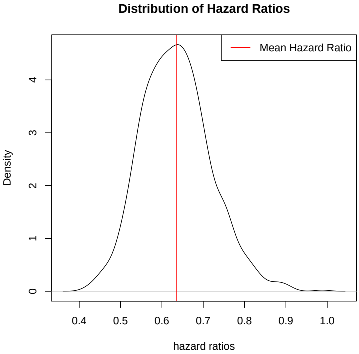
$$

Hazard ratio greater than 1 would imply that majority governments fall faster (shorter survival time) than minority governments.


Distribution of Hazard Ratios
Mean Hazard Ratio
Density
0
1
2
3
4
0.4
0.5
0.6
0.7
0.8
0.9
1.0
hazard ratios



Majority governments survive longer than minority governments.

Expected (average) Survival Time

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$$E(T|\mathbf{x}_i) = \theta_i$$

Expected (average) Survival Time

$$\begin{aligned} E(T|\mathbf{x}_i) &= \theta_i \\ &= \exp[\mathbf{x}_i\beta] \end{aligned}$$

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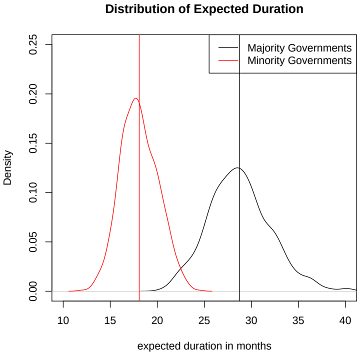

Predicted Survival Time

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Draw predicted values from the exponential distribution.

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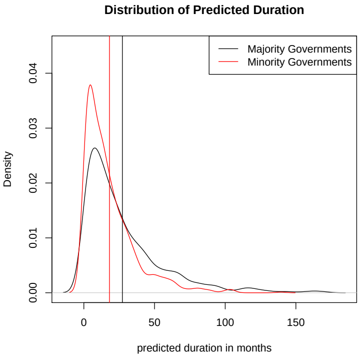

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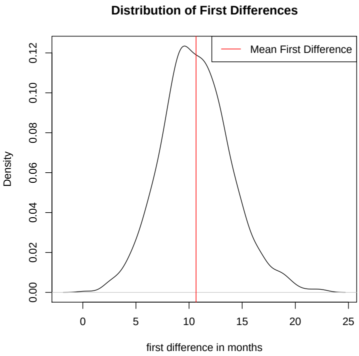
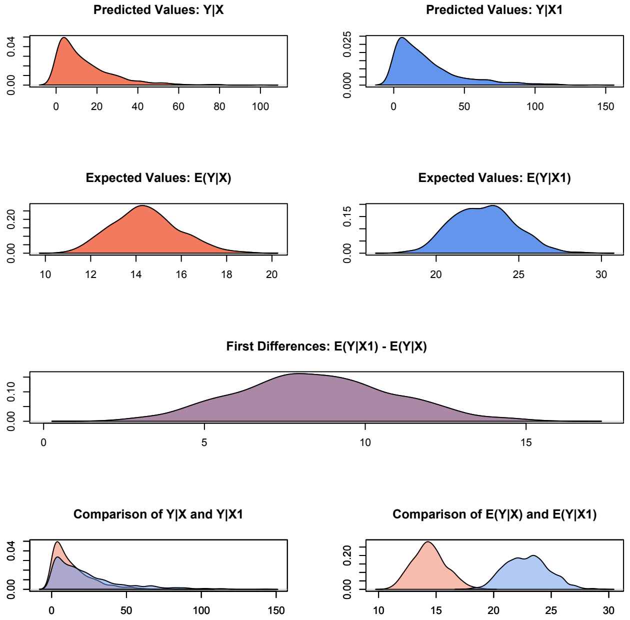


Quantities of Interest in Zelig

```
x.min <- setx(z.out,numst2=0)
x.maj <- setx(z.out,numst2=1)
s.out <- sim(z.out, x=x.min,x1=x.maj)
summary(s.out)
plot(s.out)
```


Predicted Values: Y|X
0.00
0.04
0
20
40
60
80
100
Predicted Values: Y|X1
0.000
0.025
0
50
100
150
Expected Values: E(Y|X)
0.00
0.20
10
12
14
16
18
20
Expected Values: E(Y|X1)
0.00
0.15
20
25
30
First Differences: E(Y|X1) - E(Y|X)
0.00
0.10
0
5
10
15
Comparison of Y|X and Y|X1
0.00
0.04
0
50
100
150
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0.20
10
15
20
25
30

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We can use the Weibull model.

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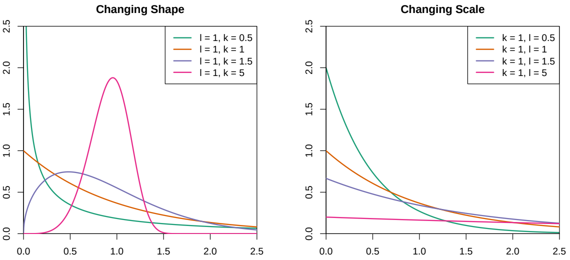

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Positive β implies that expected duration time increases as x increases.

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Hazard monotonicity assumption

$h(t_i)$ is modeled with both λ_i and α and is a function of t_i. Thus, the Weibull model assumes a **monotonic hazard**.

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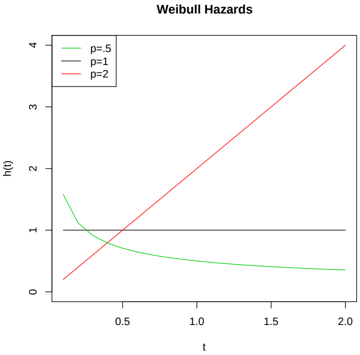

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The scale parameter given by `survreg()` is NOT the same as the scale parameter in the Weibull distribution, which should be $\theta_i = e^{\mathbf{x}_i \beta}$.

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With the Weibull model we make a **proportional hazards** assumption: hazard ratio does not depend t.

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But what if we don't want to make an assumption about the shape of the hazard?

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3. Assume no events can happen between event times.

We know that exactly one event occurred at each t_i for all non-censored i.

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For example, if $t_i = 5$ months, then all observations that do not experience the event or are not censored before 5 months are at risk.

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$h_0(t)$ is the baseline hazard, which is the same for all observations, so it cancels out.

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There is no β_0 term estimated. This implies that the shape of the baseline hazard is left unmodeled.

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How do I run a Cox proportional hazards model in R?

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Use the `coxph()` function in the `survival` package (also in the `Design` and `Zelig` packages).

Alternatives

- Survival models are cool . . .

Alternatives

- Survival models are cool . . . but hard.

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- Survival models are cool . . . but hard.
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If you encounter survival data think carefully about the process and then choose a corresponding model.

References:

Box-Steffensmeier, Janet M. and Bradford S. Jones. 2004. Event History Modeling. Cambridge University Press.

Therneau, Terry M., and Patricia M. Grambsch. 2013 Modeling survival data: extending the Cox model. Springer Science & Business Media.

Andersen, Per Kragh, et al. 2012 Statistical models based on counting processes. Springer Science & Business Media.

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How can we account for this duration dependence in a logit model?

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Year	t_k	Dyad	Y_i	T_i
1992	1	US-Iraq	0	
1993	2	US-Iraq	0	
1994	3	US-Iraq	0	
1995	4	US-Iraq	0	
1996	5	US-Iraq	0	
1997	6	US-Iraq	0	
1998	7	US-Iraq	0	12
1999	8	US-Iraq	0	
2000	9	US-Iraq	0	
2001	10	US-Iraq	0	
2002	11	US-Iraq	0	
2003	12	US-Iraq	1	

Then we end up with:

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So then we get

$$P(y_{i,t_k} = 1|\mathbf{x}_{i,t_k}) = 1 - e^{-\int_{t_{k-1}}^{t_k} h(u)du}$$

where we take the integral from t_{k-1} to t_k in order to get the conditional survival.

$$P(y_{i,t_k} = 1 | \mathbf{x}_{i,t_k}) \quad = \quad 1 - \exp\left(-\int_{t_{k-1}}^{t_k} h(u) du\right)$$

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This is equivalent to a model with a complementary log-log (cloglog) link and time dummies κ_{t_k}.

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- The use of time dummies may use up a lot of degrees of freedom, so BKT suggest using restricted cubic splines.

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- You can now interpret these models and learn new ones.

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These topics are a long-term bet on things that will be important in your career. Also a short case-study in reading into a new statistical literature.

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What is the (unique) MLE of β_2 and β_3? Different parameter values lead to the same values of μ and thus the same likelihood values:

$$\mu_i = x_{1i}\beta_1 + x_{2i}(5+3)$$
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Example 2: Non-unique Reparameterization

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So $\{\beta_2 = 2, \beta_3 = 5\}$ gives the same likelihood as $\{\beta_2 = 5, \beta_3 = 2\}$.

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(BTW, you now know how to form the likelihood for multiple equation models!)

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Also assume parametric independence, and you can estimate the equations separately.

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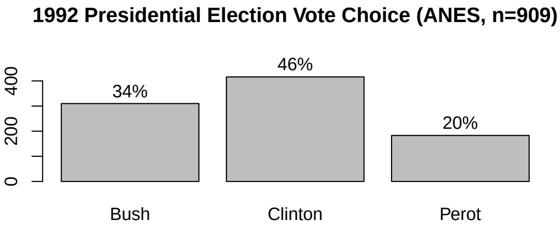

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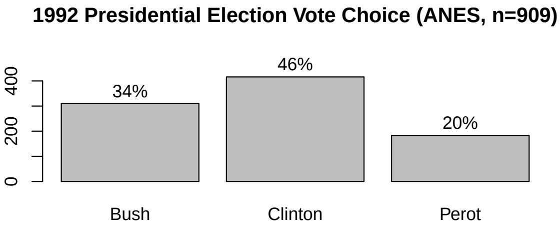

- Two types of predictors:
  - Voter-specific (V_i): age, gender, education, party, opinions, etc.
  - Candidate-varying (X_{ij}): ideological distance between voter i and candidate j

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- In that case we suppress the subscript to X_j

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- We use the names CL and MNL interchangeably from here on

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- Use `reshape` to change between wide and long
- Some estimation functions (e.g. `mlogit`) can take both formats

Latent Variable Interpretation

- Recall the random utility model:

$$Y_{ij}^* \;=\; X_{ij}^\top \beta + \epsilon_{ij},$$

where $\begin{cases} Y_{ij}^* & = & \text{latent utility from choosing } j \text{ for } i \\ \epsilon_{ij} & = & \text{stochastic component of the utility} \end{cases}$

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- Proof for $J = 2$:

$$\begin{aligned} \Pr(Y_i = 1 \mid X) &= \Pr(Y_{i1}^* \geq Y_{i2}^* \mid X) \\ &= \Pr\left(\epsilon_{i2} - \epsilon_{i1} \leq (X_{i1} - X_{i2})^\top \beta\right) \\ &= \frac{\exp\left((X_{i1} - X_{i2})^\top \beta\right)}{1 + \exp\left((X_{i1} - X_{i2})^\top \beta\right)} = \frac{\exp(X_{i1}^\top \beta)}{\exp(X_{i1}^\top \beta) + \exp(X_{i2}^\top \beta)} \end{aligned}$$

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- It can be shown that the log-likelihood is globally concave
  $\Rightarrow$ guaranteed convergence to the true (not local) MLE

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Compute a quantity that has a clear substantive interpretation!

Calculating Quantities of Interest

1. Choice probability:

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- Estimate by plugging in sample analogues (e.g. $\pi_j \rightarrow \hat{\pi}_j$, $\mathbb{E} \rightarrow \frac{1}{n}\sum$)

Example: 1992 U.S. Presidential Election

- Model specification (Alvarez and Nagler 1995):

$$\pi_{ij} = \frac{\exp(X_{ij}^{\top}\beta + V_i^{\top}\delta_j)}{\sum_{k=1}^{J}\exp(X_{ik}^{\top}\beta + V_i^{\top}\delta_k)}$$

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- Estimated coefficients:

$$\hat{\beta} = -0.11 \ (0.02)$$

$$\hat{\delta} = \begin{bmatrix} \hat{\delta}_{\text{Bush}} & \hat{\delta}_{\text{Clinton}} \end{bmatrix} = \begin{array}{c} \begin{bmatrix} 0.67\ (0.94) & -0.41\ (0.45) \\ -0.52\ (0.11) & -0.02\ (0.12) \\ 0.54\ (0.23) & 0.30\ (0.22) \\ \vdots & \vdots \end{bmatrix} \end{array} \begin{array}{l} \text{(intercept)} \\ \text{(support abortion)} \\ \text{(female)} \\ \vdots \end{array}$$

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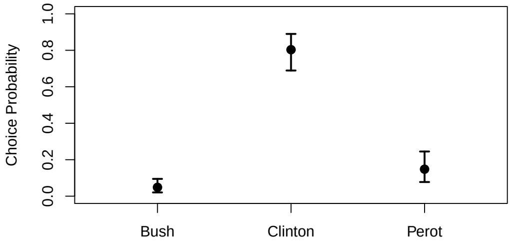
- Estimated choice probabilities for a “typical” voter from South:



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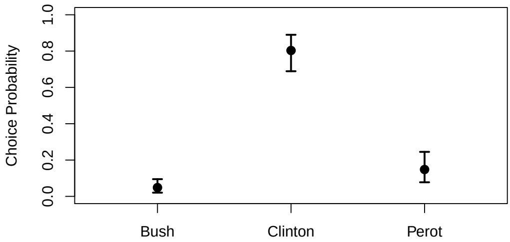
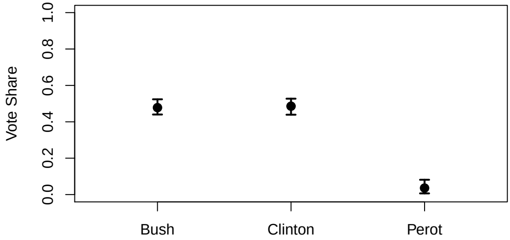


- Predicted vote shares if everyone opposed abortion:



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- That is, the multinomial choice reduces to a series of independent pairwise comparisons

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- Consequently, MNP is only feasible when J is small

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2. Calculate the generalized Pearson χ^2 statistic:

$$\mathcal{X}^2 \;\equiv\; \sum_{i=1}^{n}\frac{\omega_i(Y_i-\hat{\mu}_i)^2}{b''(\hat{\theta}_i)} \;=\; \phi\sum_{i=1}^{n}\frac{(Y_i-\hat{\mu}_i)^2}{\widehat{V}_i}$$

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When $Y_i \sim \mathcal{N}$, $D = \mathcal{X}^2 \sim \phi\chi^2_{n-k}$ exactly, and $\hat{\phi}_D$ and $\hat{\phi}_P$ are identical and MLE

Relaxing the Distributional Assumption

- Note that in the overdispersed Poisson model, Y_i *cannot* be Poisson distributed
- How can this seemingly arbitrary modification justified?

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Generalized Gamma Distribution

The Weibull and Exponential distributions are special cases of the Generalized Gamma distribution, $Y \sim GGamma(\nu, \lambda, p)$:

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When $p = 1$ and $\nu = 1$, $Y \sim Expo(\lambda)$:

$f_Y(y) = \lambda exp(-\lambda y)$

When would a Gamma regression be appropriate?

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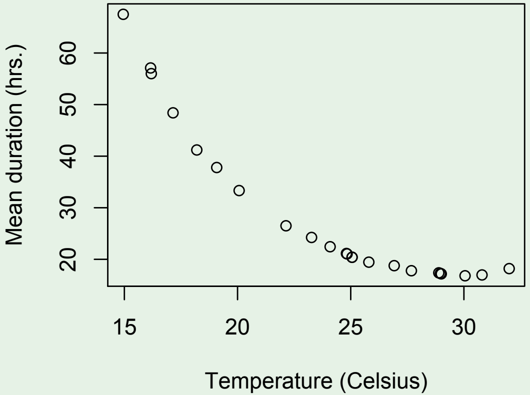
The **Gamma distribution** has this property.

When would a Gamma regression be appropriate?

Example

Mean duration of developmental period in *Drosphila melanogaster* (McCullagh & Nelder, 1989)



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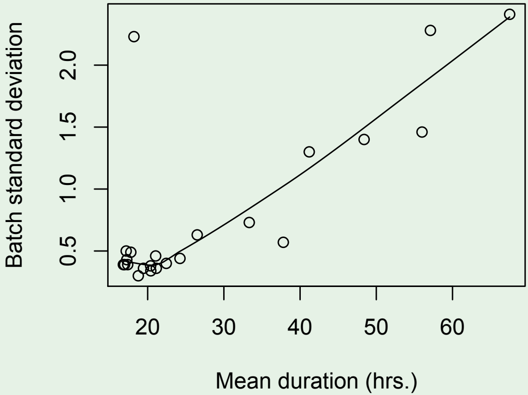
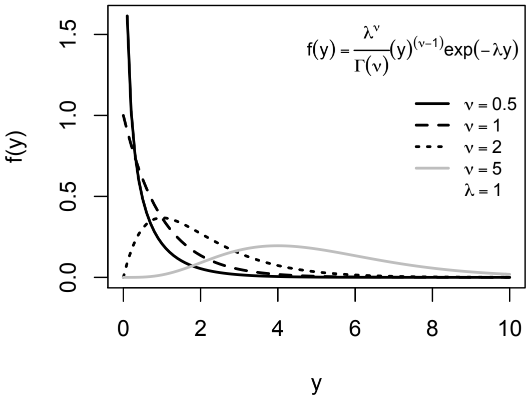


Gamma shapes

ν is the shape parameter, λ is the scale parameter



Special cases:

$$\nu = 1 \implies \textit{Exponential}$$

$$\nu \to \infty \implies \textit{Normal}$$

Gamma as an EDF

$$
\begin{aligned}
f_Y(y) &= \frac{\lambda^{\nu}}{\Gamma(\nu)} y^{\nu-1} exp(-\lambda y) \\
&= exp\left(\frac{-\frac{\lambda}{\nu} y + ln(\frac{\lambda}{\nu})}{\nu^{-1}} + \nu ln(\nu y) - ln(y) - ln(\Gamma(\nu)) \right)
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Where

$$
\begin{aligned}
\theta &= -\frac{\lambda}{\nu} \\
\phi &= \nu^{-1} = \sigma^2 \\
b(\theta) &= -ln\left(\frac{\lambda}{\nu}\right) \\
E[Y] &= b'(\theta) = \frac{\nu}{\lambda} = \mu \\
Var(Y) &= \phi b''(\theta) = \frac{1}{\nu}\frac{\nu^2}{\lambda^2} = \sigma^2 \mu^2
\end{aligned}
$$

Link functions

Canonical link

$$\eta = \theta = -\frac{1}{\mu}$$

The reciprocal transformation does not map the range of μ onto the whole real line.

The requirement that $\mu > 0$ places restrictions on β's.

The canonical link is rarely used.

Link functions

Inverse polynomial: linear

$$\eta = \mu^{-1} = \beta_0 + \beta_1/x$$

Inverse polynomial: quadratic

$$\eta = \mu^{-1} = \beta_0 + \beta_1 x + \beta_2/x$$

Inverse polynomials have appealing property that η is everywhere positive and bounded.

Application: sometimes used in plant density experiments, where yield per plant (y_i) varies inversely with plant density (x_i)

Link functions

Log link

$$\eta = ln(\mu) = \beta_0 + \beta_1 x$$

$$\eta = ln(\mu) = \beta_0 + \beta_1 x + \beta_2 / x$$

Application: useful for describing functions that have turning points, but are noticeably asymmetric around that point.

Link functions

Identity link

$$\eta = \mu = \beta_0 + \beta_1 x$$

Application: used for modeling variance components.

Maximum Likelihood Estimation

$$\mathcal{L} = \prod_{i=1}^{n} \frac{\lambda^{\nu}}{\Gamma(\nu)} y_i^{\nu-1} exp(-\lambda y_i)$$

$$\begin{aligned}
\ln\mathcal{L} &= \sum_{i=1}^{n} \ln\left[\frac{\lambda^{\nu}}{\Gamma(\nu)} y_i^{\nu-1} exp(-\lambda y_i)\right] \\
&= \sum_{i=1}^{n} \nu \ln\lambda - \ln\Gamma(\nu) + (\nu-1)\ln y_i - \lambda y_i
\end{aligned}$$

Gamma regression with weights

Suppose your data consist of n observations, each from a separate group $i \in \{1, \ldots, n\}$. Each group has n_i individuals.

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Y_i is the duration of embryonic period in n_i batches of fruit flies

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$$Y_i^s = Y_i / n_i = \text{ average duration in } i\text{-th batch}$$

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If $Y_{ij} \sim Gamma(\lambda_i, \nu)$, independent, with $\lambda_i = \nu/\mu_i$:

$$E[Y_i^s] = \frac{1}{n_i}\sum_{j=1}^{n_i} E[Y_{ij}] = \frac{1}{n_i}\sum_{j=1}^{n_i} \mu_i = \mu_i$$

$$Var(Y_i^s) = \frac{1}{n_i} Var(Y_i) = \frac{\sigma^2 \mu_i^2}{n_i} \qquad \text{weights} = n_i$$

Application: *Drosophila melanogaster*

4 models estimated:

1. $log(\text{Duration}_i) = \beta_0 + \beta_1 \text{Temp}_i$

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4. $log(\text{Duration}_i) = \beta_0 + \beta_1 \text{Temp}_i + \beta_2 / (\text{Temp}_i - \delta)$
   (weighted by batch size)

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