Precept 4 - More GLMs: Models of Binary and Lognormal Outcomes

Soc 504: Advanced Social Statistics

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[1]These slides owe an enormous debt to generations of TFs in Gov 2001 at Harvard. Many slides are directly adapted from those by Brandon Stewart and Stephen Pettigrew.

Outline

Replication Paper

Any thoughts or issues to discuss?

Generalized Linear Models

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Three elements of a GLM:

- A distribution for Y (stochastic component)
- A linear predictor $X\beta$ (systematic component)
- A link function that relates the linear predictor to a parameter of the distribution. (systematic component)

1. Specify a distribution for Y

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- Binary:

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Examples:

- Continuous and Unbounded: Normal
- Binary: Bernoulli

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Examples:

- Continuous and Unbounded: Normal
- Binary: Bernoulli
- Event Count:

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Examples:

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- Continuous and Unbounded: Normal
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Examples:

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- Continuous and Unbounded: Normal
- Binary: Bernoulli
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- Duration: Exponential
- Ordered Categories:

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- Unordered Categories: Multinomial

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$$X\beta = \beta_0 + x_1\beta_1 + x_2\beta_2 + \cdots + x_k\beta_k$$

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This is the systematic component that we've been talking about all

along.

Example Link Functions

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- Link: $\mu = X\beta$

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- Inverse Link: $\lambda = \exp(X\beta)$

4. Estimate Parameters via ML

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Use `optim` to estimate the parameters just like we have all along.

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1. Simulate parameters from multivariate normal.
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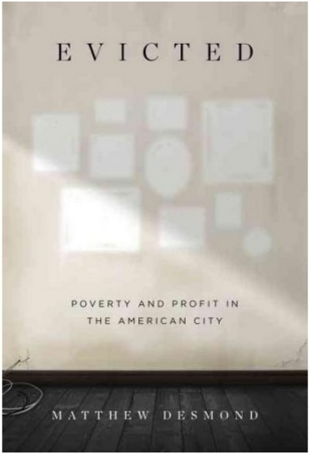
3. Draw from distribution of Y for predicted values.

Outline

1. GLMs
   - General Structure of GLMs
   - Procedure for Running a GLM
2. Complementary log-log
3. Quantities of Interest


EVICTED
POVERTY AND PROFIT IN
THE AMERICAN CITY
MATTHEW DESMOND

We will use data from the Fragile Families and Child Wellbeing Study to study the cumulative risk of eviction over child for children born in large American cities.

Fragile Families
& Child Wellbeing Study
PRINCETON | COLUMBIA

Research question and data

What is the probability of eviction in a given year for a child with a given set of covariates?

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`ffEviction.csv` is the data that we use.

Fragile Families data

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```
> head(d)
  idnum income married cm1ethrace ev
1  0001    1.5       0   Hispanic  0
2  0002    1.6       0      Black  0
3  0003    2.7       0      White  0
4  0004    1.0       0   Hispanic  0
5  0006    0.2       0      Black  0
6  0007    1.3       0   Hispanic  0

> summary(d)
   idnum               income         married          cm1ethrace         ev
 Length:12298       Min.   :0.000   Min.   :0.0000   White    :2709   Min.   :0.00000
 Class :character   1st Qu.:0.500   1st Qu.:0.0000   Black    :5911   1st Qu.:0.00000
 Mode  :character   Median :1.200   Median :0.0000   Hispanic:3225   Median :0.00000
                    Mean   :1.666   Mean   :0.2502   Other    : 453   Mean   :0.02301
                    3rd Qu.:2.400   3rd Qu.:1.0000                    3rd Qu.:0.00000
                    Max.   :5.000   Max.   :1.0000                    Max.   :1.00000
```

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`ev`: dependent variable; was this child evicted in a given year?

`income`: family income / poverty line at age 1

`married`: were the parents married at the birth?

`cm1ethrace`: mother's race/ethnicity

Outline

Binary Dependent Variable

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Our outcome variable is whether or not a child was evicted

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What's the first question we should ask ourselves when we start to model this dependent variable?

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$$X_i\beta$$

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- Scobit: $\pi_i = (1+e^{-x_i\beta})^{-\alpha}$

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Log-likelihood of the c-loglog

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$$\begin{aligned} \ell(\beta \mid Y) &= \log(L(\beta \mid Y)) \\ &= \log(p(Y \mid \beta)) \\ &= \log\left(\prod_{i=1}^{n} p(Y_i \mid \beta)\right) \end{aligned}$$

Log-likelihood of the c-loglog

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&= \sum_{i=1}^{n} (Y_i \log(1 - \exp(-\exp(X_i\beta))) + (1 - Y_i) \log[\exp(-\exp(X_i\beta))])
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&= \sum_{i=1}^{n} (Y_i \log(1 - \exp(-\exp(X_i\beta))) + (1 - Y_i)\log[\exp(-\exp(X_i\beta))]) \\
&= \sum_{i=1}^{n} (Y_i \log(1 - \exp(-\exp(X_i\beta))) - (1 - Y_i)\exp(X_i\beta))
\end{aligned}
$$

Coding our log likelihood function

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cloglog.loglik <- function(par, X, y) {
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cloglog.loglik <- function(par, X, y) {

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     log.lik <- sum(y * log(1 - exp(-exp(X %*% beta))) -
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  return(log.lik)
}
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X <- model.matrix(~married + cm1ethrace + income,
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opt <- optim(par = rep(0, ncol(X)),
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```

Point estimate of the MLE:

```
opt$par
[1] -2.704 -1.211 -0.526 -0.620 -0.272 -0.348
```

Standard errors of the MLE

[2] Credit to Stephen Pettigrew for including this figure in slides.

Standard errors of the MLE

Recall that the standard errors are defined as the diagonal of:

$$\sqrt{-\left[\frac{\partial^2 \ell}{\partial \beta^2}\right]^{-1}}$$

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Variance-covariance matrix:

```
-solve(opt$hessian)
       [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
[1,]  0.026 -0.002 -0.021 -0.021 -0.019 -0.006
[2,] -0.002  0.063  0.004  0.002 -0.001 -0.004
[3,] -0.021  0.004  0.026  0.019  0.018  0.002
[4,] -0.021  0.002  0.019  0.033  0.018  0.002
[5,] -0.019 -0.001  0.018  0.018  0.128  0.002
[6,] -0.006 -0.004  0.002  0.002  0.002  0.004
```

Standard errors of the MLE

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       [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
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[4,] -0.021  0.002  0.019  0.033  0.018  0.002
[5,] -0.019 -0.001  0.018  0.018  0.128  0.002
[6,] -0.006 -0.004  0.002  0.002  0.002  0.004
```

Standard errors:

```
sqrt(diag(-solve(opt$hessian)))
[1] 0.162 0.252 0.160 0.183 0.358 0.064
```

Outline

Interpreting c-loglog coefficients

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Here's a nicely formatted table with your regression results from our model:

Variable	Coefficient	SE
Intercept	-2.70	0.16
Married	-1.21	0.25
Black	-0.53	0.16
Hispanic	-0.62	0.18
Other	-0.27	0.36
Income / poverty line	-0.35	0.06

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Intercept	-2.70	0.16
Married	-1.21	0.25
Black	-0.53	0.16
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All else constant, children of married parents have -1.21 points lower log rate of eviction.

And what are log rates? Nobody thinks in terms of log odds, or

probit coefficients, or exponential rates.

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We'll spend the rest of today looking at how to do that.

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7. Repeat steps 2 through 6 thousands of times
8. Use the results to make fancy graphs and informative tables

Simulate from the sampling distribution of $\hat{\beta}_{MLE}$

By the central limit theorem, we assume that
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So we account for this uncertainty by simulating βs from the multivariate normal distribution defined above

Simulate from the sampling distribution of $\hat{\beta}_{MLE}$

Simulate one draw from $\text{mvnorm}(\hat{\beta}, \hat{V}(\hat{\beta}))$

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Simulate one draw from $\mathrm{mvnorm}(\hat{\beta}, \hat{V}(\hat{\beta}))$
Install the mvtnorm package if you need to

```
install.packages("mvtnorm")
require(mvtnorm)


sim.betas <- rmvnorm(n = 1,
                     mean = opt$par,
                     sigma = -solve(opt$hessian))
sim.betas
       [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
[1,] -2.825 -1.424 -0.478 -0.358 -0.123 -0.237
```

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Let's make predictions for one white child born to married parents with family income at the poverty line Recall that our predictors (in order) are:

```
> colnames(X)
[1] "(Intercept)"        "married"                 "cm1ethraceBlack"    "cm1ethraceH
[5] "cm1ethraceOther"    "income"
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We can set the values of X as:

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We can set the values of X as:

```
setX <- c(1, 1, 0, 0, 0, 1)
```

Untransform $X\beta$

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Now we multiply our covariates of interest by our simulated parameters:

```
setX %*% t(sim.betas)
          [,1]
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1. -4.486287 is not the predicted probability (obviously - it's negative!), it's the predicted log rate
2. We haven't done a very good job of accounting for the uncertainty in the model

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To turn $\tilde{X}\tilde{\beta}$ into a predicted probability we need to plug it back into our link function, which was $1 - \exp(-\exp(X_i\beta))$

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```
> sim.p <- 1 - exp(-exp(setX %*% t(sim.betas)))
> sim.p
          [,1]
[1,] 0.0111992
```

Simulate from the stochastic function

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Is 0.0117 our best guess at the predicted probability of eviction? Nope

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To do this we'd need to loop over all the steps I just went through and get the full distribution of predicted probabilities for this case.

Speeding up the process

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Or, instead of using a loop, let's just vectorize our code:

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```
sim.betas <- rmvnorm(n = 10000,
                     mean = opt$par,
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Or, instead of using a loop, let's just vectorize our code:

```
sim.betas <- rmvnorm(n = 10000,
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head(sim.betas)
       [,1]   [,2]   [,3]   [,4]   [,5]   [,6]
[1,] -2.800 -0.955 -0.545 -0.531 -0.217 -0.319
[2,] -2.410 -0.879 -0.772 -0.896 -0.301 -0.485
[3,] -2.715 -1.672 -0.483 -0.741 -0.365 -0.333
[4,] -2.718 -0.993 -0.588 -0.545 -0.094 -0.389
[5,] -2.479 -1.161 -0.721 -0.794 -0.601 -0.413
[6,] -2.625 -1.227 -0.654 -0.586 -0.496 -0.351
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dim(sim.betas)
[1] 10000     6
```

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```
pred.xb <- setX %*% t(sim.betas)
```

And untransform them

```
> pred.prob <- 1 - exp(-exp(pred.xb))
> pred.prob[,1:5]
[1] 0.016858874 0.022674923 0.008876607 0.016426627 0.017223131
```

Look at Our Results in Tabular Form

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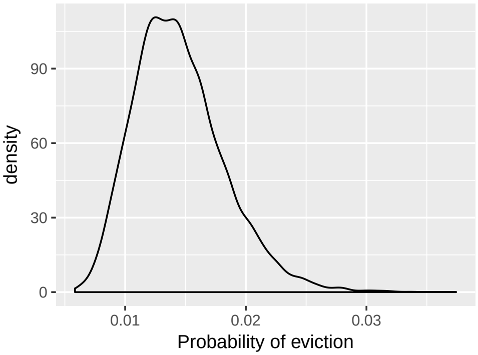

Look at Our Results

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```
> mean(pred.prob)
[1] 0.01446521
> quantile(pred.prob, prob = c(.025, .975))
      2.5%      97.5%
0.00844883 0.02295435

> mean(pred.prob > .02)
[1] 0.0828
```

Different QOI

[3] Note: We assume independence between eviction in each year, and a constant risk over time. This corresponds to an Exponential survival model.

Different QOI

What if our QOI was the chance of any eviction from birth to age 9?

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Different QOI

What if our QOI was the chance of any eviction from birth to age 9?

$$
\begin{aligned}
P(\text{Ever evicted}) &= 1 - P(\text{Never evicted}) \\
&= 1 - \prod_{i=1}^{9}(1 - P(\text{Evicted at age } i)) \\
&= 1 - (1-p)^9
\end{aligned}
$$

All that will change is the very last step[3]

[3] Note: We assume independence between eviction in each year, and a constant risk over time. This corresponds to an Exponential survival model.

Different QOI

We already estimated the sampling distribution of p and stored samples from this distribution in the vector `predprob`. Now we can just transform them!

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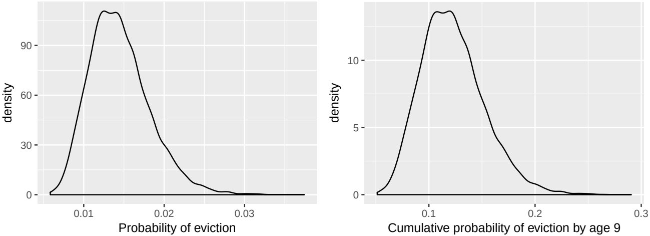
$$P(\text{Ever evicted}_i) = 1 - (1 - p_i)^9$$

```
> cum.prob <- 1 - (1 - pred.prob) ^ 9
> mean(cum.prob)
[1] 0.1224441
```

12%! The probability of eviction looks much higher than we had thought!

Look at Our Results For Both QOIs

Look at Our Results For Both QOIs



Quantities of interest matter in continuous cases as well

Example: Modeling log income

Modeling log income

We want to model the effect of college (D) on earnings (Y), net of age (X).

Let's get some data! `http://cps.ipums.org`

Causal identification

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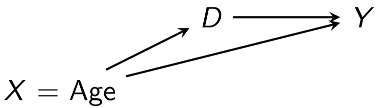

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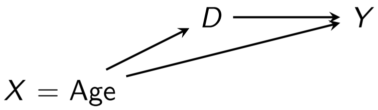

This is our **identification strategy**

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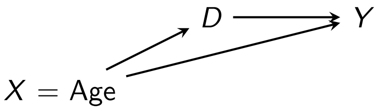

This is our **identification strategy**
but it says nothing about **estimation**.

Estimation via GLMs

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4. Estimate parameters via maximum likelihood
5. Simulate quantities of interest

Likelihood

$$
\begin{aligned}
L(\beta, \sigma^2 \mid Y) &\propto f(Y \mid \beta, \sigma^2) \\
&= \prod_{i=1}^{n} f(Y_i \mid \beta, \sigma^2) \\
&= \prod_{i=1}^{n} \frac{1}{Y_i \sigma \sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - \mu)^2}{2\sigma^2}\right)
\end{aligned}
$$

Log likelihood

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$$= \ln\left[\prod_{i=1}^{N} \frac{1}{Y_i\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - X_i\beta)^2}{2\sigma^2}\right)\right]$$

$$= \sum_{i=1}^{N} \ln\left[\frac{1}{Y_i\sigma\sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - X_i\beta)^2}{2\sigma^2}\right)\right]$$

Log likelihood

$$
\begin{aligned}
\ell(\beta, \sigma^2 \mid \mathbf{Y}) &= \ln \left[\prod_{i=1}^{N} f(Y_i|\beta, \sigma^2) \right] \\
&= \ln \left[\prod_{i=1}^{N} \frac{1}{Y_i \sigma \sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - X_i\beta)^2}{2\sigma^2} \right) \right] \\
&= \sum_{i=1}^{N} \ln \left[\frac{1}{Y_i \sigma \sqrt{2\pi}} \exp\left(-\frac{(\ln(Y_i) - X_i\beta)^2}{2\sigma^2} \right) \right] \\
&= \sum_{i=1}^{N} \left(-\ln(Y_i) - \ln(\sigma) - \ln(\sqrt{2\pi}) + \ln \left[\exp\left(-\frac{(\ln(Y_i) - X_i\beta)^2}{2\sigma^2} \right) \right] \right)
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$$\sum_{i=1}^{N}\left(-\ln(\sigma)-\frac{(\ln(Y_i)-X_i\beta)^2}{2\sigma^2}\right)$$

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  return(log.lik)
}
```

Finding the MLE

```
X <- model.matrix(~ college + age,
                  data = d)
y <- d$incwage

opt <- optim(par = rep(0, ncol(X) + 1),
             fn = logNormal.log.lik,
             y = y,
             X = X,
             control = list(fnscale = -1),
             method = "BFGS",
             hessian = TRUE)
```

Extract the MLE

```
> opt$par
[1]  8.84550213  0.72904517  0.03270657 -0.03216774
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> coef(lm.fit)
(Intercept) collegeTRUE         age
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Why is that last term negative?

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```

Why is that last term negative? Because it's the **log** of σ^2!

See how σ^2 matches

```
> summary(lm.fit)

Call:
lm(formula = log(incwage) ~ college + age, data = d)
.......other output....
Residual standard error: 0.9841 on 68932 degrees of freedom
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We estimated

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We estimated $\sigma^2 = e^{\gamma} = e^{-0.03216774}$

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> exp(opt$par[4])
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It matches!

Variance-covariance matrix matches

We know how to calculate the variance-covariance matrix - the inverse of the negative Hessian!

```
> vcov.optim <- -solve(opt$hessian)
> vcov.optim
              [,1]          [,2]          [,3]         [,4]
[1,]  1.938105e-04 -6.974235e-06 -4.762674e-06 1.171351e-13
[2,] -6.974235e-06  6.311970e-05 -4.031325e-07 1.714362e-14
[3,] -4.762674e-06 -4.031325e-07  1.316824e-07 2.028694e-14
[4,]  1.171351e-13  1.714362e-14  2.028694e-14 2.901283e-05
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And we can compare that to the canned version...

```
> vcov(lm.fit)
              (Intercept)   collegeTRUE           age
(Intercept)  1.938189e-04 -6.974537e-06 -4.762880e-06
collegeTRUE -6.974537e-06  6.312244e-05 -4.031500e-07
age         -4.762880e-06 -4.031500e-07  1.316881e-07
```

....which matches!

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Using the naive approach of exponentiating things, we would find an effect of college on earnings of **$25,181.27**.

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Why the discrepancy? (Draw on the board).

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No! Effect sizes depend on the values chosen for the rest of the covariates.

Different effect sizes for different groups!

	Effect	2.5%	97.5%
20-year-olds	23,276.88	19,466.92	27,494.30
50-year-olds	62,319.15	51,498.74	73,129.50

Since 50-year-olds have higher predicted earnings to begin with, multiplying by a factor of 2.208 increases their earnings by more dollars.

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Which of those goes away as the sample size grows?

Estimation uncertainty disappears in large samples

Each row section below shows the difference in earnings (college - noncollege), for 50 year olds.

```
$`N = 100`
                                           2.5%    97.5%
Difference in any two 50-year-olds -186146.3 890301.0
Average difference                    53387.9 249905.7

$`N = 1,000`
                                            2.5%     97.5%
Difference in any two 50-year-olds -169635.63 479623.19
Average difference                    51505.21  85801.36

$`N = 50,000`
                                            2.5%     97.5%
Difference in any two 50-year-olds -177369.77 459907.68
Average difference                    52190.08  73289.31
```

The expected difference is more precisely estimated with larger sample sizes, but fundamental uncertainty makes it always difficult to make precise predictions about the difference between any two actual individuals.

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8. Use the results to make fancy graphs and informative tables