Week 2: Random Variables

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Princeton

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[1] These slides are heavily influenced by Adam Glynn, Justin Grimmer and Jens Hainmueller. Many illustrations by Shay O'Brien.

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  - welcome and outline of course
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Questions?

Admin

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- Notation guide

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- Notation guide
- Using the slides (links, what's contained in a single deck etc.)

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- Any logistical hiccups?

1. Random Variables and Distributions
   - What is a Random Variable?
   - Discrete Distributions
   - Continuous Distributions
2. Characteristics of Distributions
   - Central Tendency
   - Measures of Dispersion
3. Conditional Distributions
4. Fun with Averages
5. Fun with Sensitive Questions
6. Appendix: Why the Mean?
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We will do this by introducing a random variable X to be Barack Obama's position on the 2008 New Hampshire primary ballot.

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- To do this we need to understand random variables

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- $X(\{heads, heads\}) = 2$
- $X(\{heads, tails\}) = 1$
- $X(\{tails, heads\}) = 1$
- $X(\{tails, tails\}) = 0$

A Visual Example

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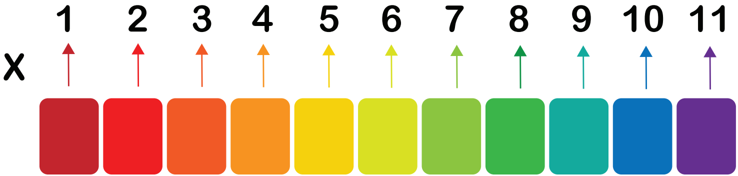

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- Sometimes the sample space is already numeric so its more obvious (e.g. how long until the train arrives)

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  yep. seriously. let's do an example!

NH Ballot Order Example

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
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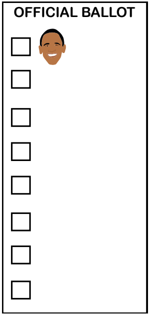
- Bill Richardson

$$X = \begin{cases} 1 \\ \\ \\ \end{cases}$$



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

X is a random variable indicating Obama's position on the ballot. Highlighted letters are those leading to a given ballot position. Highlighted individual is first.

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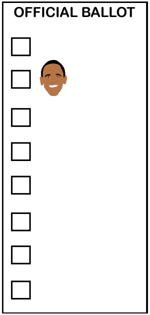

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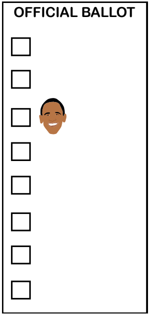
$$X = \begin{cases} 1 \\ 2 \\ 3 \end{cases}$$



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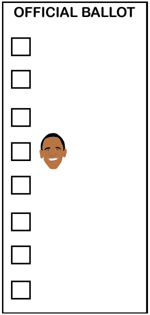
$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ \\ \\ \\ \end{cases}$$



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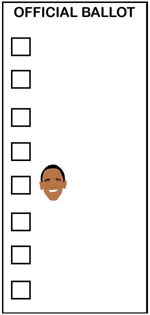
$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{cases}$$



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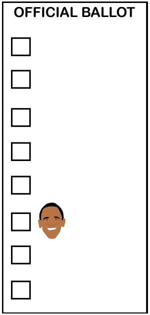
$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{cases}$$



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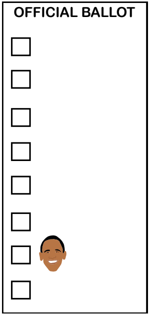
$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ \end{cases}$$



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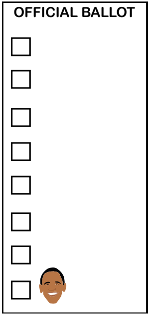

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- Probability mass functions provide a compact way to represent information about how likely various outcomes are.

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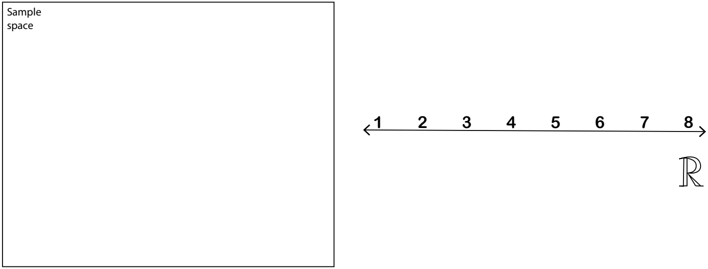

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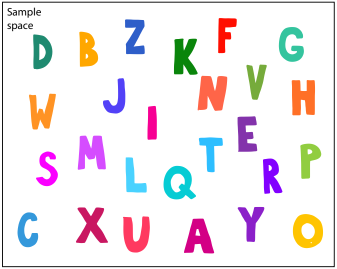

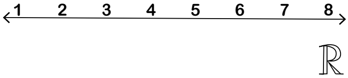

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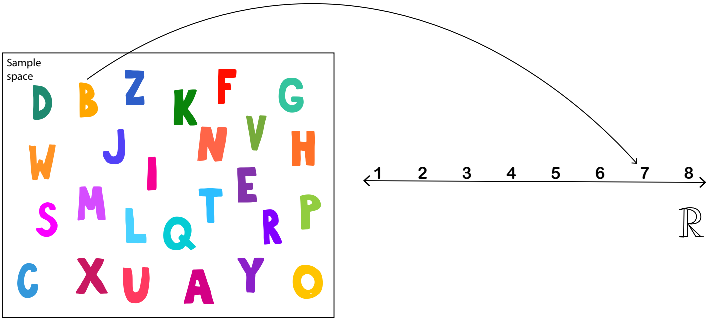

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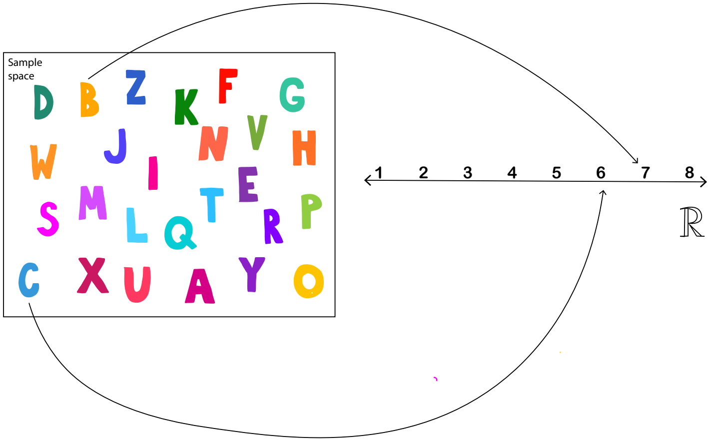

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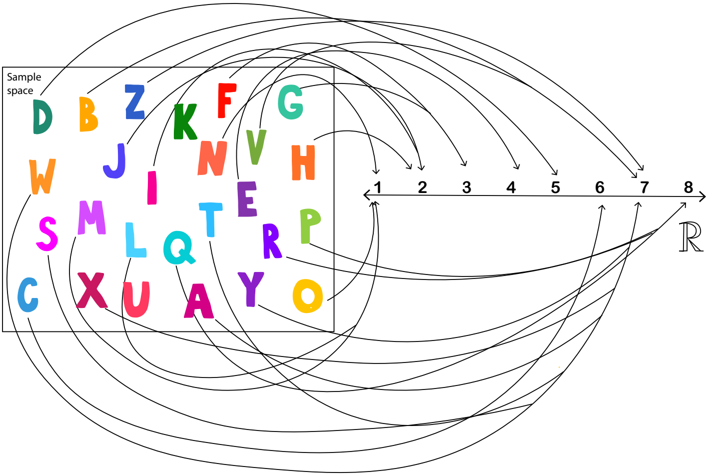

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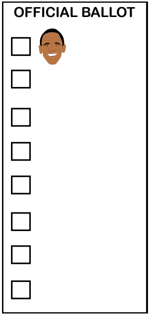
$$f(x) = \begin{cases} 4/26 & x = 1 \\ & \end{cases}$$



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Example: New Hampshire

Candidates:

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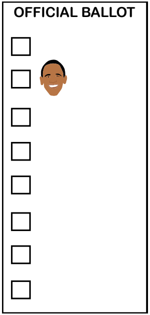

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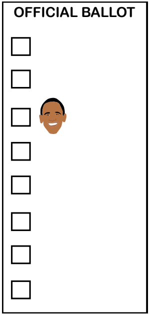

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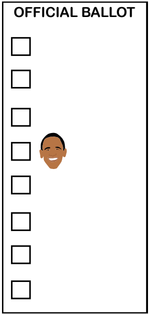

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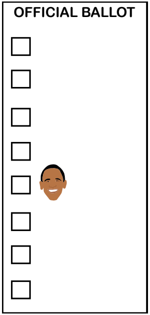

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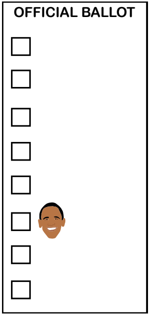

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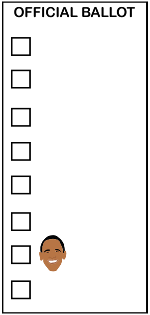

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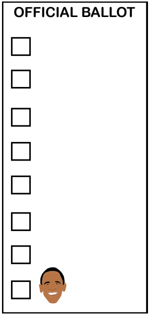

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Discrete Probability Mass Functions

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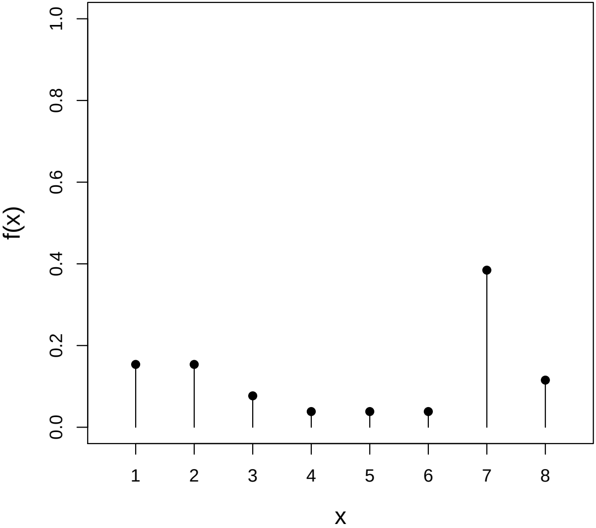
A probability mass function $f(x)$ of a random variable X is a non-negative function that gives the probability that $X = x$ and $\sum_x f(x) = 1$.

NH Obama Ballot Position PMF Plot



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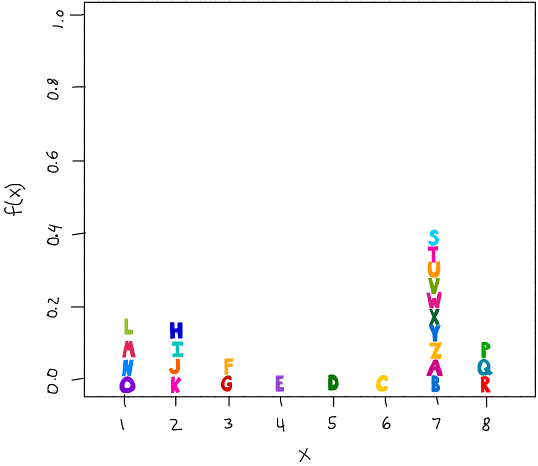
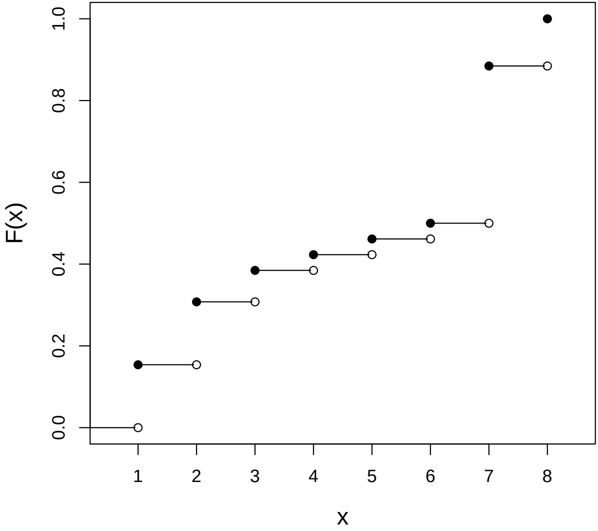


Discrete Cumulative Distribution Function

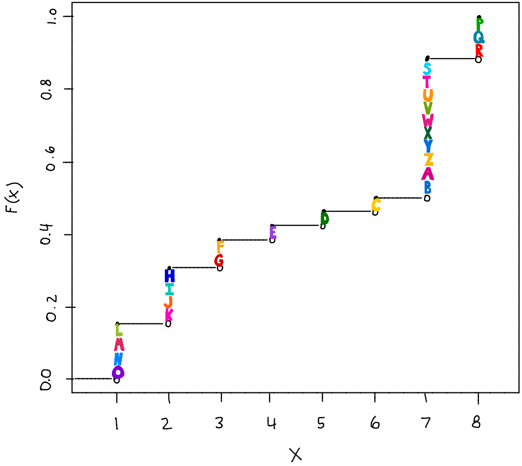
Discrete Cumulative Distribution Function

A cumulative distribution function $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$.

NH Obama Ballot Position CDF Plot



NH Obama Ballot Position CDF Plot



Some Important Discrete Distributions

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- We can summarize these distributions with one number (e.g. the probability of variables being 1)

Empirical Distributions

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An empirical mass function $\widehat{f}(x)$ of a variable X is a non-negative function that gives the frequency of the value x from data on X.

An empirical cumulative distribution function $\widehat{F}(x)$ of a variable X is a non-decreasing function that gives the frequency of values of X less than x.

Example: Assessing Racial Prejudice

- We often want to ask sensitive questions which a survey respondent is unlikely to honestly answer
- A list experiment asks respondents how many items on a list they agree with
  - for example, what proportion of people would be upset by a black family moving in next door to them (Kuklinski et al 1997).
  - randomly split survey into two halves
  - first half ask how many of the following items upset you:
    1. the federal government increasing the tax on gasoline
    2. professional athletes getting million-dollar salaries
    3. large corporations polluting the environment.
  - second half, add a fourth item
    4. a black family moving in next door
  - use the answers to infer the proportion upset by the fourth item.
- To do this we need to understand random variables

Racial Prejudice Example (Kuklinski et al, 1997)

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$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3
$f(x)$	?	?	?	?
$\widehat{f}(x)$	0.02	0.27	0.43	0.28
$\widehat{F}(x)$	0.02	0.29	0.72	1.00

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$\widehat{F}(x)$	0.02	0.29	0.72	1.00

$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4
$f(y)$	?	?	?	?	?
$\widehat{f}(y)$	0.02	0.20	0.40	0.28	0.10
$\widehat{F}(y)$	0.02	0.22	0.62	0.90	1.00

Continuous Distributions

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- A probability density function (pdf) and a cumulative distribution function (cdf) are two common ways to define the distribution for a continuous RV.

Example: Age in the Racial Prejudice Example

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Let X be the age of a randomly selected individual from the Kuklinski et al. (1997) data set.

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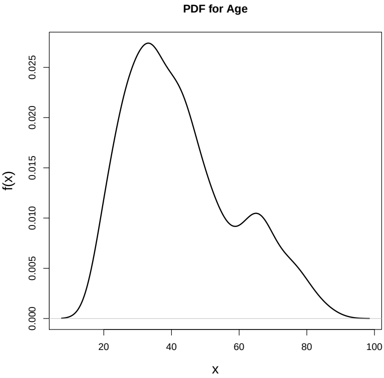
The probability distribution for this variable is well approximated by a probability density function.

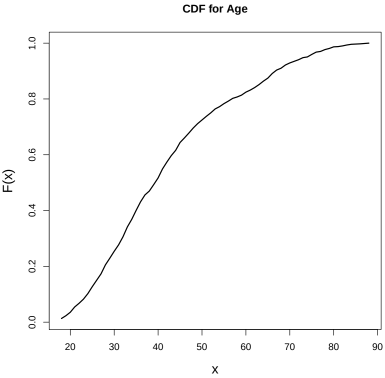


Continuous Cumulative Distribution Functions

Continuous Cumulative Distribution Functions

A cumulative distribution function $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$. For a continuous RV, the cdf is continuous.

$$F(x) = \int_{-\infty}^{x} f(z)dz$$



From PDFs to CDFs

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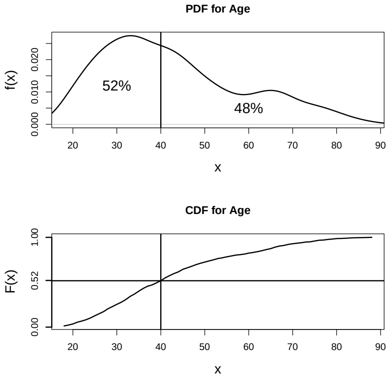
$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(z)dz$$

$$.52 = P(X \leq 40) = \int_{-\infty}^{40} f(z)dz$$



From CDFs to PDFs

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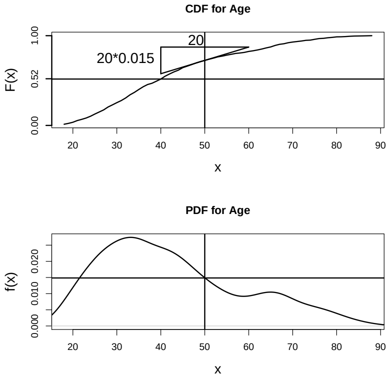
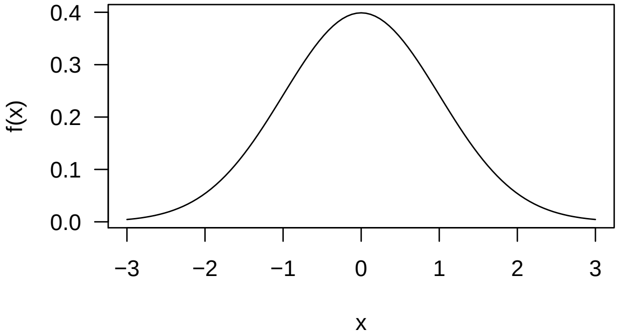
$$f(x) = \frac{dF(x)}{dx}$$

$$.015 = \frac{dF(50)}{dx}$$



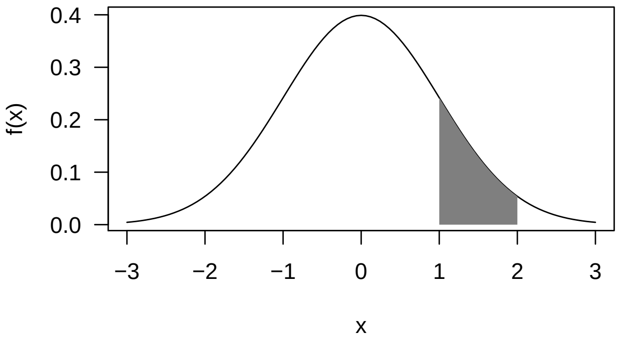
Subtleties of Continuous Densities

Remember- the height of the curve is not the probability of x occurring.



Subtleties of Continuous Densities

Remember- the height of the curve is not the probability of x occurring. To get the probability that X will fall in some region, you need the area under the curve.



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   - What is a Random Variable?
   - Discrete Distributions
   - Continuous Distributions
2. Characteristics of Distributions
   - Central Tendency
   - Measures of Dispersion
3. Conditional Distributions
4. Fun with Averages
5. Fun with Sensitive Questions
6. Appendix: Why the Mean?
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Expectation

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The expected value of a random variable X is denoted by $E[X]$ and is a measure of **central tendency** of X. Roughly speaking, an expected value is like a weighted average of all of the values weighted by probability of occurrence.

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What did we expect for Obama's NH position?

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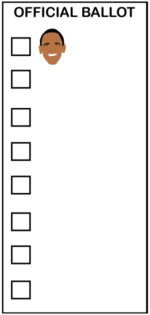
$4/26 \quad \times 1$

$+$



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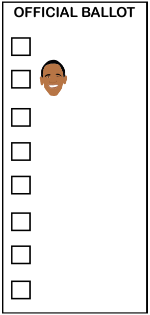

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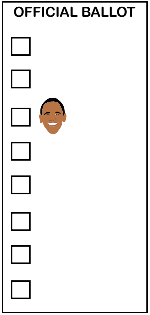

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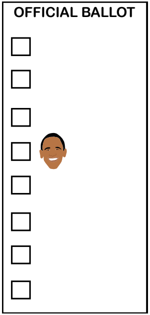

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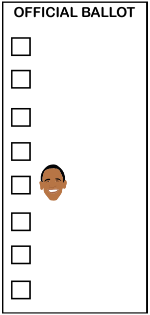

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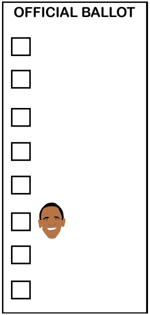

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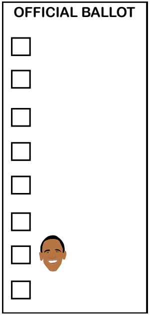

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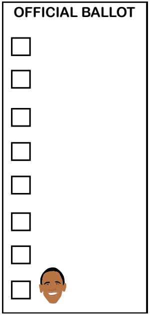

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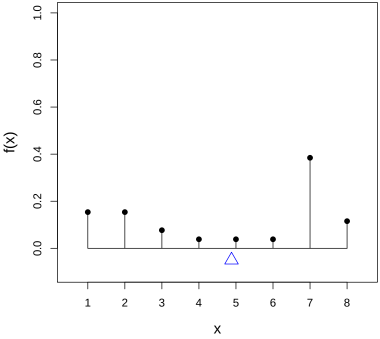
A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

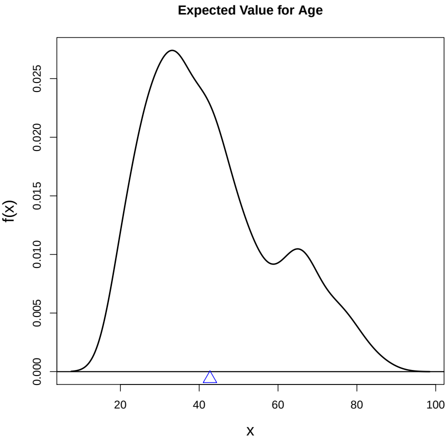
Interpreting Discrete Expected Value

The expected value for a discrete random variable is the balance point of the mass function.



Interpreting Continuous Expected Value

The expected value for a continuous random variable is the balance point of the density function.



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Appendix

Population Mean as an Expected Value

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Note how this resembles the definition of discrete expected value. If all values distinct (i.e. $x_i \neq x_j$ for all $i \neq j$).

$$\bar{x} = \sum_{\text{all } x_i} x_i f(x_i), \text{ where } f(x_i) = \frac{1}{N}$$

Property 1 of Expected Value: Additivity

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Together properties 1 and 2 are linearity (and this is sometimes presented as Linearity of Expectations).

Property 3 of Expected Value: LOTUS

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We will come back to this later. But it means that we can can calculate the expected value of $g(X)$ without explicitly knowing the distribution of $g(X)$!

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- $E[XY] \neq E[X]E[Y]$ unless X and Y are independent

Racial Prejudice Example

$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3	Sum
$\widehat{f}(x)$	0.02	0.27	0.43	0.28	1.00
$x \cdot \widehat{f}(x)$	0.00	0.27	0.86	0.84	1.97

$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4	Sum
$\widehat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$y \cdot \widehat{f}(y)$	0.00	0.20	0.80	0.84	0.40	2.24

Identifying the Percent Angry

Assume that $Y = X + A$, where for a randomly sampled respondent,

- $Y =$ the number of total angering items
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- What properties and assumptions were necessary?

Variance

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The variance is a special case of this, and the variance of a random variable X (a measure of its dispersion) is given by

$$V[X] = E[(X - E[X])^2]$$

It is the expectation of the squared distances from the mean.

For a discrete random variable X

$$V[X] = \sum_{\text{all } x} (x - E[X])^2 f_X(x)$$

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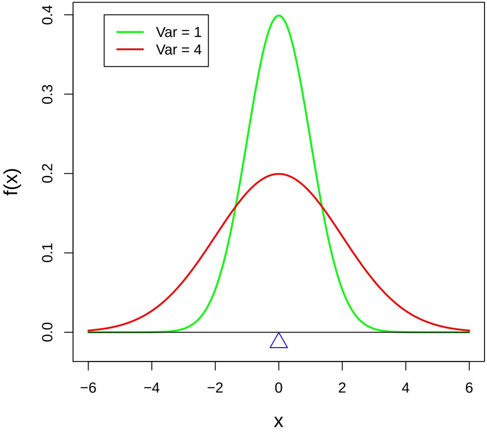
For a continuous random variable X

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

Variance Measures the Spread of a Distribution



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- The square root of the variance is the standard deviation.
- The variance and standard deviation have some useful properties.

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Suppose we have k independent random variables $X_1, \ldots, X_k$. If $V[X_i]$ exists for all $i = 1, \ldots, k$, then

$$V\left[\sum_{i=1}^{k} X_i\right] = V[X_1] + \cdots + V[X_k]$$

NB: Technically independence is sufficient but not necessary.

What was the variance of Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$4/26 \quad \times (1 - 4.88)^2$$

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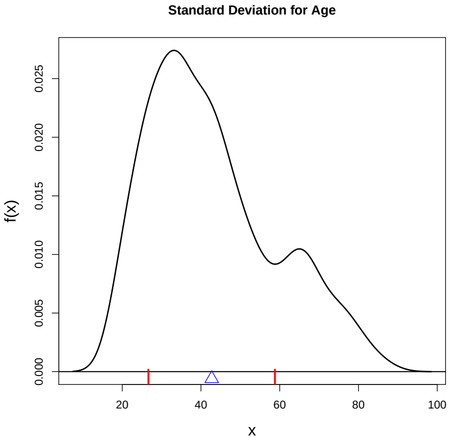
Does variance matter for fairness?

Interpreting Continuous Standard Deviation

The standard deviation for a continuous random variable is a measure of the spread of the pdf.



Do we lose anything when we use the list experiment?

$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4	Sum
$\widehat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$(y - 2.24)^2 \cdot \widehat{f}(y)$	0.15	0.31	0.02	0.16	0.31	0.95

More on this next week when we talk about estimator properties!

1. Random Variables and Distributions
   - What is a Random Variable?
   - Discrete Distributions
   - Continuous Distributions
2. Characteristics of Distributions
   - Central Tendency
   - Measures of Dispersion
3. Conditional Distributions
4. Fun with Averages
5. Fun with Sensitive Questions
6. Appendix: Why the Mean?
7. Joint Distributions
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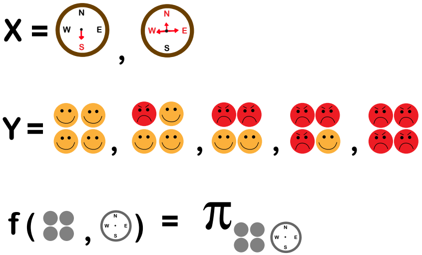

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$f(y,x)$	x		
y	0	1	$f(y)$
0	π_{00}	π_{01}	$\pi_{00} + \pi_{01}$
1	π_{10}	π_{11}	$\pi_{00} + \pi_{01}$
2	π_{20}	π_{21}	$\pi_{00} + \pi_{01}$
3	π_{30}	π_{31}	$\pi_{00} + \pi_{01}$
4	π_{40}	π_{41}	$\pi_{00} + \pi_{01}$
$f(x)$	$\sum_{y=0}^{4} \pi_{y0}$	$\sum_{y=0}^{4} \pi_{y1}$	

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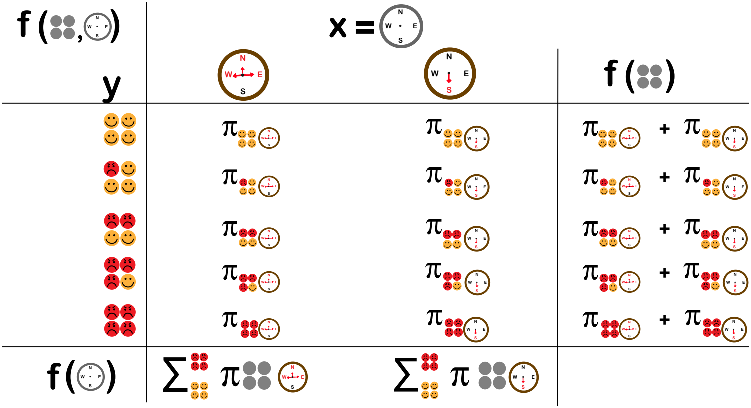

Discrete Conditional Distribution

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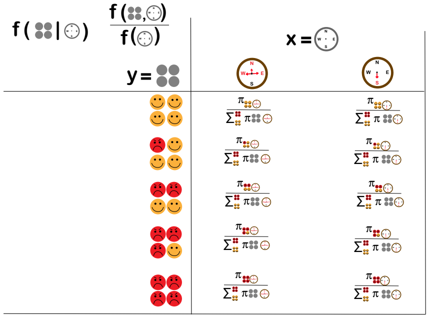

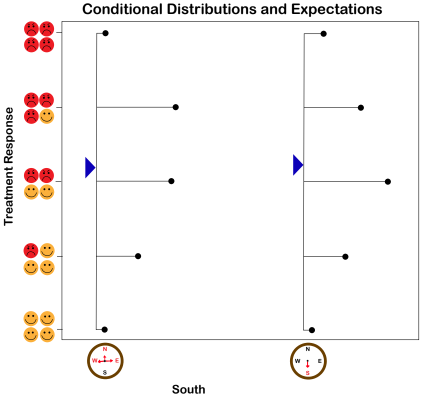

(More on conditional expectations on Wednesday)

Example: Conditional Distribution with "Continuous" Y

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Suppose we define $X =$ "number of angering items" and $Y =$ "age" for a randomly selected respondent receiving the treatment list.

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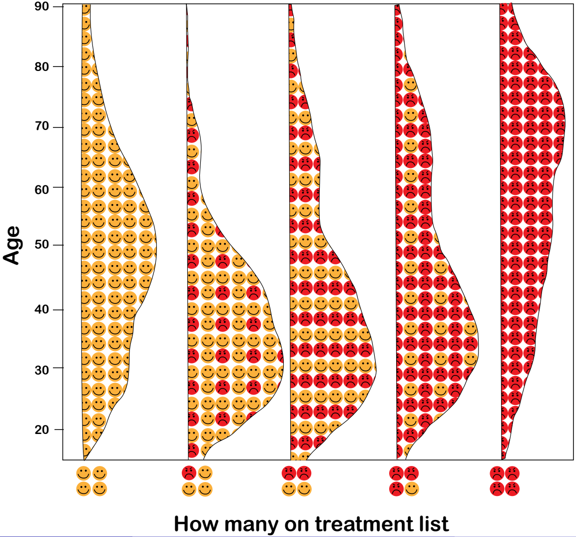

Conditional Expectation Function (CEF)

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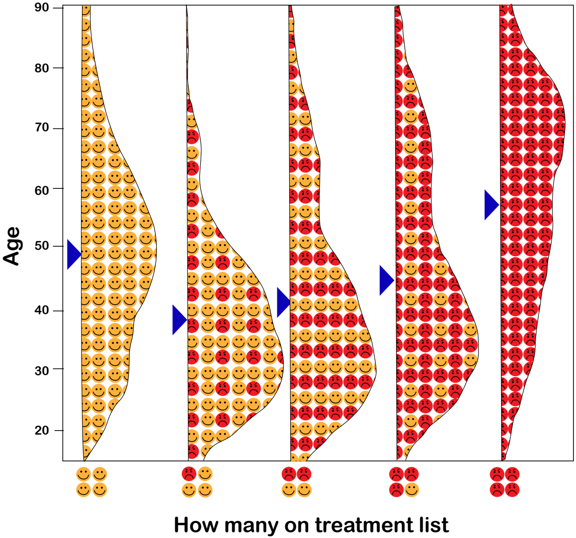
The conditional expectations form a CEF: $E[Y|X = x] = h(x)$



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Often we will assume that the CEF is linear: $E[Y|X = x] = \beta_0 + \beta_1 x$

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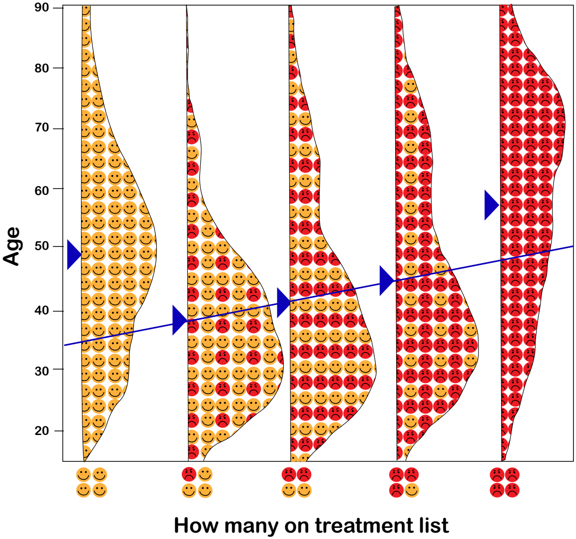
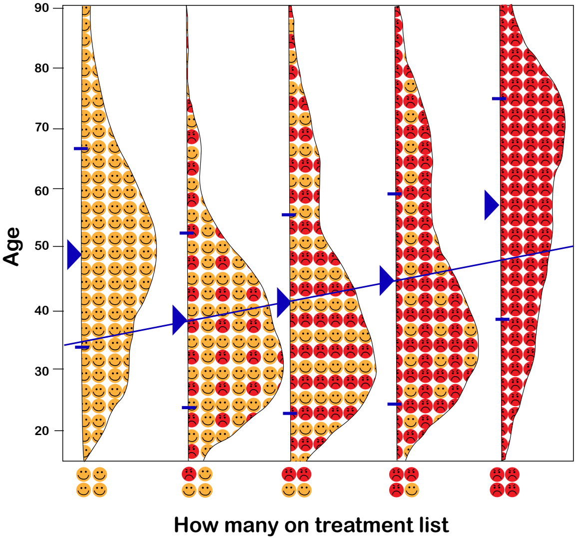


Conditional Variance and Standard Deviation

Conditional Variance and Standard Deviation

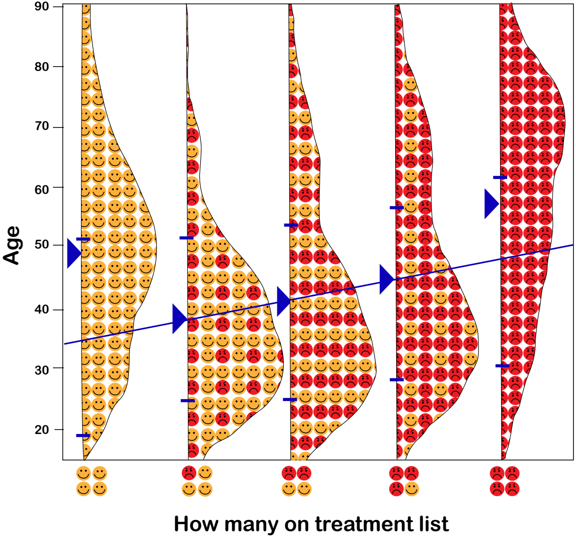
Similarly, we can assess the conditional standard deviation



Linear CEF and Constant Variance Assumptions

Linear CEF and Constant Variance Assumptions

Often, we assume that variance is the same for all values of x.



Preview: Interpreting the CEF

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Because the CEF is defined merely in terms of the larger population and not in terms of a causal effect (e.g., the causal effect of "number of angering items" on Age), we will utilize a descriptive interpretation of β_0 and β_1.

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- For this example, β_0 is the expected age for an individual that is angered by zero items
- β_1 is the expected difference in age between two individuals that have a one unit difference in the number of angering items.

Summary

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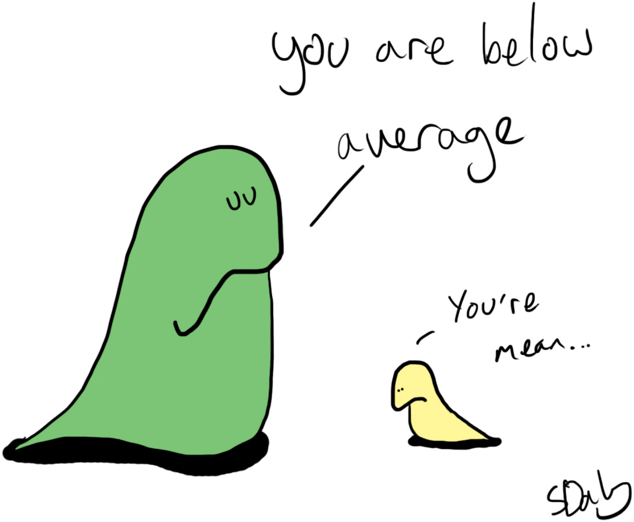
- Expected value and variance are two useful characteristics of the probability distributions associated with random variables.
- These concepts can be extended by conditioning on other variables.
- Next class we will cover joint distributions and conditional expectations in more depth.

Fun with

Fun with Averages

Fun with Averages

Central Tendency



The Story of Averages

Measurements

MESURES de la POITRINE.	NOMBRE d'hommes.	NOMBRE PROPORTIONNEL.	PROBABILITÉ d'après L'OBSERVATION.	RANG dans LA TABLE.	RANG d'après le CALCUL.	PROBABILITÉ d'après LA TABLE.	NOMBRE D'OBSERVATIONS calculé.
Pouces.							
33	3	5	0,5000			0,5000	7
34	18	31	0,4995	52	50	0,4993	29
35	81	141	0,4964	42,5	42,5	0,4964	110
36	185	322	0,4823	33,5	34,5	0,4854	323
37	420	732	0,4501	26,0	26,5	0,4531	732
38	749	1305	0,3769	18,0	18,5	0,3799	1333
39	1073	1867	0,2464	10,5	10,5	0,2466	1838
			0,0597	2,5	2,5	0,0628	
40	1079	1882	0,1285	5,5	5,5	0,1359	1987
41	934	1628	0,2913	13	13,5	0,3034	1675
42	658	1148	0,4061	21	21,5	0,4130	1096
43	370	645	0,4706	30	29,5	0,4690	560
44	92	160	0,4866	35	37,5	0,4911	221
45	50	87	0,4953	41	45,5	0,4980	69
46	21	38	0,4991	49,5	53,5	0,4996	16
47	4	7	0,4998	56	61,8	0,4999	3
48	1	2	0,5000			0,5000	1
	5738	1,0000					1,0000

Social Physics

Social Physics

The determination of the average man is not merely a matter of speculative curiosity; it may be of the most important service to the science of man and the social system. It ought necessarily to precede every other inquiry into social physics, since it is, as it were, the basis. The average man, indeed, is in a nation what the centre of gravity is in a body; it is by having that central point in view that we arrive at the apprehension of all the phenomena of equilibrium and motion
- Quetelet

The Military Takes to the Idea

The Problem with Averages

The Average Man

THE EDGE (HEALTH)

Stacking Up

Ever wonder how you measure up against the average American Joe? As a weight-training and fitness enthusiast, some of these stats may be a bit surprising...and pathetic.

69% Percentage of men who consider themselves "physically fit"

13% Percentage who actually are fit

$874 Amount spent on vitamins and supplements annually

45% Percentage of men who use their fitness equipment as a place to hang their clothes

52 Beats per minute of a fit man's resting heart rate

2.87 Hours per week of exercise

1.1 lbs. Amount of fat gained each year

1 Average number of pull-ups

77 yrs. Average life expectancy

23 Age when the average guy is in the best shape of his life

12:17 Average time to run a mile

1 Pounds of muscle the average sedentary guy loses each year

Size Yourself Up

Age (yrs.)	20s	30s	40s	50s	60s
Weight (lbs.)	168	179	182	185	184
Sit-ups (1 min.)	40	36	31	26	20
Push-ups (1 min.)	33	27	21	15	15
Bench Press (lbs.) (one time)	180	158	143	128	116

5'9.2" The average American male height

40" Size of the average guy's chest (inhaled) This is also the bodypart men spend the most time trying to develop

17.6% Average bodyfat

13" Size of the average guy's biceps (flexed)

34" Average waist size

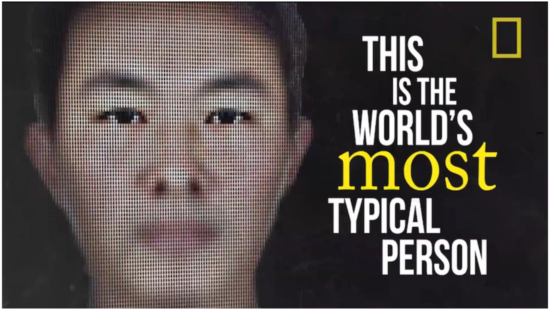
5.08 The average penis length in inches, according to *The Journal of Urology*

The *International Journal of Impotence Research* finds it slightly longer at 5.35 inches.

Sex Stats

- The average **sex session** lasts 3-10 minutes.
- The average **hotel porn-viewer** watches 12 minutes.
- The average **male orgasm** lasts six seconds.
- Number of men who would **rather work out than have sex**: 1 in 7

The Face of the Average Man



Fun with Sensitive Questions

Fun with Sensitive Questions

Graeme Blair
(slides that follow from Graeme)

Fun with Sensitive Questions

Cannot ask direct questions when there are **incentives to conceal sensitive responses**

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1. Social pressure

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Cannot ask direct questions when there are **incentives to conceal sensitive responses**

1. Social pressure
2. Physical retaliation
3. Legal jeopardy

How to Address Incentives to Conceal

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Develop trust with respondents, ask directly

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Survey experimental methods

1. **Endorsement experiment** Evaluation bias
2. **List experiment** Aggregation

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3. **Randomized response** Random noise

Bias in Direct Questions on Vote Buying

Estimated rate of vote buying from direct survey item

2.4%

Gonzalez-Ocantos et al. 2011, *AJPS*

Question text: "they gave you a gift or did you a favor"

Bias in Direct Questions on Vote Buying

Estimated rate of vote buying from direct survey item

2.4%

Estimate using list experiment

24.3%

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Question text: "they gave you a gift or did you a favor"

Survey

- Survey of 2,448 civilians in the Niger Delta

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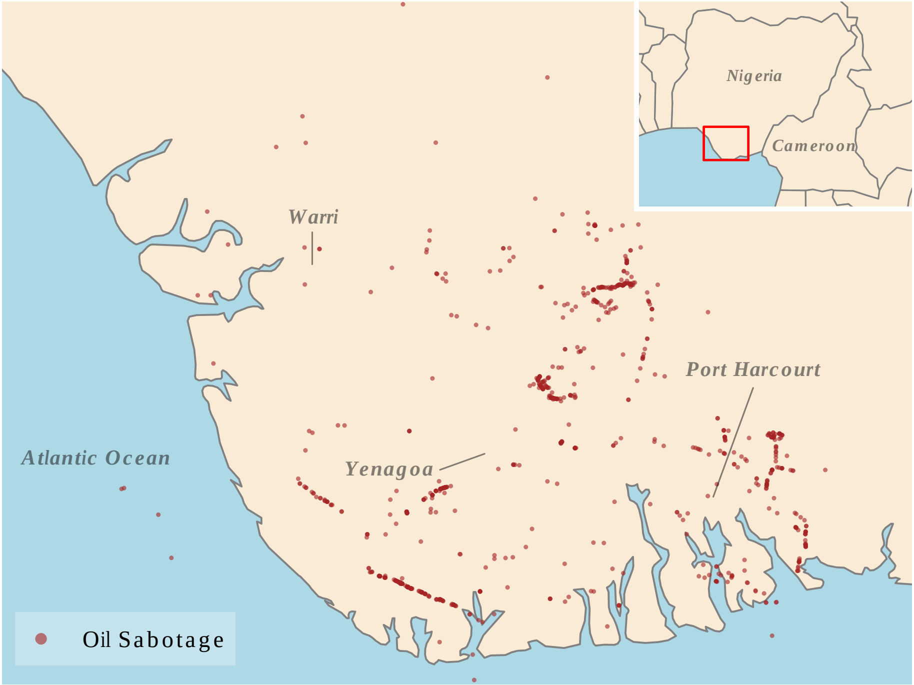
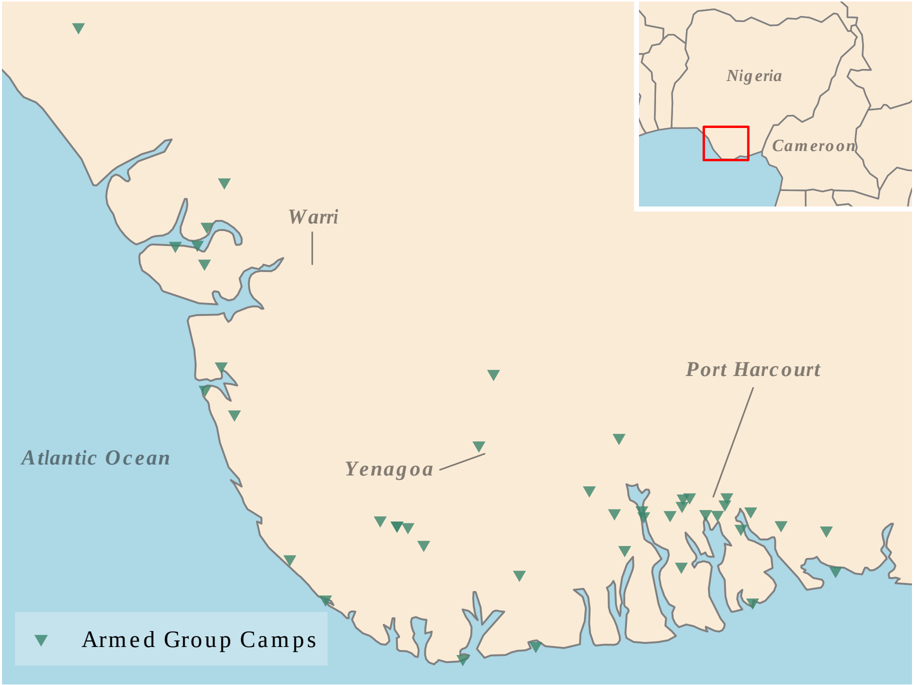
- Randomly sampled 204 communities near oil interruption sites and camps of armed groups


Nigeria
Cameroon
Warri
Port Harcourt
Atlantic Ocean
Yenagoa
Oil Sabotage


Nigeria
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Armed Group Camps

Survey

- Survey of 2,448 civilians in the Niger Delta
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Survey

- Survey of 2,448 civilians in the Niger Delta
- Random sample of 204 communities near and far from oil interruption sites and armed group camps
- Interviewed 12 people per community
  Random walk pattern to select households; Kish grid within household

Funded by the International Growth Centre

Outcome

"Did you share information with **militants** about their enemies in the community, state counterinsurgency forces, or oil facility activities?"

Problems with using list or endorsement experiments

Too sensitive for list experiment

Often difficult to define "control" condition in endorsement experiment for behaviors

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Alternative: **Randomized response technique**

Randomized response technique

How? Introducing random noise

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- Roll the dice in private

Randomized response technique

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- Roll the dice in private
- If you roll a 1, tell me "no"

Randomized response technique

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- Roll the dice in private
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Randomized response technique

How? Introducing random noise

- Roll the dice in private
- If you roll a 1, tell me "no"
- If you roll a 6, tell me "yes"
- Otherwise, answer: "Did you share information with armed groups"

Analysis of the randomized response technique

1. Used fair dice, and actually rolled it.

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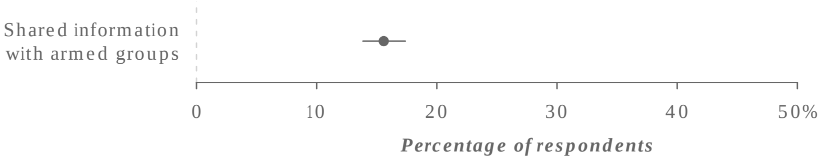
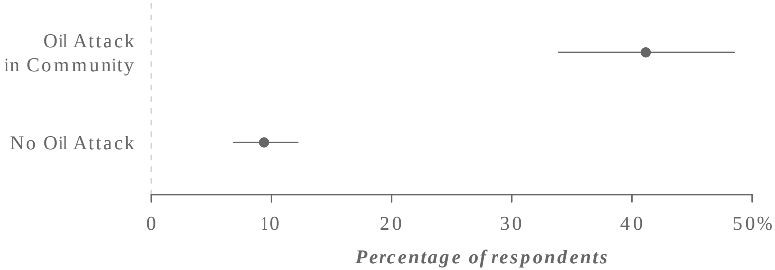
Proportion yes to sensitive item

$= 3/2 \cdot ($ Proportion answered yes $- 1/6)$

1. Civilians share information regularly with armed groups



2. Civilians near oil interruptions dominate collaboration



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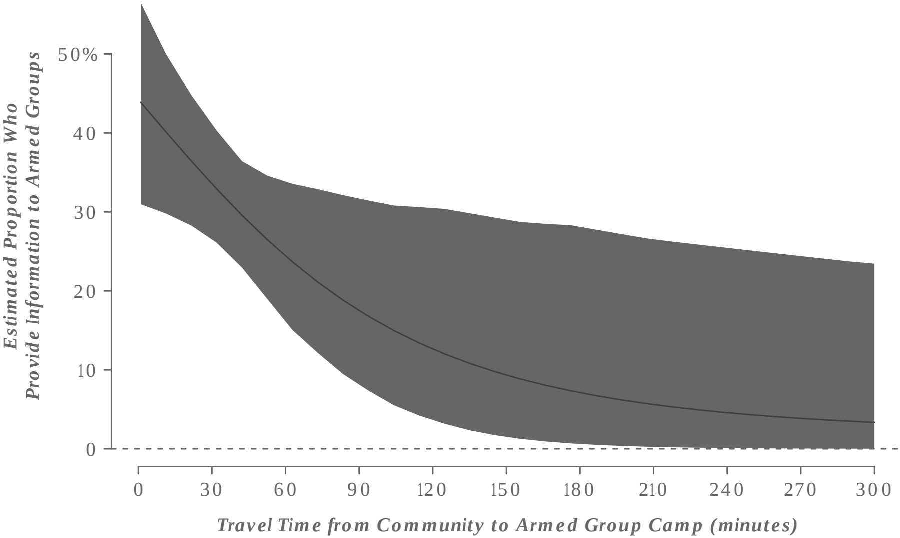

Software

- `rr` package in R for randomized response

  Blair with Yang-Yang Zhou and Kosuke Imai

- `list` package in R for list experiments

  Blair with Kosuke Imai

- `endorse` package in R for endorsement experiments

  Yuki Shiraito and Kosuke Imai

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Expected Value as Mean Square Error Minimizer

Back

Suppose we want to pick a single number (c) that summarizes a random variable X. What we mean by summarizes determines the best choice of c.

Generally speaking we want a summary that is in the "center" of the data, i.e. that is as close as possible to all possible datapoints. Again though, the choice turns on what we mean by close.
Let's say we want to minimize:

- Mean Squared Error: $E(X-c)^2$
  This leads to choosing the mean of X: μ
- Mean Absolute Error: $E[|X-c|]$
  This leads to choosing the median of X: m

Let's prove the first result (see Blitzstein and Hwang 2014 Theorem 6.1.4 on pg 245 for this proof and the proof on mean absolute error).

Proof of Mean as Mean Square Error Minimizer

Let X be a random variable with mean μ. We want to show that the value of c that minimizes the mean squared error $E(X-c)^2$ is the mean, μ (Blitzstein and Hwang Theorem 6.1.4).

We will prove the following identity below:

$$E(X-c)^2 = \text{Var}(X) + (\mu - c)^2 \tag{1}$$

We are trying to choose c to minimize this term. The choice cannot affect $\text{Var}(X)$. Setting $c = \mu$ sets $(\mu - c)^2 = 0$ and any other choice makes $(\mu - c)^2 > 0$. Therefore (assuming the identity holds), $c = \mu$ minimizes Eq 1.

Now to prove the identity:

$$\begin{aligned}
\text{Var}(X) &= \text{Var}(X - c) && \text{(Prop 1 of Variance)}\\
&= E(X-c)^2 - (E[X-c])^2 && \text{(Defn of Variance)}\\
&= E(X-c)^2 - (\mu - c)^2 && \text{(Linearity of Exp)}\\
\text{Var}(X) + (\mu - c)^2 &= E(X-c)^2
\end{aligned}$$

▸▸ Back

References

- Kuklinski et al. 1997 "Racial prejudice and attitudes toward affirmative action" *American Journal of Political Science*
- Glynn 2013 "What can we learn with statistical truth serum? Design and analysis of the list experiment"
- All the Blair papers above.

Where We've Been and Where We're Going...

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  - in regression we think about how the distribution of one variable changes under different values of another variable
  - e.g. does running more negative ads decrease election turnout?
- The joint distribution of two (or more) variables describes the pairs of observations that we are more or less likely to see.

Understanding Joint Distributions

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- Consider two r.v.s now, X and Y, each on the real line, $\mathbb{R}$.

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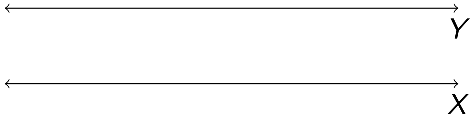

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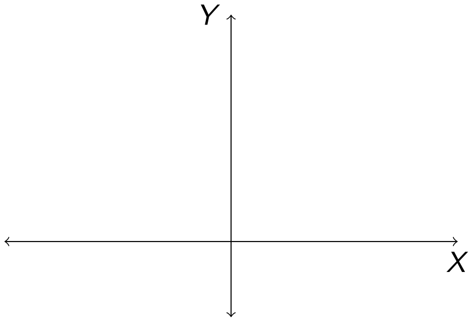

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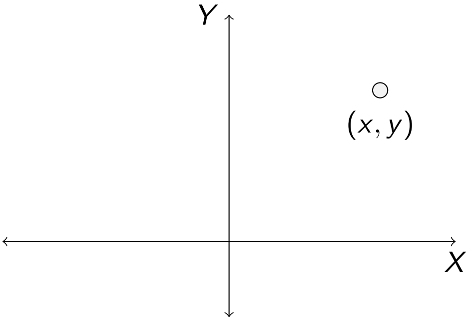
- One realization of the r.v. is a point in that space



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- Often, we are interested in two random variables that are qualitatively different:
  - Y (response, outcome, dependent variable, etc.)
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  - X (predictor, explanatory/independent variable, covariate, etc.)
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Joint Probability Mass Function

Definition

For two discrete random variables X and Y the joint PMF $P_{X,Y}(x, y)$ gives the probability that $X = x$ and $Y = y$ for all x and y:

$$P_{X,Y}(x, y) \; = \; \Pr(X = x \text{ and } Y = y)$$

Restrictions:

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- $P_{X,Y}(x,y) \geq 0$ and $\sum_x \sum_y P_{X,Y}(x,y) = 1$.

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Should the U.S. allow more immigrants to come and live here?

		X: Education			
		less HS	HS	College	BA
Y: Support	oppose	0.07	0.22	0.18	0.15
	neutral	0.02	0.06	0.05	0.05
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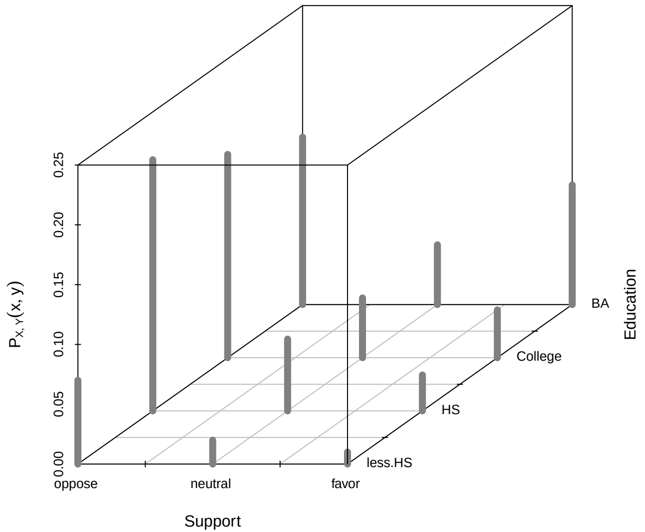
With discrete r.v.s this is very similar to thinking about a cross-tab, with frequencies/ probabilities in the cells instead of raw numbers.

Joint Probability Mass Function



From Joint to Marginal PMF

Given the joint PMF $P_{X,Y}(x,y)$ can we recover the marginal PMF $P_Y(y)$ (distribution over a single variable)?

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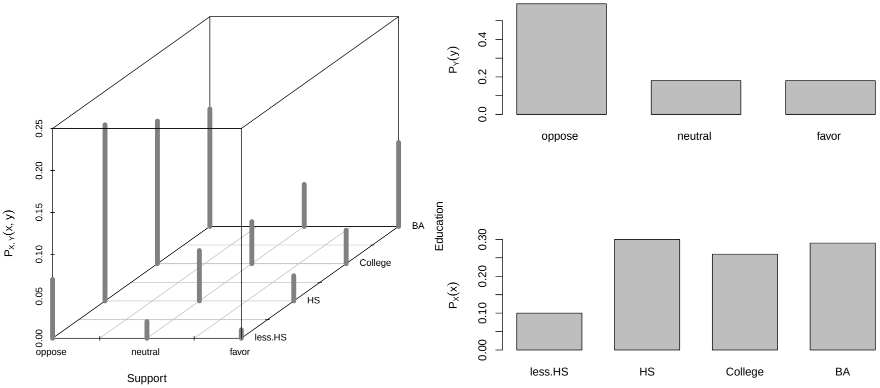
		X: Education				
		less HS	HS	College	BA	$P_Y(y)$
Y: Support	oppose	0.07	0.21	0.17	0.14	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.10	0.19

To obtain $P_Y(y)$ we marginalize the joint probability function $P_{X,Y}(x,y)$ over X:

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \sum_x \Pr(X = x, Y = y)$$

Joint and Marginal Probability Mass Functions



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Marginalizing over y to get $p(x)$ is then,

$$p(x_j) = \sum_{i=1}^{N} p(x_j|y_i)p(y_i)$$

A Table

	$Y = 0$	$Y = 1$	
$X = 0$	$p(0,0)$	$p(0, 1)$	$p_X(0)$
$X = 1$	$p(1,0)$	$p(1,1)$	$p_X(1)$
	$p_Y\ (0)$	$p_Y\ (1)$	

A Table

	Y = 0	Y = 1	
X = 0	0.01	0.05	?
X = 1	0.25	0.69	?
	0.26	0.74	

$$
\begin{aligned}
p_X(0) &= p(0|y=0)p(y=0) + p(0|y=1)p(y=1) \\
&= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\
&= 0.06
\end{aligned}
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p_X(1) &= p(1|y=0)p(y=0) + p(1|y=1)p(y=1) \\
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&= 0.94
\end{aligned}
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The conditional PMF of Y given X, $P_{Y|X}(y|x)$, is the PMF of Y when X is known to be at a particular value $X = x$:

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Key relationships:

- $P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x)$ (multiplicative rule)
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Conditional distributions are key in statistical modeling because they inform us how the distribution of Y varies across different levels of X.

From Joint to Conditional: $P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$

Table: Joint PMF $P_{X,Y}(x,y)$ and Marginal PMFs $P_X(x), P_Y(y)$

		Education				
	$P_{X,Y}(x,y)$	less HS	HS	College	BA	$P_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
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	$P_X(x)$	0.11	0.32	0.27	0.31	1.00

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Table: Conditional PMF $P_{Y|X}(y|x)$

		Education				
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Support	oppose	0.70	0.70	0.65	0.48	0.62
	neutral	0.20	0.20	0.19	0.17	0.19
	favor	0.10	0.10	0.15	0.34	0.19
		1.00	1.00	1.00	1.00	1.00

Joint and Conditional Probability Mass Functions

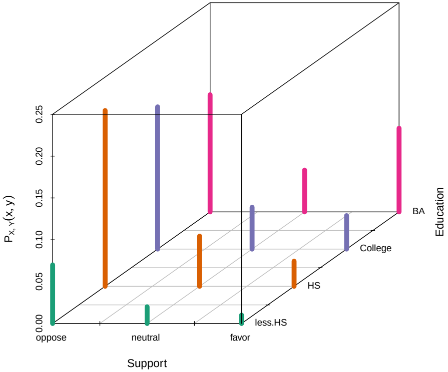

Figure: Joint

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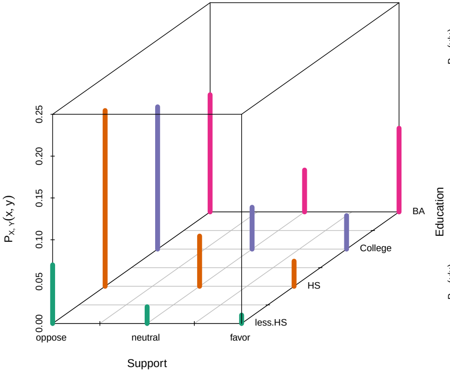

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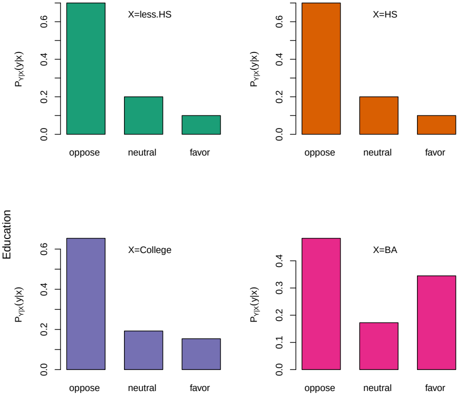

Figure: Conditional

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The multiplicative rule:

$$f_{X,Y}(x, y) = f_{Y|X}(y|x) f_X(x)$$

where

- $f_{Y|X}(y|x)$: Conditional PDF of Y given $X = x$
- $f_X(x)$: Marginal PDF of X

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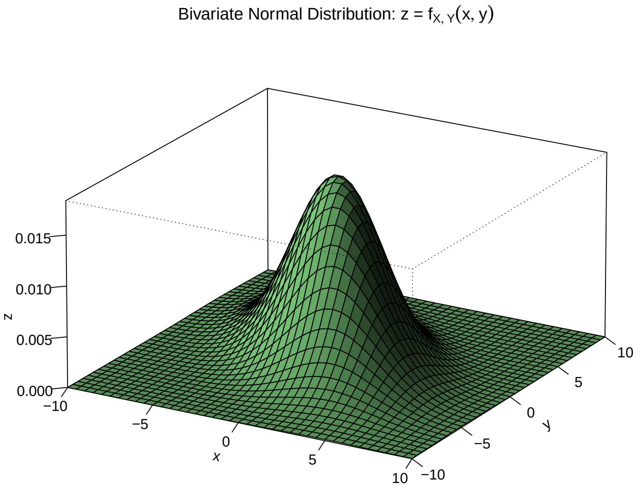
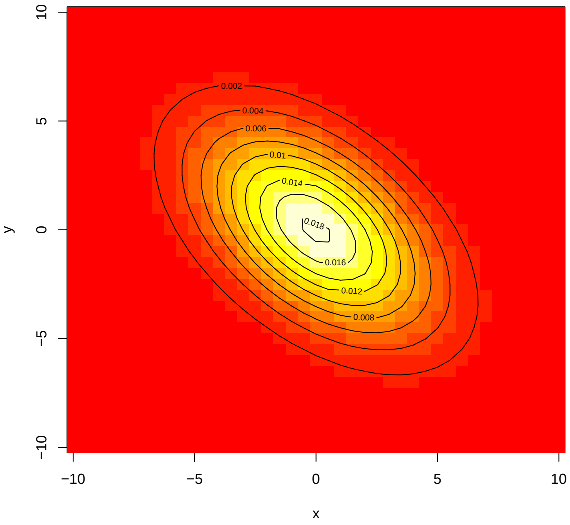
Restrictions:

- $\int_x \int_y f_{X,Y}(x,y)\, dy\, dx = 1$

3D Plot of a Joint Probability Density Function



Contour Plot of a Joint Probability Density Function



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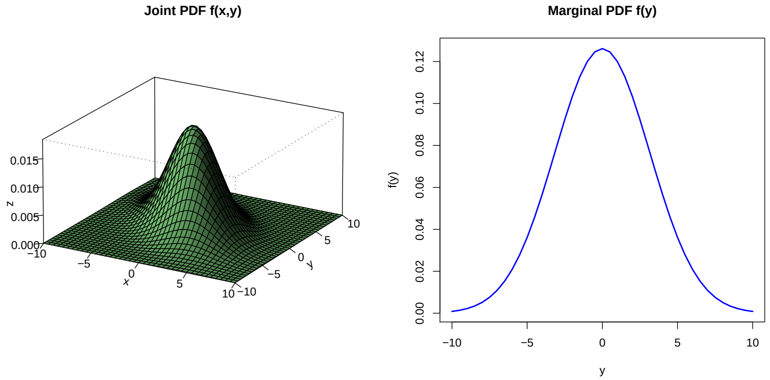
We marginalize the joint probability function $f_{X,Y}(x,y)$ over X:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$



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- Goal in statistical modeling is often to characterize the conditional distribution of the outcome variable $f_{Y|X}(y|x)$ across different levels of $X = x$.
- Typically, we summarize the conditional distributions with a few parameters such as the conditional mean of $E[Y|X = x]$ and the conditional variance $V[Y|X = x]$
- Moreover, we are often interested in estimating $E[Y|X]$, i.e. the conditional expectation function that describes how the conditional mean of Y varies across all possible values of X (we sometimes call this the population regression function)

Conditional Expectation

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

$$E[Y|X=x] = \sum_y y \Pr(Y=y|X=x) = \sum_y y\, P_{Y|X}(y|x)$$

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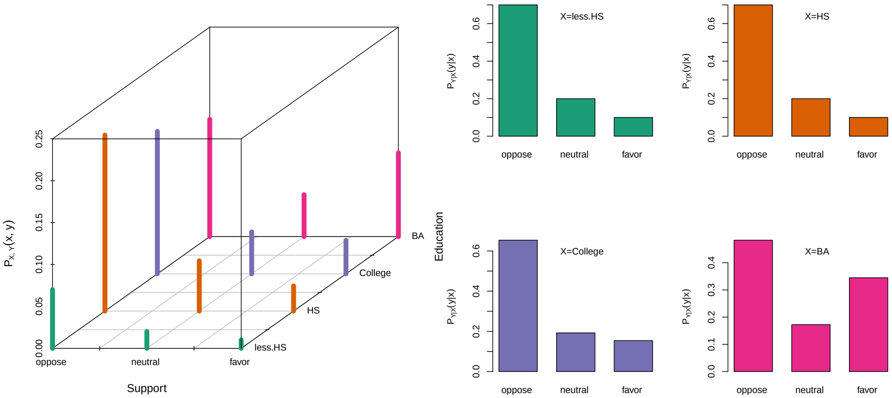
Definition (Conditional Expectation (Continuous))

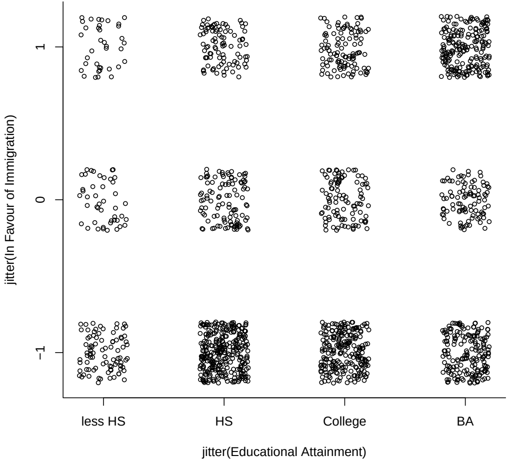
Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) dy$$

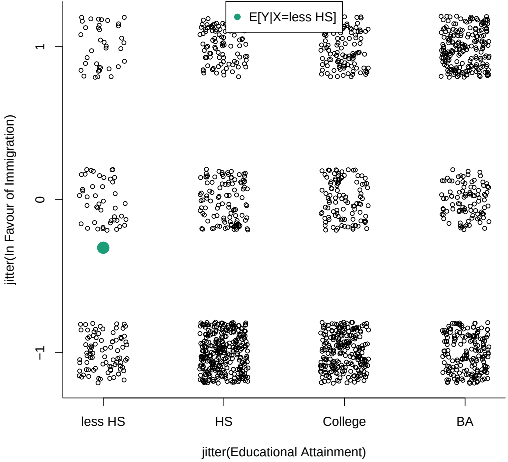
Joint and Conditional Probability Mass Functions



Conditional PMF $P_{Y|X}(y|x)$



Conditional Expectation $E[Y|X = 1]$



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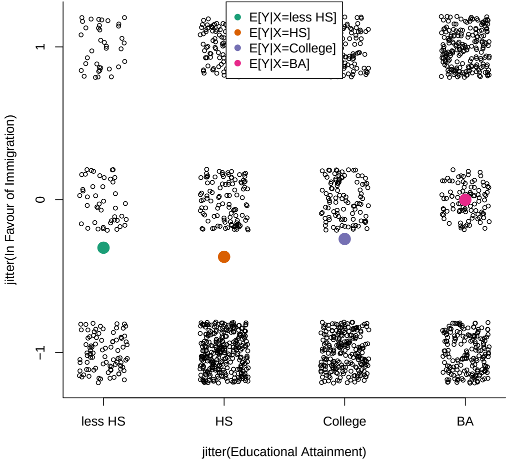

Law of Iterated Expectations

Theorem (Law of Iterated Expectations)

For two random variables X and Y,

$$E[Y] = E[E[Y|X]] = \begin{cases} \displaystyle\sum_{\text{all } x} E[Y|X=x] \cdot P_X(x) & (\text{discrete } X) \\ \displaystyle\int_{-\infty}^{\infty} E[Y|X=x] \cdot f_X(x)dx & (\text{continuous } X) \end{cases}$$

Note that the outer expectation is taken with respect to the distribution of X.

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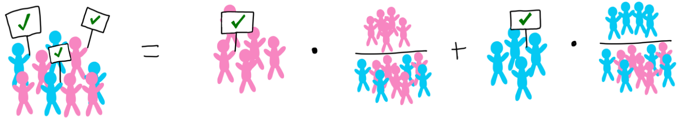
Note that the outer expectation is taken with respect to the distribution of X.

Example: Y (support) and $X \in \{1, 0\}$ (gender). Then, the LIE tells us:

$$\underbrace{E[Y]}_{\text{Average Support}} = E[E[Y|X]]$$

$$= \underbrace{E[Y|X=1]}_{\text{Average Support}|\text{Woman}} \cdot \underbrace{P_X(1)}_{\text{Pr(Woman)}} + \underbrace{E[Y|X=0]}_{\text{Average Support}|\text{Man}} \cdot \underbrace{P_X(0)}_{\text{Pr(Man)}}$$



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   - Basically, any function of X is a constant with regard to the conditional expectation. If we know X, then we also know X^2, for instance.
2. If $E[Y^2] < \infty$ and $E[g(X)^2] < \infty$ for some function g, then $E[(Y - E[Y|X])^2|X] \leq E[(Y - g(X))^2|X]$ and $E[(Y - E[Y|X])^2] \leq E[(Y - g(X))^2]$

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The second property is quite important. It says that the conditional expectation is the function of X that minimizes the squared prediction error for Y across any possible function of X.

Conditional Variance

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Remember, the conditional distribution of $Y|X$ is basically like any other probability distribution, so we are going to want to summarize the center *and* spread.

Conditional Variance

Definition

The conditional variance of Y given $X = x$ is defined as:

$$V[Y|X=x] = \begin{cases} \sum_{\text{all } y} (y - E[Y|X=x])^2 P_{Y|X}(y|x) & \text{(discrete } Y\text{)} \\ \int_{-\infty}^{\infty} (y - E[Y|X=x])^2 f_{Y|X}(y|x) dy & \text{(continuous } Y\text{)} \end{cases}$$

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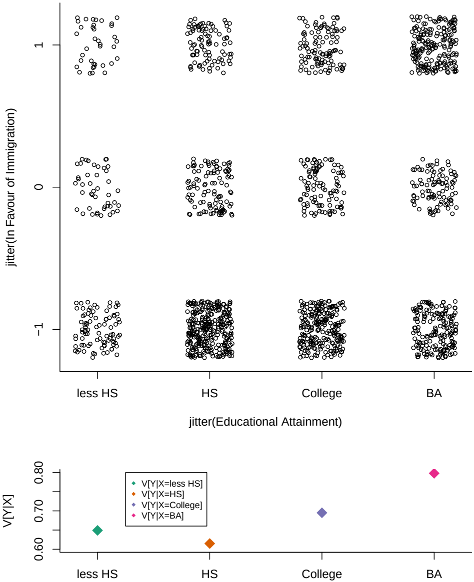
Example: Y (support) and $X \in \{1, 0\}$ (gender). The LTV says that the total variance in support can be decomposed into two parts:

1. On average, how much support varies within gender groups (within variance)
2. How much average support varies between gender groups (between variance)

Conditional Variance Function $V[Y|X]$



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Let's look at this in pictures.
(If you want to know more: Blitzstein and Hwang pg 392-393 is great.)

Important Subtleties in Pictures

Sample space

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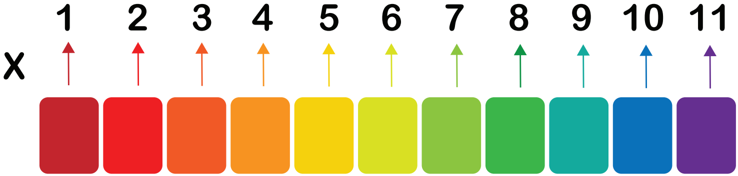
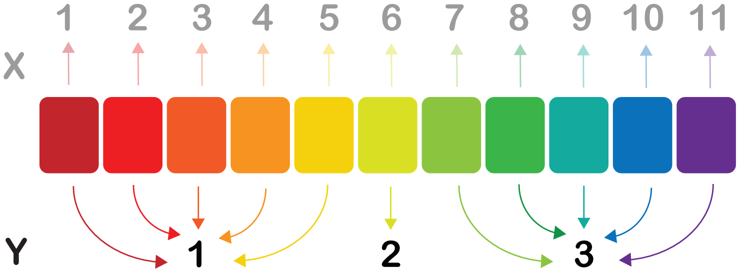


Random variable

Important Subtleties in Pictures



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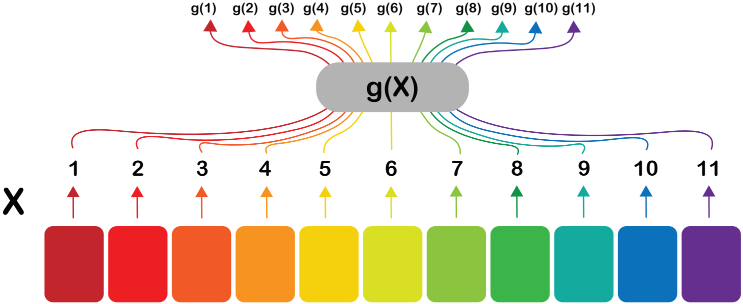

Function of a random variable is a random variable

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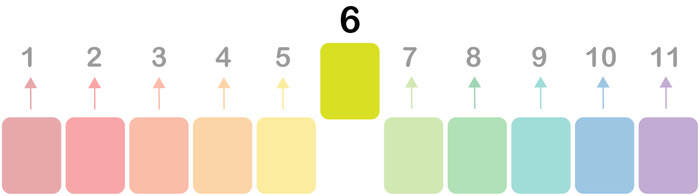
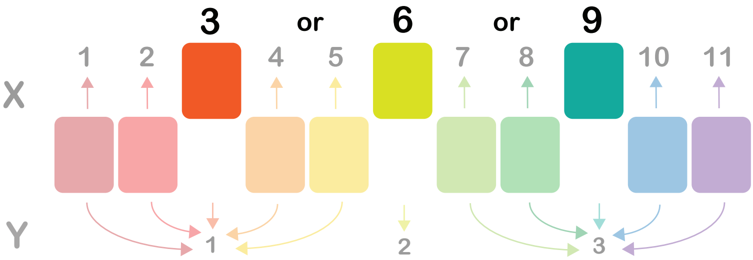


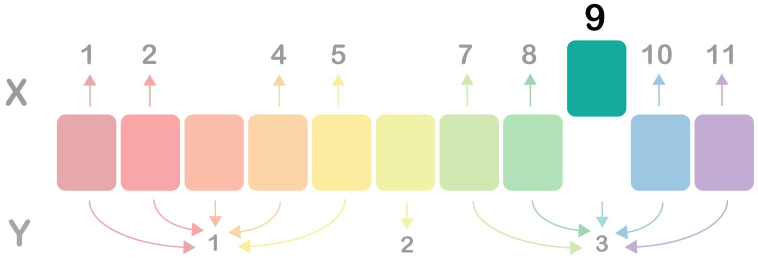
$E[X]$

Important Subtleties in Pictures



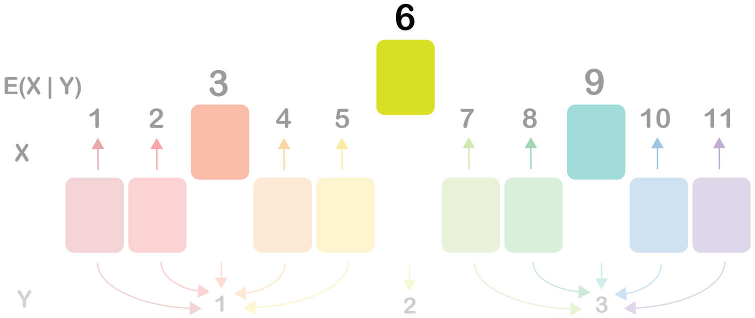
$E[X|Y]$

Important Subtleties in Pictures



$$E[X|Y=3]$$

Important Subtleties in Pictures



$$E[E[X|Y]] = E[X]$$

1. Random Variables and Distributions
   - What is a Random Variable?
   - Discrete Distributions
   - Continuous Distributions
2. Characteristics of Distributions
   - Central Tendency
   - Measures of Dispersion
3. Conditional Distributions
4. Fun with Averages
5. Fun with Sensitive Questions
6. Appendix: Why the Mean?
7. Joint Distributions
   - Discrete Random Variable
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8. Conditional Expectation
9. Properties
   - Independence
   - Covariance and Correlation
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10. Famous Distributions
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Independence

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all x and y. We write this as $Y \perp\!\!\!\perp X$.

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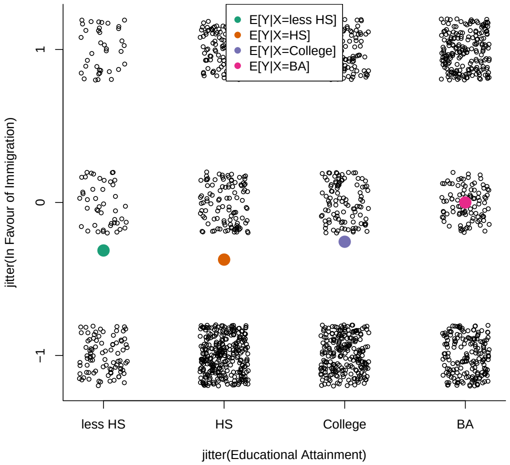
Is $Y \perp\!\!\!\perp X$?



Expected Values with Independent Random Variables

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We can prove the continuous case by following the same steps, with $\sum$ replaced by $\int$.

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- If $Cov[X, Y] > 0$, observing an X value greater than $E[X]$ makes it more likely to also observe a Y value greater than $E[Y]$, and vice versa.

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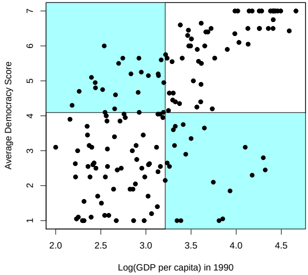
  - Points in upper right and lower left quadrants (relative to the means) add to the covariance.
  - Points in the upper left and lower right quadrants subtract from the covariance.



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$$\begin{aligned} Cov[X, Y] &= E[XX^2] - E[X]E[X^2] = E[X^3] - E[X]E[X^2] \\ &= E[X] - E[X]E[X^2] = 0 - 0 \cdot E[X^2] = 0. \end{aligned}$$

Covariance and Independence

Does $X \perp\!\!\!\perp Y$ imply $Cov[X, Y] = 0$? Yes!

Proof:

$$\begin{aligned} Cov[X, Y] &= E[XY] - E[X]E[Y] \\ &= E[X]E[Y] - E[X]E[Y] \quad \text{(independence)} \\ &= 0. \end{aligned}$$

Does $Cov[X, Y] = 0$ imply $X \perp\!\!\!\perp Y$? No!

Counterexample: Suppose $X \in \{-1, 0, 1\}$ with $P_X(x) = 1/3$ and $Y = X^2$.

Is $X \perp\!\!\!\perp Y$? No, because $P_{Y|X}(y \mid x) \neq P_Y(y)$
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Therefore, $X \perp\!\!\!\perp Y \implies Cov[X, Y] = 0$, but not vice versa.

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1. For random variables X and Y and constants a, b and c,

$$V[aX + bY + c] = a^2 V[X] + b^2 V[Y] + 2ab\, Cov[X, Y]$$

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Proof: Plug in to the definition of variance and expand (try it yourself!)

Correlation

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Definition (Correlation)

The correlation between two random variables X and Y is defined as

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- Always satisfies: $-1 \leq Cor[X, Y] \leq 1$.

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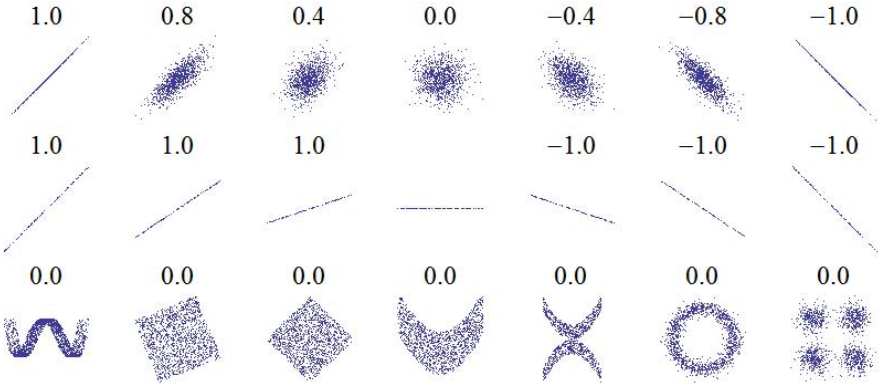

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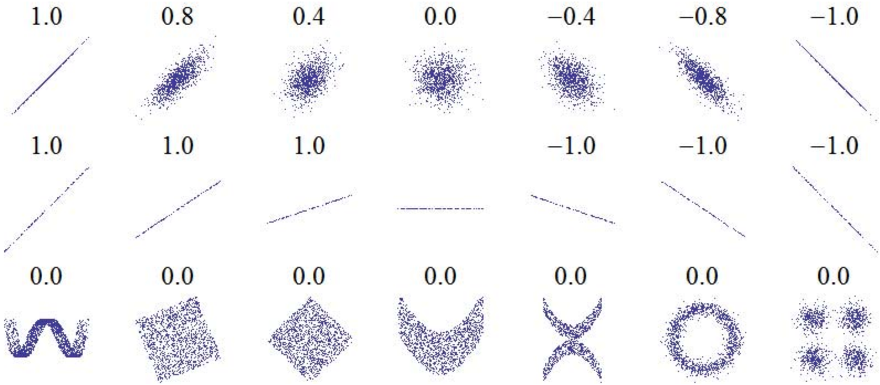

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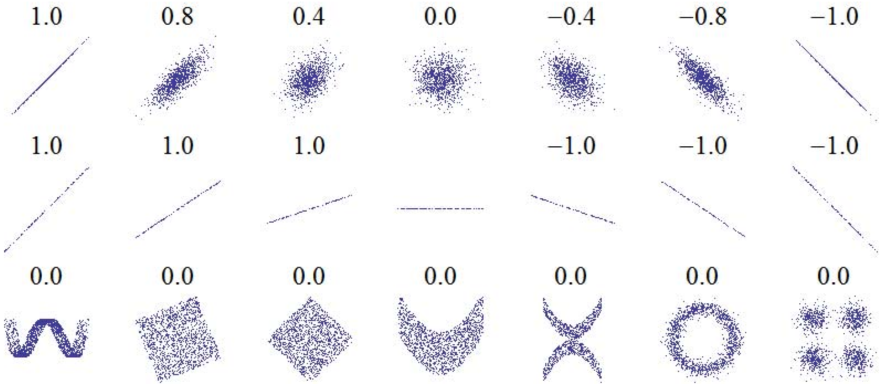

- $Cor[X, Y] = \pm 1$ iff $Y = aX + b$ where $a \neq 0$.
- Like covariance, correlation measures the linear association between X and Y.

Conditional Independence

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Definition (Conditional Independence of Random Variables)

Random variables Y and X are conditionally independent given Z iff

$$f_{X,Y|Z}(x,y|z) \;=\; f_{Y|Z}(y|z) \cdot f_{X|Z}(x|z)$$

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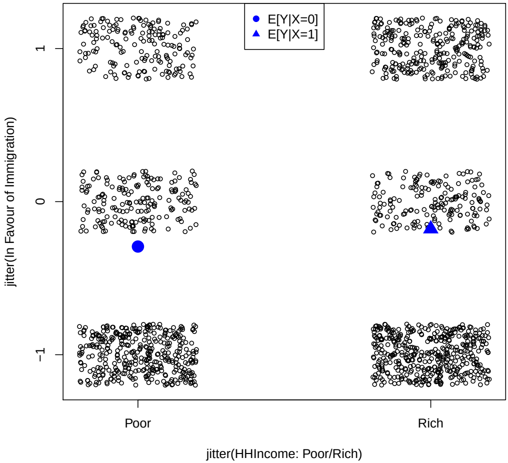
Is $Y \perp\!\!\!\perp X$?

Example: X = wealth, Y = support for immigration, Z = education.



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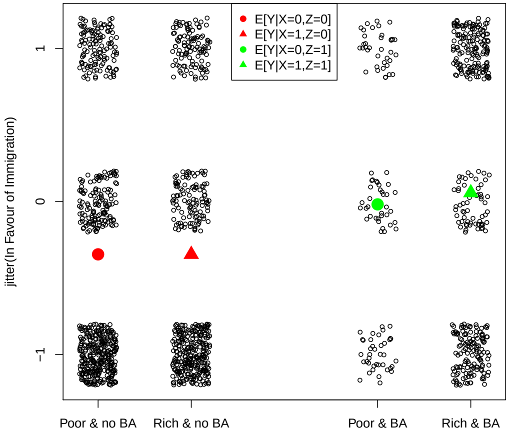

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    - What is a Random Variable?
    - Discrete Distributions
    - Continuous Distributions
2. Characteristics of Distributions
    - Central Tendency
    - Measures of Dispersion
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- When we can work with an existing set of distributions, it makes calculations simpler
- Examples: Bernoulli, Binomial, Gamma, Normal, Poisson, t-distribution

Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$.
Then we will say that X is Bernoulli random variable,

$$p(X = x) \quad = \quad \pi^x (1 - \pi)^{1-x}$$

for $x \in \{0, 1\}$ and $p(X = x) = 0$ otherwise.
We will (equivalently) say that

$$X \quad \sim \quad \text{Bernoulli}(\pi)$$

$\sim$ means equality in distribution (not values!). Often $X \sim$ Bernoulli(π) would be read 'X is distributed Bernoulli with parameter π'

Bernoulli Random Variable Mean and Variance

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$$\begin{aligned} E[X] &= 1 \times P(X = 1) + 0 \times P(X = 0) \\ &= \pi + 0(1 - \pi) = \pi \end{aligned}$$

$E[X] = \pi$

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&= \pi \\
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&= \pi(1 - \pi)
\end{aligned}$$

$E[X] = \pi$
$\text{var}(X) = \pi(1 - \pi)$
Importantly, we can also just look this up!

Normal/Gaussian Random Variables

Definition

Suppose X is a random variable with $X \in \mathbb{R}$ and density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a normally distributed random variable with parameters μ and σ^2.
Equivalently, we'll write

$$X \sim \text{Normal}(\mu, \sigma^2)$$

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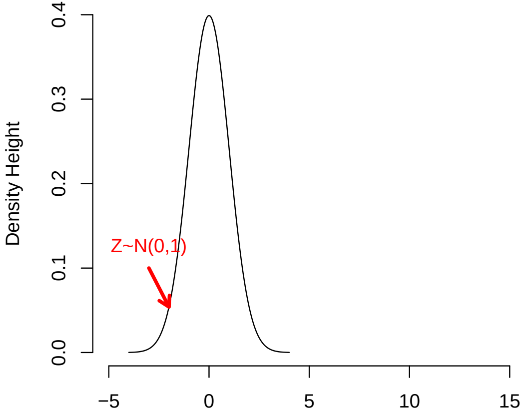
Proposition

Scale/Location. If $Z \sim N(0, 1)$, then $X = aZ + b$ is,

$$X \sim Normal(b, a^2)$$

Intuition

Suppose $Z \sim \text{Normal}(0, 1)$.



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Suppose $Z \sim \text{Normal}(0, 1)$.
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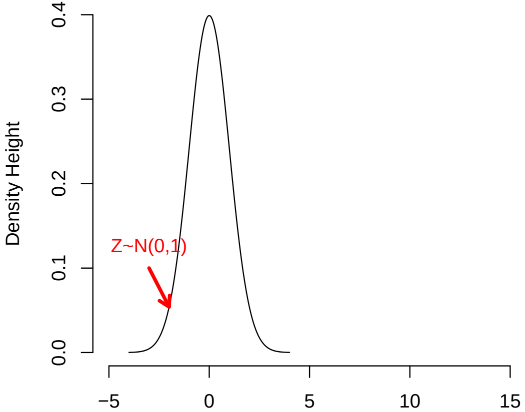

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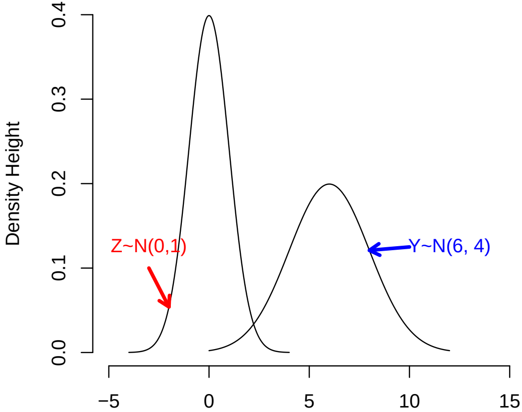
$Y \sim \text{Normal}(6,4)$



Proof: $Z \sim N(0, 1)$ and $Y = aZ + b$, then $Y \sim N(b, a^2)$

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&= \frac{1}{\sqrt{2\pi}a}\exp\left[-\frac{(x-b)^2}{2a^2}\right] \\
&= \text{Normal}(b, a^2)
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This implies that, for $Y \sim \text{Normal}(\mu, \sigma^2)$

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Multivariate Normal

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ is a vector of random variables. If $\boldsymbol{X}$ has pdf

$$f(\boldsymbol{x}) \quad = \quad (2\pi)^{-N/2} \det(\boldsymbol{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}(\boldsymbol{x} - \boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ has a Multivariate Normal Distribution,

$$\boldsymbol{X} \quad \sim \quad \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

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$$f(x_1, x_2) = (2\pi)^{-2/2} 1^{-1/2} \exp\left(-\frac{1}{2}\left((\boldsymbol{x} - \mathbf{0})' \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} (\boldsymbol{x} - \mathbf{0})\right)\right)$$

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$\rightsquigarrow$ product of univariate standard normally distributed random variables

Properties of the Multivariate Normal Distribution

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$

$$
\begin{aligned}
E[\boldsymbol{X}] &= \boldsymbol{\mu} \\
\text{cov}(\boldsymbol{X}) &= \boldsymbol{\Sigma}
\end{aligned}
$$

So that,

$$
\boldsymbol{\Sigma} = \begin{pmatrix}
\text{var}(X_1) & \text{cov}(X_1, X_2) & \ldots & \text{cov}(X_1, X_N) \\
\text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N)
\end{pmatrix}
$$

One Step Deeper: Exponential Family

Nearly every distribution we will discuss is in the exponential family. An exponential family distribution has the density of the following form:

$$f_Y(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

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Many other examples, including: Normal, Bernoulli/binomial, Gamma, multinomial, exponential, negative binomial, beta, uniform, chi-squared, etc.

This slide and the following based on material from Teppei Yamamoto

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  $\Rightarrow \mathbb{E}(Y_i) = \frac{db(\theta_i)}{d\theta_i} = \exp(\theta_i) = \mu_i$ and $\mathbb{V}(Y_i) = \frac{d^2 b(\theta_i)}{d\theta_i^2} = \exp(\theta_i) = \mu_i$

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- We can characterize distributions in terms of their expectation (location) and variance (spread).
- Joint and conditional distributions capture the relationship between random variables.
- There is a common set of famous distributions such as the Normal distribution.

Where We've Been and Where We're Going . . .

This week:

- Monday:
  - summarize one random variable using expectation and variance
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  - properties of joint distributions
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New reading:

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New reading:

- Aronow and Miller Chapter 3.1-3.2.6, 3.4.1
- Optional: Fox Chapter 3: Examining Data

1. Random Variables and Distributions
   - What is a Random Variable?
   - Discrete Distributions
   - Continuous Distributions
2. Characteristics of Distributions
   - Central Tendency
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3. Conditional Distributions
4. Fun with Averages
5. Fun with Sensitive Questions
6. Appendix: Why the Mean?
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- Then apply model to new data, classify those observations

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This is called a Naïve Bayes classifier.

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Simple intuition about Naïve Bayes:

- Learn what documents in class j look like
- Find class k that document i is most similar to

Example: Building a Spam Filter

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Scoring the algorithm is easy.

$$p(C_k|\boldsymbol{x}_i) \propto p(C_k) \prod_{j=1}^{J} p(x_{i,j}|C_k)^{x_{ij}}$$

which is simply the probability of the class multiplied by the product of the probabilities for the words that are observed in the test document.

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- More on estimators next week!