# Precept 2: Likelihood inference

## Soc 504: Advanced Social Statistics

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# Outline

# Replication Paper

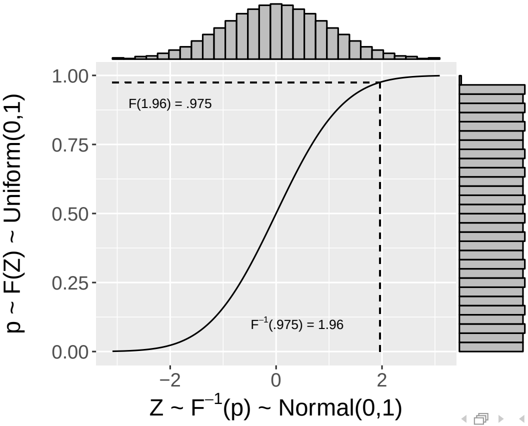
Any thoughts or issues to discuss?

# Universality of the Uniform

aka Probability Integral Transform, PIT

## Theorem

- *Regardless of the distribution of* $X$, $F(X) \sim Uniform(0,1)$
- *For a r.v.* $X$ *with CDF* $F$ *and a Uniform r.v.* $U$, $F^{-1}(U) \sim X$



# Integral of PDF

All PDFs integrate to 1. Why?

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Because the probability of observing a value somewhere in the support of the random variable is 1!

# Expected value as an integral

$$E(X) =$$

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$$E(X) = \int_{-\infty}^{\infty} x f_X(x) dx$$

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Suppose

$$f_Y(y) = \begin{cases} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1} & \text{for } y \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

This is called the **beta distribution**.

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&= \left(\frac{\Gamma(\alpha+1)\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\alpha+\beta+1)}\right) \underbrace{\int_0^1 \left(\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+1)\Gamma(\beta)}\right) y^{(\alpha+1)-1}(1-y)^{\beta-1} dy}_{\text{Beta pdf integrates to 1}}
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&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha+\beta+1)} \\
&= \frac{\alpha}{\alpha+\beta}
\end{aligned}
$$

# Numeric integration

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integrate(f = function(x) x * beta.pdf(1,2,x),
          lower = 0, upper = 1)
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integrate(f = function(x) x * beta.pdf(1,2,x),
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0.3333333 with absolute error < 3.7e-15
```

# Variance as an integral

$$V(X) =$$

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$$V(X) = E(X - E[X])^2$$
$$=$$

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$$\begin{aligned} V(X) &= E(X - E[X])^2 \\ &= E(X^2) - (E[X])^2 \\ &= \end{aligned}$$

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$$
\begin{aligned}
V(X) &= E(X - E[X])^2 \\
&= E(X^2) - (E[X])^2 \\
&= \int_{-\infty}^{\infty} x^2 f_X(x) dx - \left( \int_{-\infty}^{\infty} x f_X(x) dx \right)^2
\end{aligned}
$$

# Likelihood

Steps of likelihood inference:

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1. Assume a data generating process.

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1. Assume a data generating process.
2. Derive the likelihood.
3. Maximize the likelihood to get the MLE.
4. Derive standard errors from the inverse of the Fisher information

# Binomial example

Suppose we would like to know the probability that a Princeton Ph.D. student in sociology who submits a paper to a major journal is offered the chance to revise and resubmit the paper. We have data on several students who each submit 5 papers over the course of the program. For each student, we observe the number of these papers that receive a revise and resubmit on the first submission.

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Can we translate this into a data generating process?

For $i = 1, \ldots, n$,

$$Y_i \sim \text{Binomial}(5, p)$$

We **assume** that the response is binomial, with each paper independent, and with all submissions from all students having the same probability $p$ of success.

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What does the model help us to learn?

The model helps us to learn the value of the parameter $\hat{p}$ that makes the observed data the most likely.

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What is the systematic component?
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**Systematic component:** The probability $p$
**Stochastic component:** The outcome $Y_i$

# $P(y \mid p)$

What is the probability of one $y_i$ given $p$?

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$$=$$

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# $L(p \mid y_1, \ldots, y_n)$

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$$=$$

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# Review of log rules

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$$\log(ab) = \log(a) + \log(b)$$

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$$
\begin{aligned}
\log(ab) &= \log(a) + \log(b) \\
\log(e^a) &= a
\end{aligned}
$$

# $\ell(p \mid y_1, \ldots, y_n)$

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$$\ell(p \mid y_1, \ldots, y_n) = \log L(p \mid y_1, \ldots, y_n)$$

$$\doteq$$

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$$
\begin{aligned}
\ell(p \mid y_1, \ldots, y_n) &= \log L(p \mid y_1, \ldots, y_n) \\
&\doteq \log \left( \prod_{i=1}^{n} p^{y_i} (1-p)^{5-y_i} \right) \\
&=
\end{aligned}
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&= \sum_{i=1}^{n} \log \left( p^{y_i} (1-p)^{5-y_i} \right) \\
&= \sum_{i=1}^{n} \Big( \log \left( p^{y_i} \right) + \log \left( (1-p)^{5-y_i} \right) \Big) \\
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\end{aligned}
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&= \sum_{i=1}^{n} \Big( \log \left( p^{y_i} \right) + \log \left( (1-p)^{5-y_i} \right) \Big) \\
&= \sum_{i=1}^{n} \Big( y_i \log p + (5 - y_i) \log(1-p) \Big)
\end{aligned}
$$

# $\ell(p \mid y_1, \ldots, y_n)$ (continued)

(we can go further)

$$\ell(p \mid y_1, \ldots, y_n) = \sum_{i=1}^{n} \Big( y_i \log p + (5 - y_i) \log(1 - p) \Big)$$

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# $\ell(p \mid y_1, \dots, y_n)$ (continued)

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\ell(p \mid y_1, \dots, y_n) &= \sum_{i=1}^{n} \Big( y_i \log p + (5 - y_i) \log(1-p) \Big) \\
&= \log p \sum_{i=1}^{n} y_i + 5n \log(1-p) - \log(1-p) \sum_{i=1}^{n} y_i \\
&=
\end{aligned}$$

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&= (\log p - \log[1 - p]) \sum_{i=1}^{n} y_i + 5n \log(1 - p)
\end{aligned}
$$

So...the data $y_1, \ldots, y_n$ only enter the likelihood through their sum.

# $\ell(p \mid y_1, \ldots, y_n)$ (continued)

(we can go further)

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So...the data $y_1, \ldots, y_n$ only enter the likelihood through their sum. We call $\sum_{i=1}^{n} y_i$ a **sufficient statistic** since it's all you need to compute the likelihood.

# Sufficient statistic discussion

Does it seem reasonable that we could compute the likelihood only knowing the number of graduate student submissions that are given R&Rs?

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This makes sense because we assumed every submission had the same probability of success, so it's like we had 5n Bernoulli trials. There is no need to distinguish who submitted them!

# Sufficient statistic discussion

Does it seem reasonable that we could compute the likelihood only knowing the number of graduate student submissions that are given R&Rs?

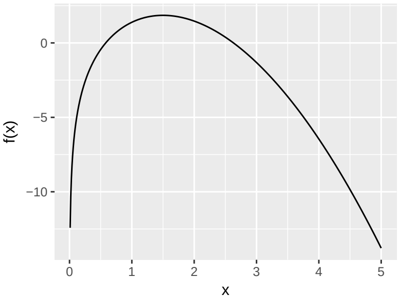
This makes sense because we assumed every submission had the same probability of success, so it's like we had 5n Bernoulli trials. There is no need to distinguish who submitted them!

Sufficient statistics can save disk space in more complex problems - no need to store all the data!

# Calculus review: Derivatives

Suppose we have a function

$$f(x) = x - x^2 + \log(x) + \log(3x^2)$$




What is the derivative?

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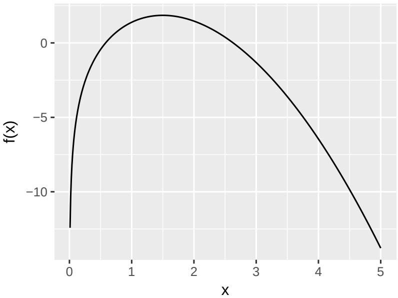
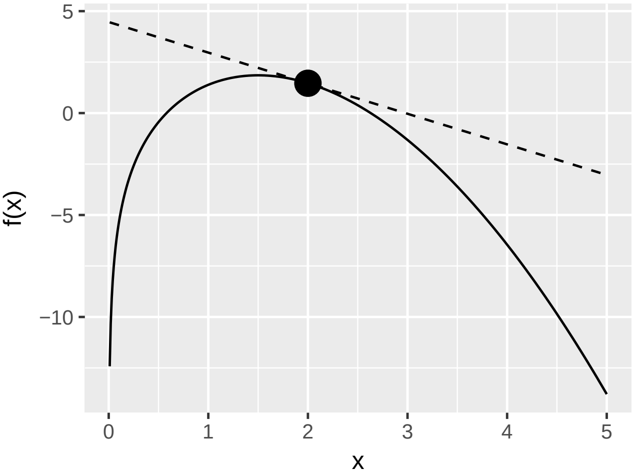



What is the derivative? **It is just the slope.**

# Calculus review: Derivatives



# Calculus review: A few derivative rules

$$\frac{\partial}{\partial x} x^a = a x^{a-1}$$

$$\frac{\partial}{\partial x} \log x = \frac{1}{x}$$

$$\frac{\partial}{\partial x} f(x) \text{ is often denoted } f'(x)$$

$$\frac{\partial}{\partial x} f(g[x]) = f'(g[x]) g'(x) \text{ (often called the chain rule)}$$

# Calculus review: Derivatives

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The derivative is

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# Calculus review: Derivatives

$$f(x) = x - x^2 + \log(x) + \log(3x^2)$$

The derivative is

$$\frac{\partial}{\partial x} f(x) = 1 + 2x + \frac{1}{x} + \frac{6x}{3x^2}$$

$$=$$

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The derivative is

$$\begin{aligned}\frac{\partial}{\partial x} f(x) &= 1 + 2x + \frac{1}{x} + \frac{6x}{3x^2} \\ &= 1 - 2x + \frac{1}{x} + \frac{2}{x} \\ \\ &= \end{aligned}$$

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Let's evaluate the derivative at $x = 2$

# Calculus review: Derivatives

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Let's evaluate the derivative at $x = 2$

$$f'(2) = 1 - 2 \times 2 + \frac{3}{2} = -1.5$$

# Calculus review: Maximizing a function

$$f(x) = x - x^2 + \log(x) + \log(3x^2)$$
$$f'(x) = 1 - 2x + \frac{3}{x}$$

How do we maximize this?

# Calculus review: Maximizing a function

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How do we maximize this?

Set the derivative equal to 0 and solve!

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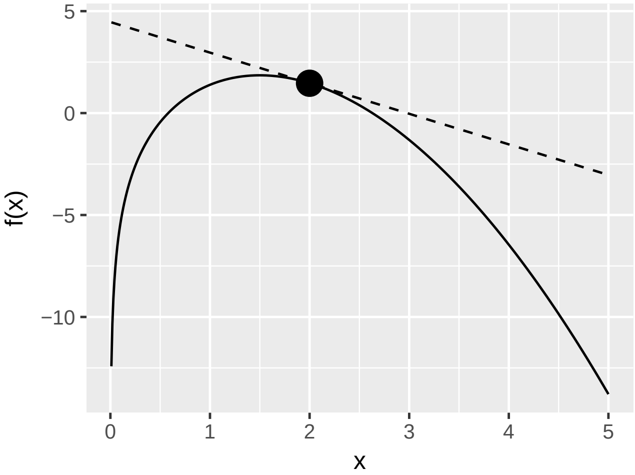
Set the derivative equal to 0 and solve!

(Then check that you find a maximum)

# Calculus review: Maximizing a function



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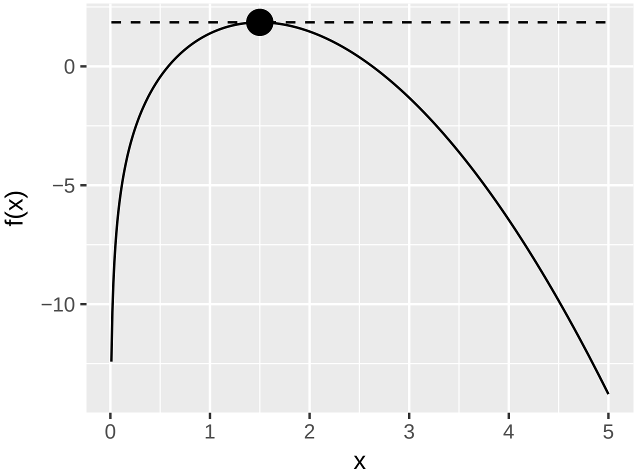

# Calculus review: Maximizing a function

Set the derivative equal to 0

$$f'(x^*) = 0$$
$$1 - 2x^* + \frac{3}{x^*} = 0$$

# Calculus review: Maximizing a function

Set the derivative equal to 0

$$
\begin{aligned}
f'(x^*) &= 0 \\
1 - 2x^* + \frac{3}{x^*} &= 0 \\
\frac{3}{x^*} &= 2x^* - 1
\end{aligned}
$$

# Calculus review: Maximizing a function

Set the derivative equal to 0

$$
\begin{aligned}
f'(x^*) &= 0 \\
1 - 2x^* + \frac{3}{x^*} &= 0 \\
\frac{3}{x^*} &= 2x^* - 1 \\
3 &= 2x^{*2} - x^*
\end{aligned}
$$

# Calculus review: Maximizing a function

Set the derivative equal to 0

$$
\begin{aligned}
f'(x^*) &= 0 \\
1 - 2x^* + \frac{3}{x^*} &= 0 \\
\frac{3}{x^*} &= 2x^* - 1 \\
3 &= 2x^{*2} - x^* \\
0 &= 2x^{*2} - x^* - 3
\end{aligned}
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# Calculus review: Maximizing a function

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3 &= 2x^{*2} - x^* \\
0 &= 2x^{*2} - x^* - 3 \\
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# Calculus review: Maximizing a function

Set the derivative equal to 0

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x^* &= \{-1, 1.5\}
\end{aligned}
$$

These are our **critical values**.

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The second derivative captures the curvature of the function.

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# Back to our example: Maximizing the log likelihood

We had

$$\ell(p \mid y_1, \ldots, y_n) = (\log p - \log[1-p]) \sum_{i=1}^{n} y_i + 5n \log(1-p)$$

How do we maximize this?

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&=
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p^* &= \frac{\sum_{i=1}^{n} y_i}{5n}
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This $p^*$ such that $\ell'(p \mid y) = 0$ is the **critical value**.

# Back to our example: Maximizing the log likelihood

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This $p^*$ such that $\ell'(p \mid y) = 0$ is the **critical value**. Is it a maximum?

# Back to our example: Maximizing the log likelihood

$$\frac{\partial^2}{\partial p^2}\ell(p \mid y) = \frac{\partial}{\partial p}\left(\frac{\sum_{i=1}^{n} y_i}{p} - \frac{5n - \sum_{i=1}^{n} y_i}{1-p}\right)$$

# Back to our example: Maximizing the log likelihood

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Since the first derivative is 0 and the second derivative is negative, the critical value $p^* = \frac{\sum_{i=1}^n y_i}{5n}$ is a maximum.

$$\hat{p}_{\text{MLE}} = \frac{\sum_{i=1}^n y_i}{5n}$$

# Plotting the log likelihood

$$\ell(p \mid y_1, \ldots, y_n) = (\log p - \log[1-p]) \sum_{i=1}^{n} y_i + 5n \log(1-p)$$

Let's define a function in R that returns the log likelihood given a vector `y`

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log.lik <- function(p) {
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```
log.lik <- function(p) {
     (log(p) - log(1 - p)) * sum(y) + 5 * length(y) * log(1 - p)
}
```

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Define some data and make a plot.

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y <- rbinom(100,5,.2)
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```

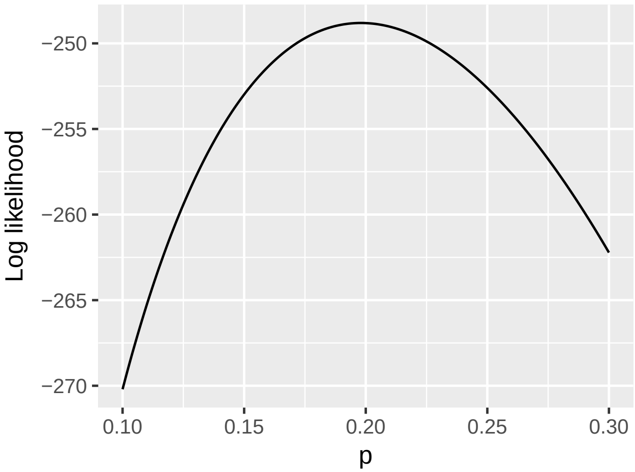
−250
−255
−260
−265
−270
Log likelihood
0.10
0.15
0.20
0.25
0.30
p

# Finding the maximum numerically

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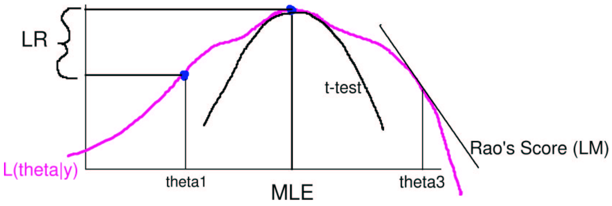
```
> y <- rbinom(100,5,.2)
> optimize(f = log.lik,
+          interval = c(0,1),
+          maximum = T)
$maximum
[1] 0.2020157

$objective
[1] -251.5813
```

The next 3 slides are exactly copied from lecture so we can discuss uncertainty.

# Uncertainty: Likelihood Ratios for nested models



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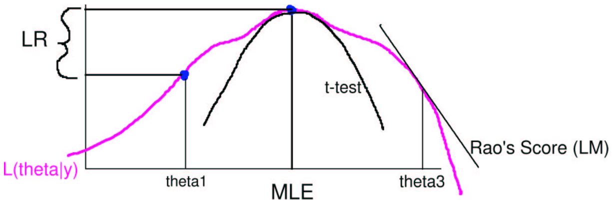


- $L^*$ is the likelihood value for the **unrestricted** model

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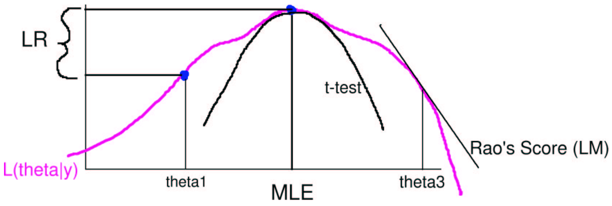


- $L^*$ is the likelihood value for the **unrestricted** model
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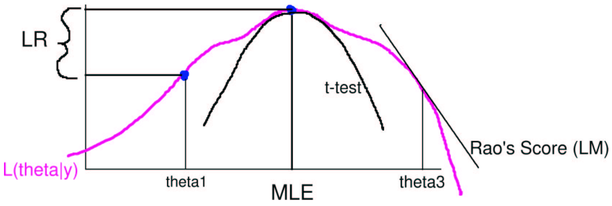


- $L^*$ is the likelihood value for the **unrestricted** model
- $L_R^*$ is the likelihood value for the (nested) **restricted** model
- $\implies L^* \geq L_R^* \implies \frac{L_R^*}{L^*} \leq 1$

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where $r$ is the observed value of $R$ and $m$ is the number of restricted parameters.

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From lecture slides

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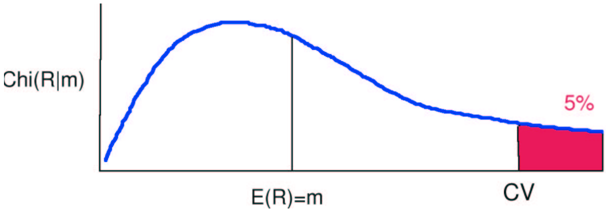

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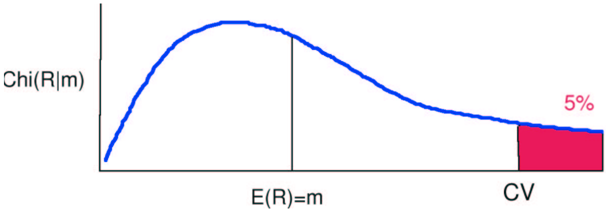


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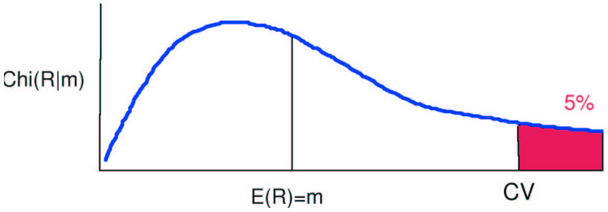


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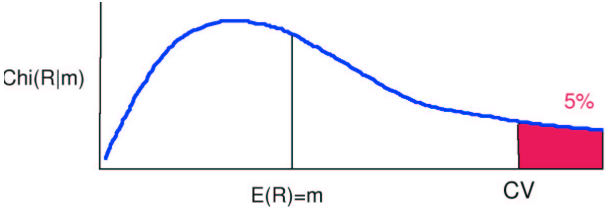
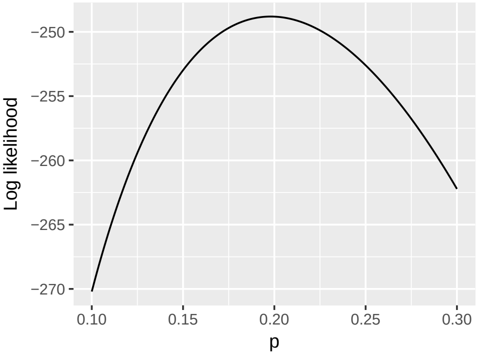



- If restrictions have no effect, $E(R) = m$.
- So only if $r >> m$ will the test parameters be clearly different from zero.
- Disadvantage: Too many likelihood ratio tests may be required to test all points of interest

# Uncertainty: Curvature at the maximum

Because of the logic of likelihood ratio tests, we can think of uncertainty as curvature around the MLE.



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The expectation is taken over the distribution of possible samples $X$. In practice, we often use the **observed fisher information** from our one sample as an estimate.

The variance is the inverse of the Fisher information:

$$V(\hat{\theta}_{\text{MLE}}) = \frac{1}{\mathcal{I}_n(\theta)}$$

Now, we can all practice the whole process on a different distribution: the **Poisson distribution**.

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Now, we can all practice the whole process on a different distribution: the **Poisson distribution**.

$$p(y \mid \lambda) = \frac{\lambda^k e^{-\lambda}}{k!}$$

The Poisson is a discrete distribution for count variables: its support is all nonnegative integers. You can learn more on Wikipedia!

# Remember the steps for likelihood inference!

1. Assume a data generating process.
2. Derive the likelihood.
3. Maximize the likelihood to get the MLE.
4. Derive standard errors from the inverse of the Fisher information

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$$\frac{\partial^2}{\partial \lambda^2} \ell(y_1, \ldots, y_n \mid \lambda) = -\frac{\sum_{i=1}^{n} y_i}{\lambda^2} < 0$$

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Keep it up!