Week 5: Simple Linear Regression

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Princeton

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller. Illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

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  - what is regression

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  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Questions?

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Review session timing.

1. Mechanics of OLS
2. Properties of the OLS estimator
3. Example and Review
4. Properties Continued
5. Hypothesis tests for regression
6. Confidence intervals for regression
7. Goodness of fit
8. Wrap Up of Univariate Regression
9. Fun with Non-Linearities
10. Appendix: r^2 derivation

The population linear regression function

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- β_0, β_1 = population intercept and population slope (what we want to estimate)

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- You can think of the residuals as the prediction errors of our estimates.

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- Interpretation: how do we interpret our estimates?

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- In words, the OLS estimates are the intercept and slope that minimize the sum of the squared residuals.

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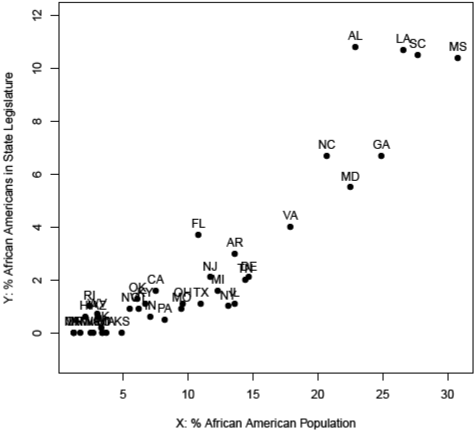

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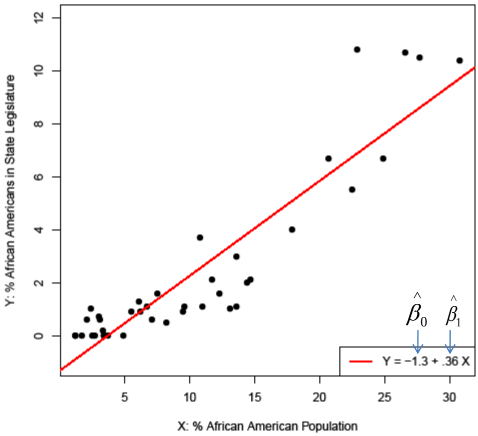

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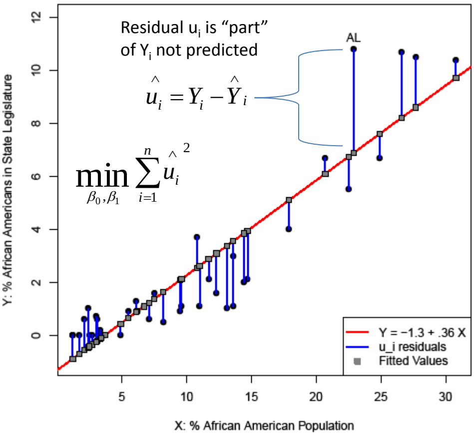

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- To the board we go!

The OLS estimator

- Now we're done! Here are the **OLS estimators**:

$$\widehat{\beta}_0 = \overline{Y} - \widehat{\beta}_1 \overline{X}$$

$$\widehat{\beta}_1 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})(Y_i - \overline{Y})}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

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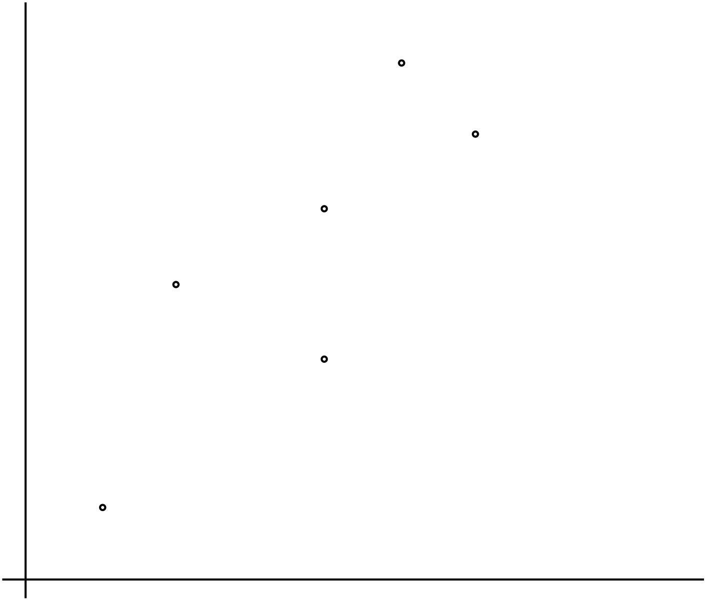

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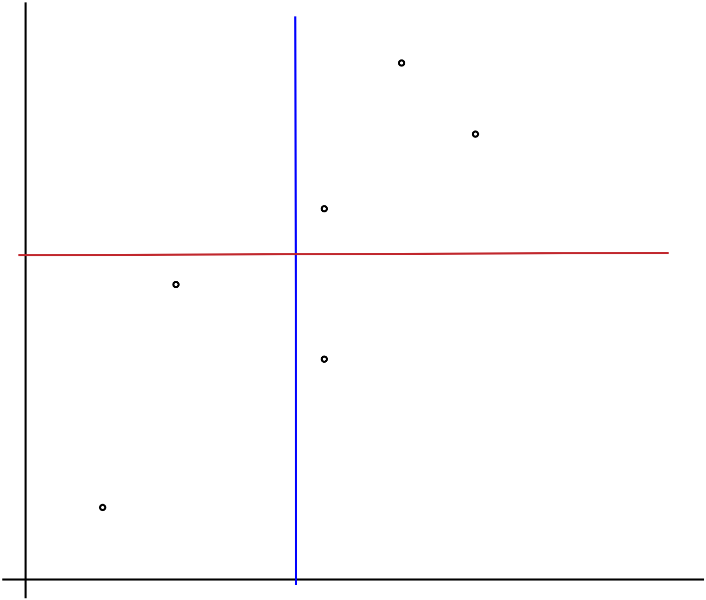

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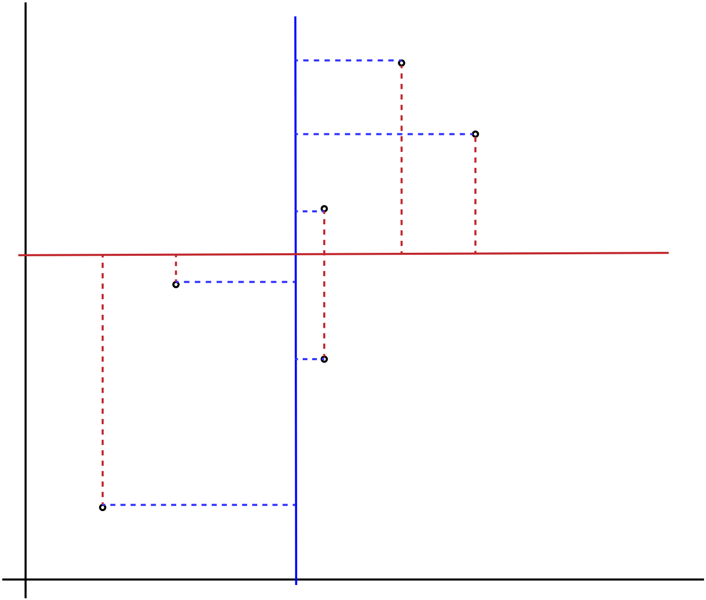

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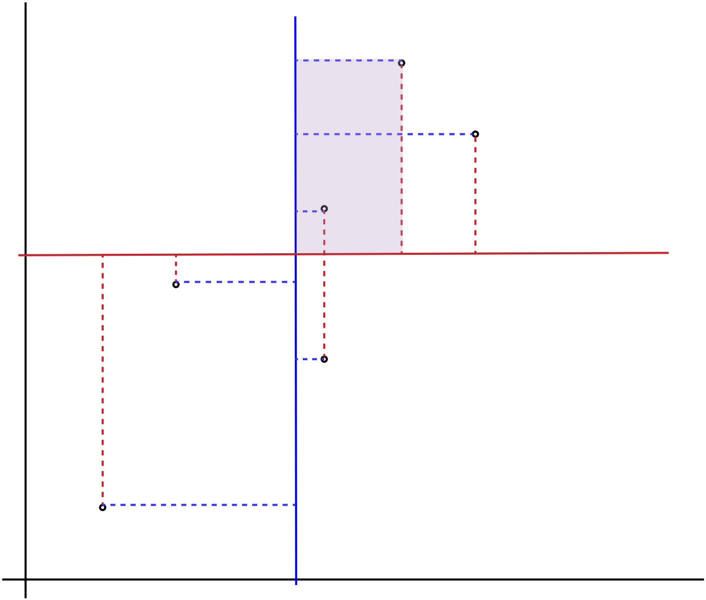

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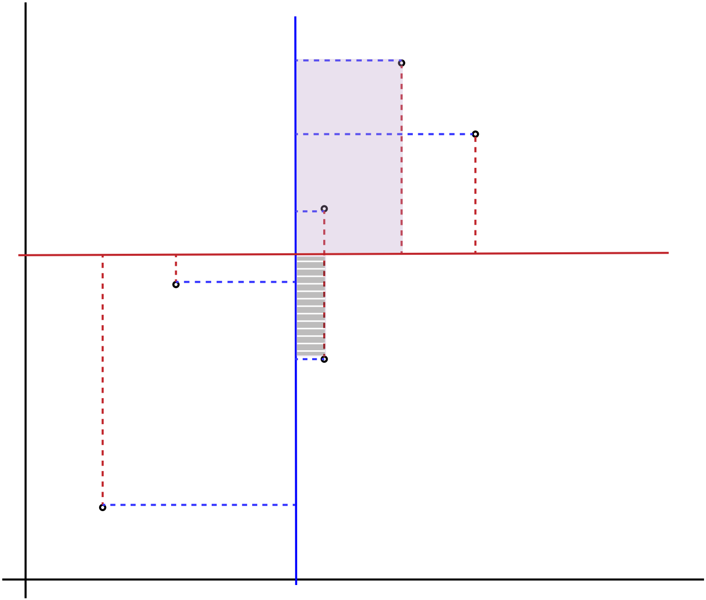

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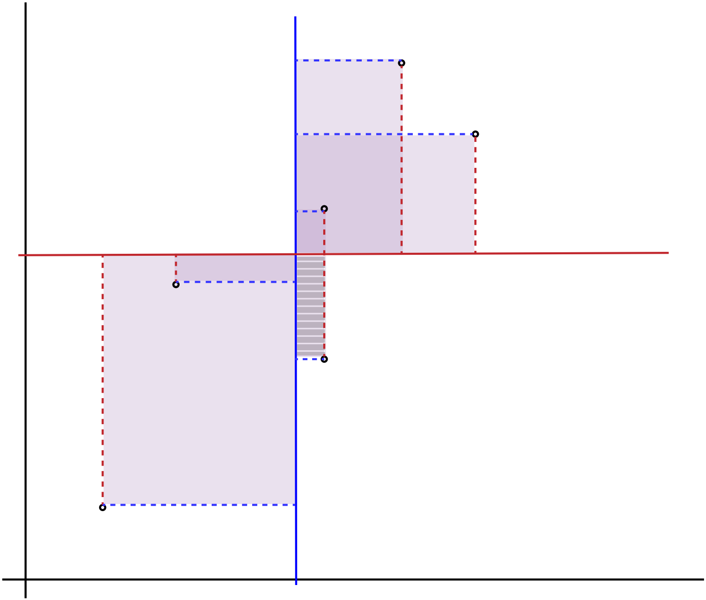

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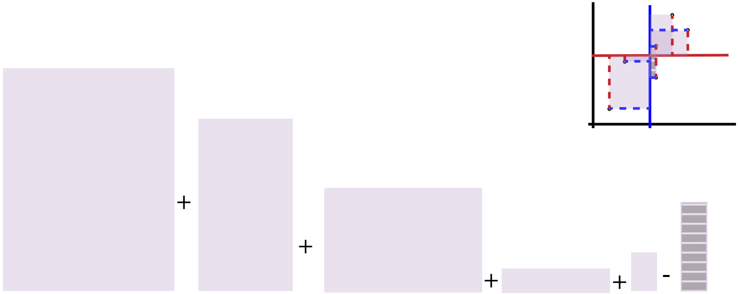

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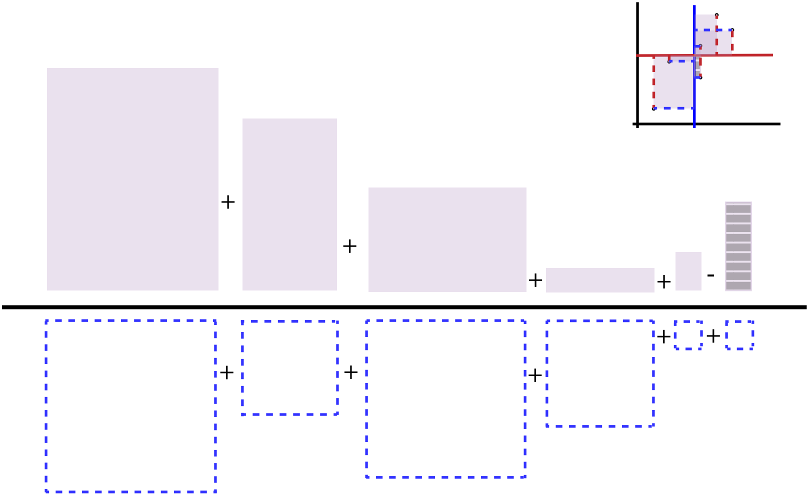

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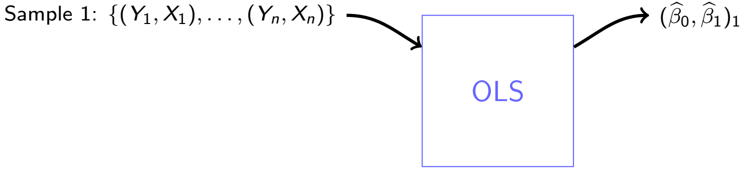

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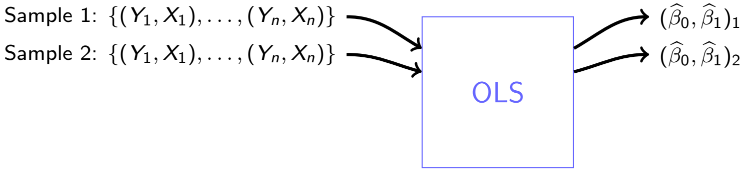

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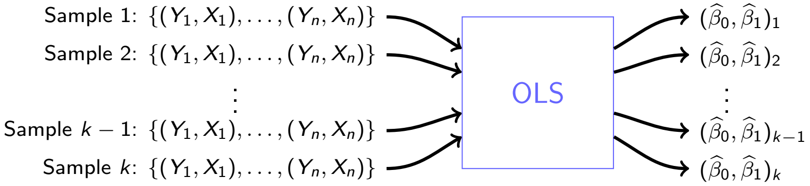

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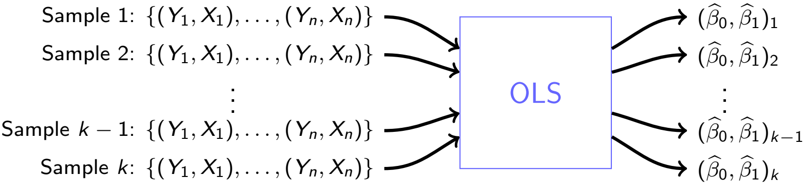

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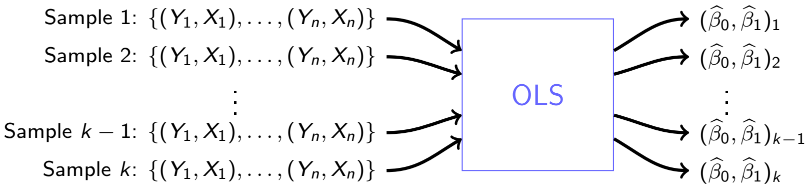

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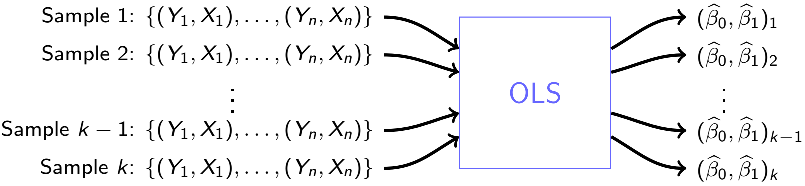

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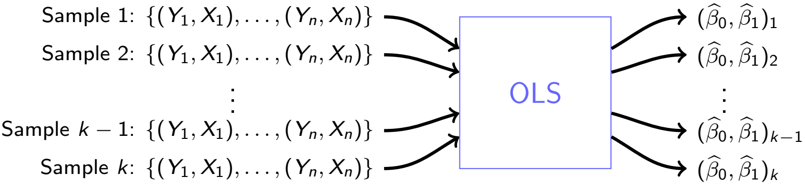

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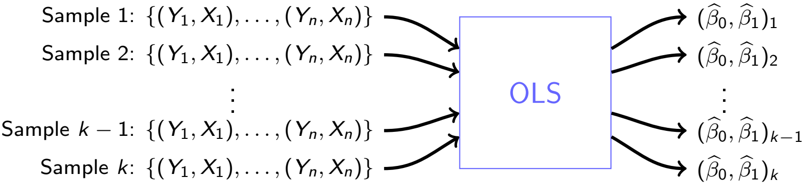

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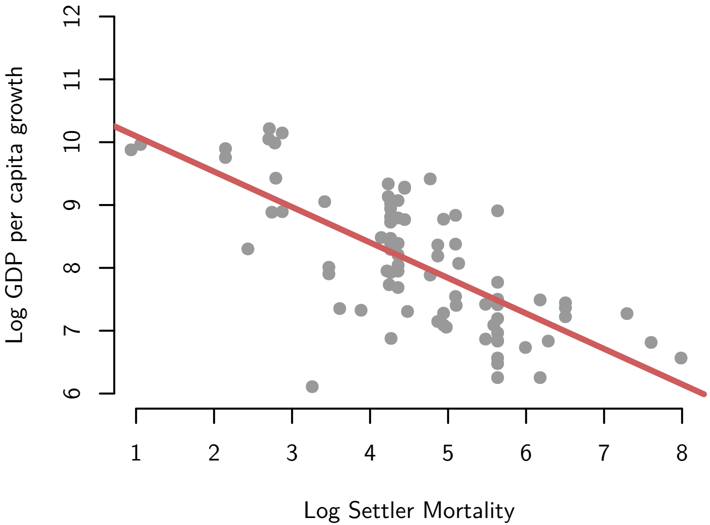
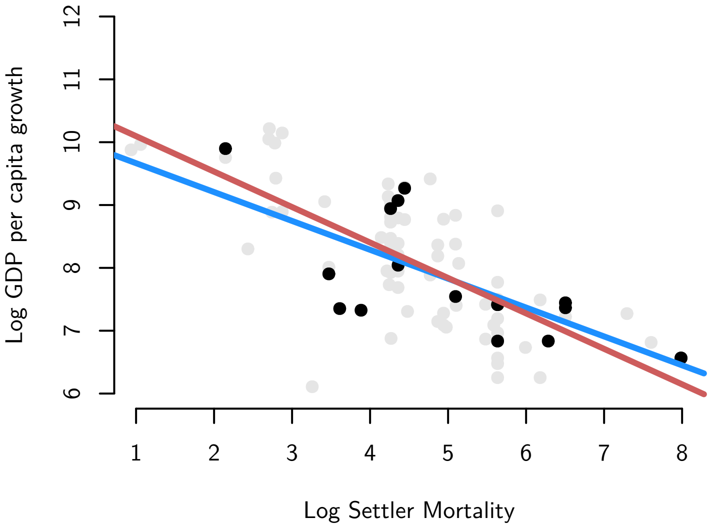


Randomly sample from AJR



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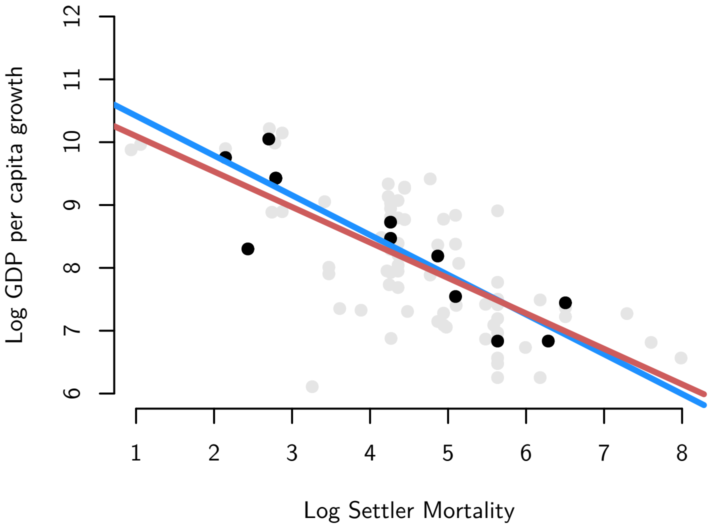

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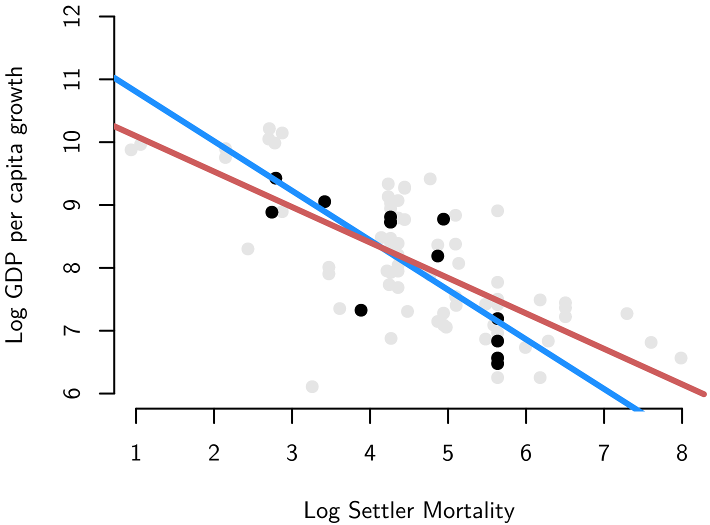

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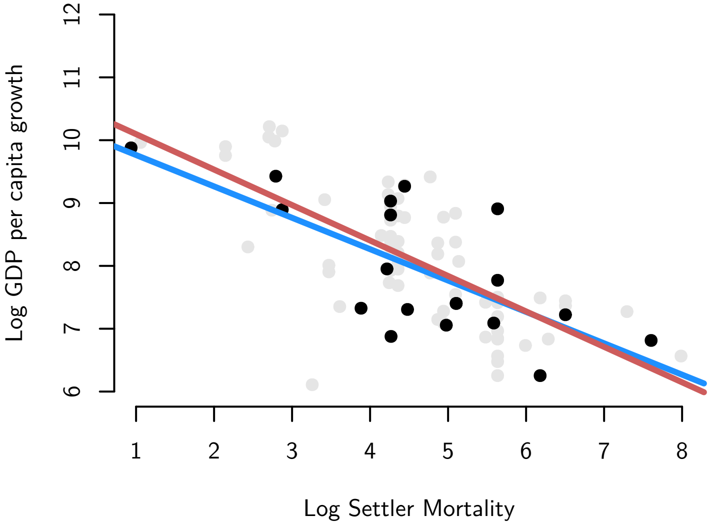

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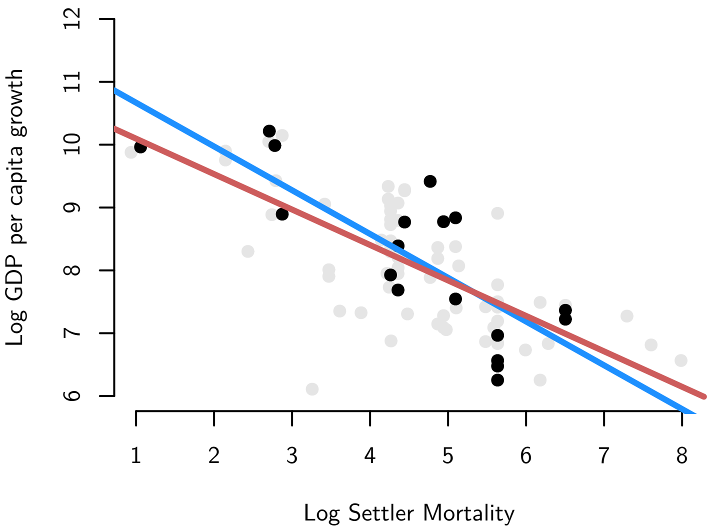

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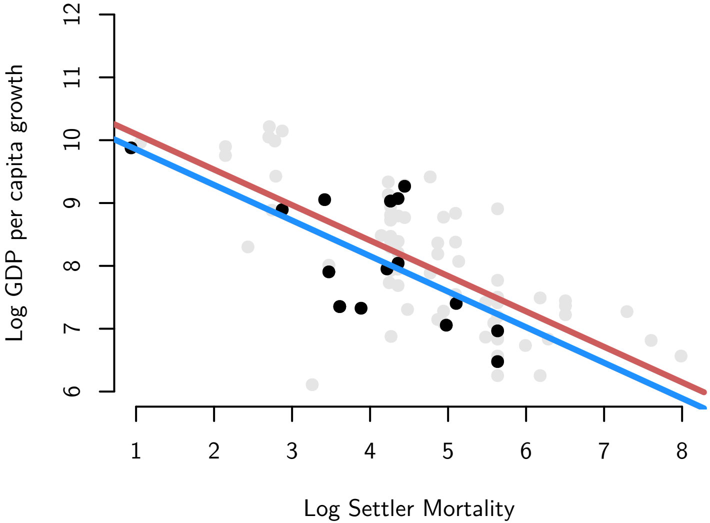

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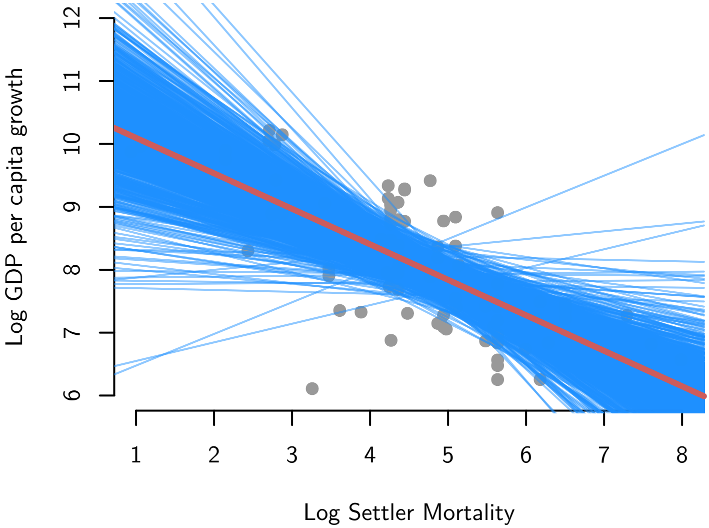

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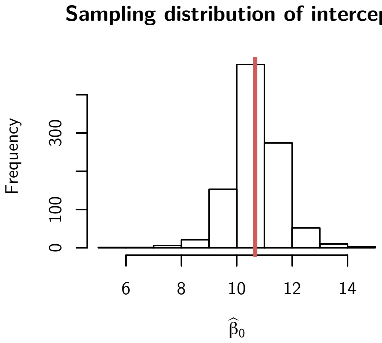

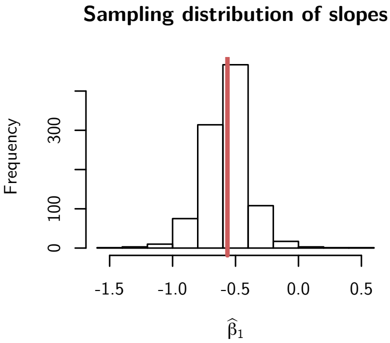

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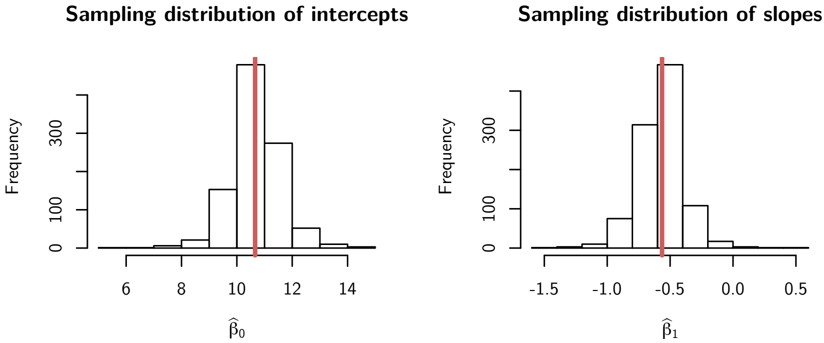

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Assumptions for unbiasedness of the sample mean

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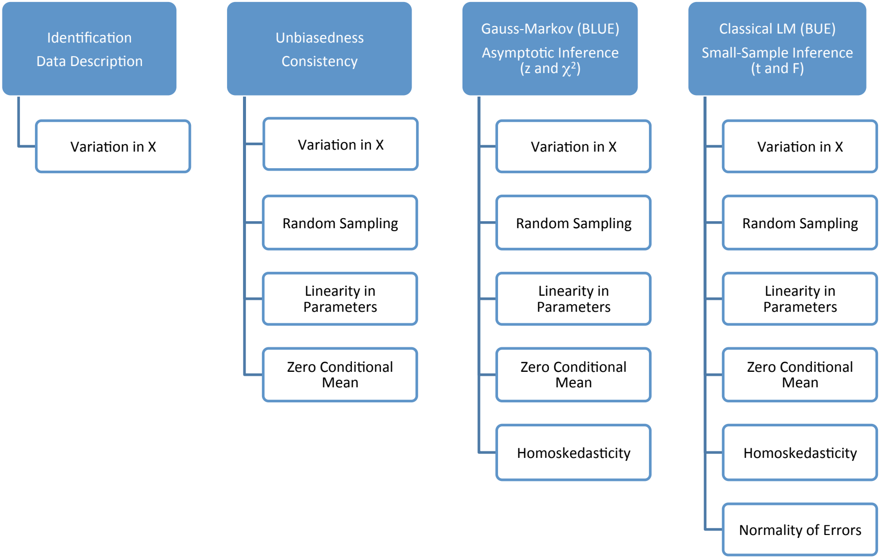

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Only assumption needed for using OLS as a pure data summary.

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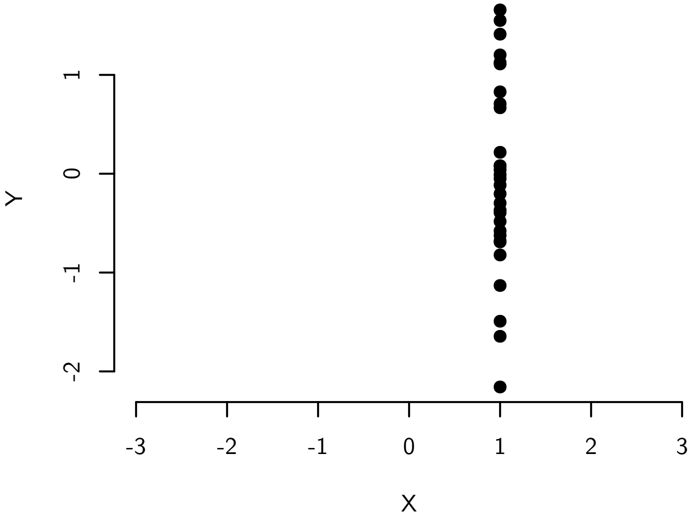
- Why does this matter? How would you draw the line of best fit through this scatterplot, which is a violation of this assumption?



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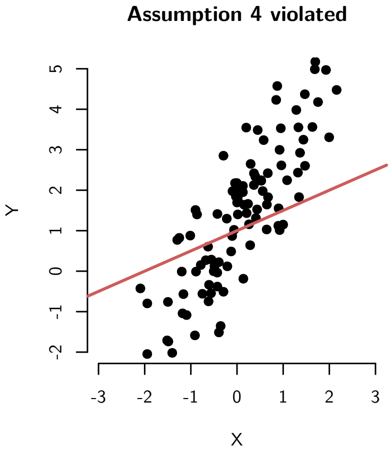
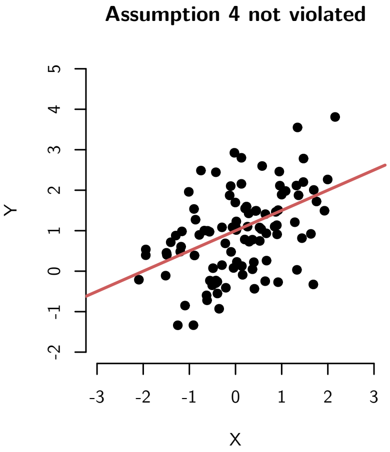
2. No relationship between them (satisfies the assumption)

Violating the zero conditional mean assumption





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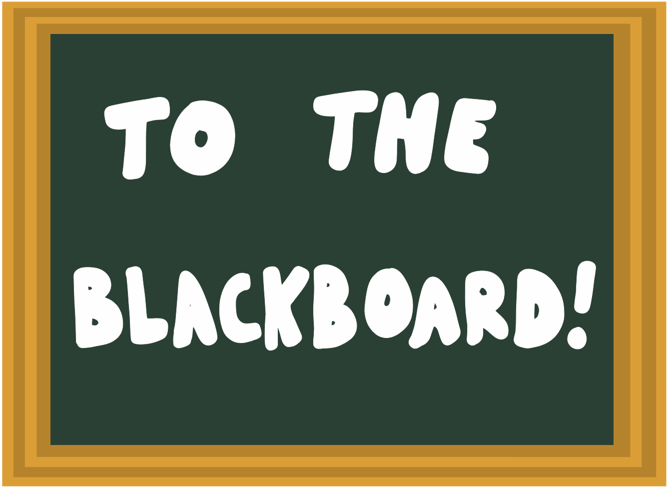

Unbiasedness of OLS

Theorem (Unbiasedness of OLS)

Given OLS Assumptions I–IV:

$$E[\hat{\beta}_0] = \beta_0 \quad \textit{and} \quad E[\hat{\beta}_1] = \beta_1$$

The sampling distributions of the estimators $\hat{\beta}_1$ *and* $\hat{\beta}_0$ *are centered about the true population parameter values* β_1 *and* β_0.

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- That is we know that the sampling distribution is centered on the true population slope, but we don't know the population variance.

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  $\rightarrow$ all errors have an identical error variance ($\sigma_{u_i}^2 = \sigma_u^2$ for all i)

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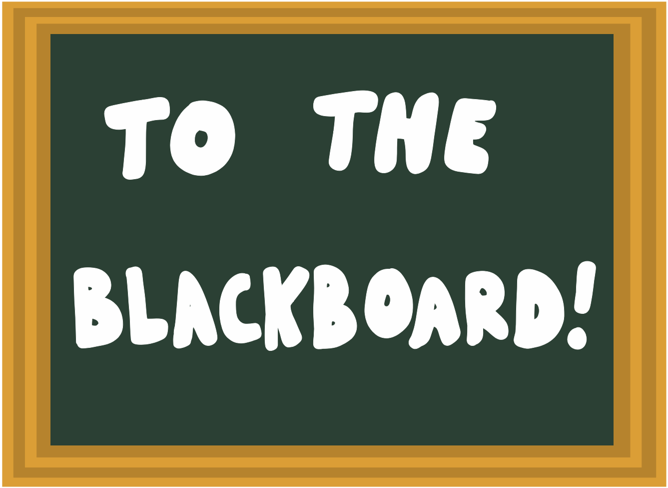

Variance of OLS Estimators

Theorem (Variance of OLS Estimators)

Given OLS Assumptions I–V (Gauss-Markov Assumptions):

$$Var[\hat{\beta}_1 \mid X] = \frac{\sigma_u^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

$$Var[\hat{\beta}_0 \mid X] = \sigma_u^2 \left\{ \frac{1}{n} + \frac{\bar{x}^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2} \right\}$$

where $Var[u \mid X] = \sigma_u^2$ *(the error variance).*

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- What drives the sampling variability of the OLS estimator?

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- But, this formula depends upon an unobserved term: σ_u^2

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Intuitively, which line is likely to be closer to the observed sample values on X and Y, the true line $y_i = \beta_0 + \beta_1 x_i$ or the fitted regression line $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_i$?

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- Thus, an unbiased estimator for the error variance is:

$$\hat{\sigma}_u^2 = \frac{n}{n-2}MSD(\hat{u}) = \frac{n}{n-2}\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2$$

We plug this estimate into the variance estimators for $\hat{\beta}_0$ and $\hat{\beta}_1$.

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Questions?

Example: Epstein and Mershon SCOTUS data

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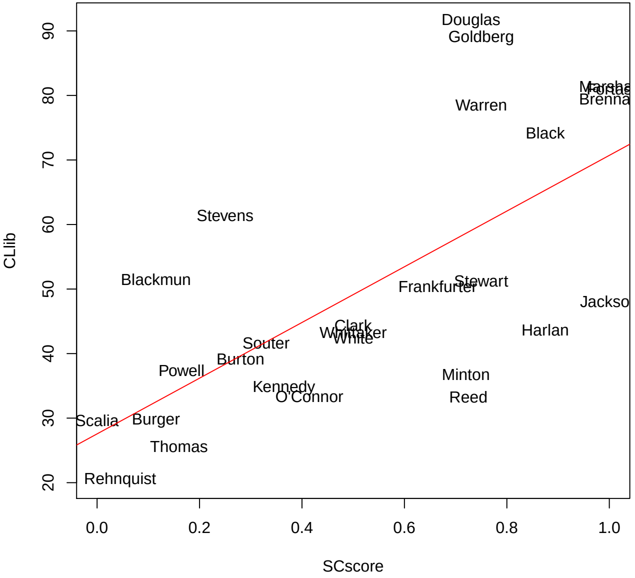
Douglas
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Warren
Brenna
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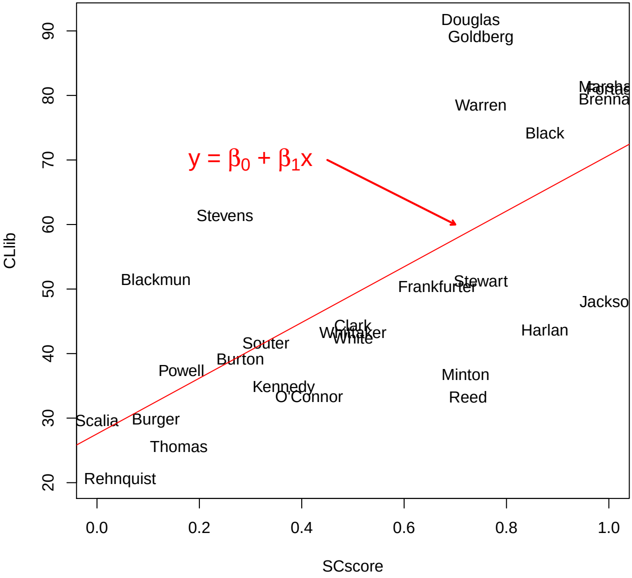
$y = \beta_0 + \beta_1 x$
Douglas
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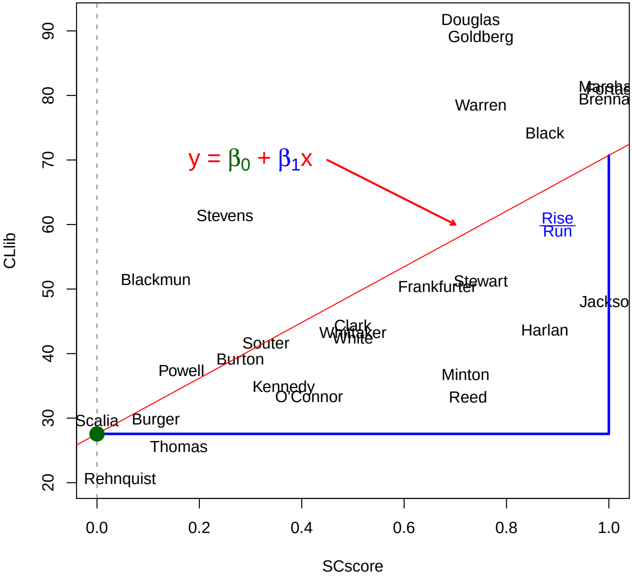

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Rise
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Douglas
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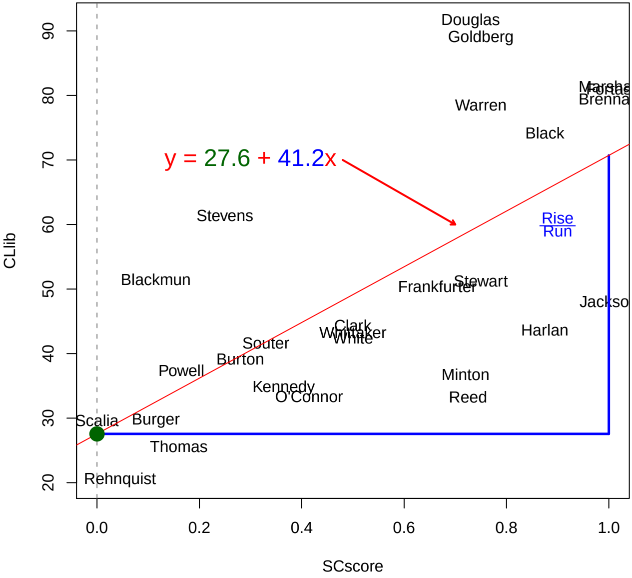
y = 27.6 + 41.2x
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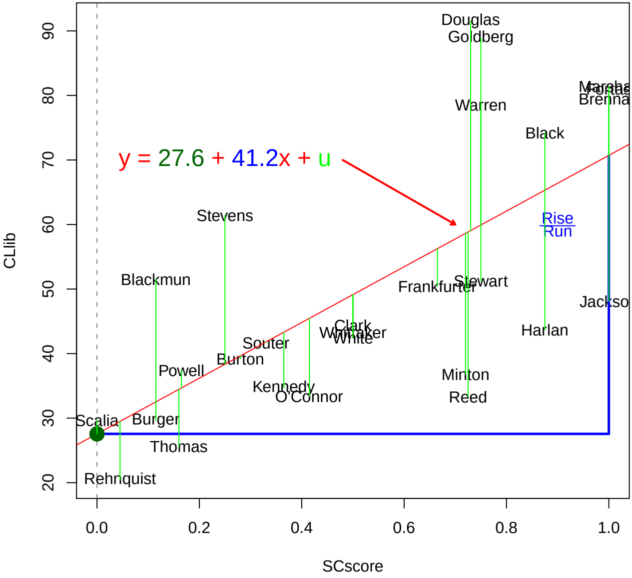

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How to get β_0 and β_1

$$\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}.$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sum_{i=1}^{n}(x_i - \bar{x})^2}.$$

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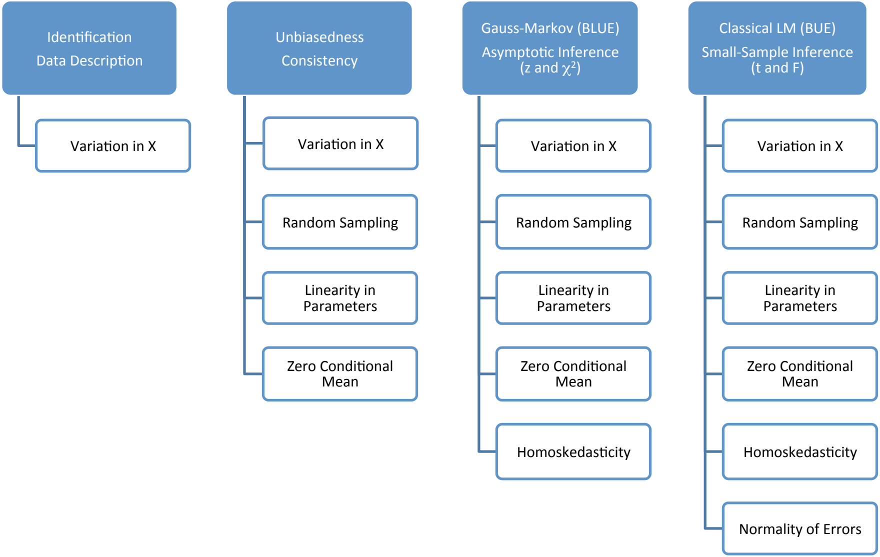
Identification
Data Description
Variation in X
Unbiasedness
Consistency
Variation in X
Random Sampling
Linearity in Parameters
Zero Conditional Mean
Gauss-Markov (BLUE)
Asymptotic Inference
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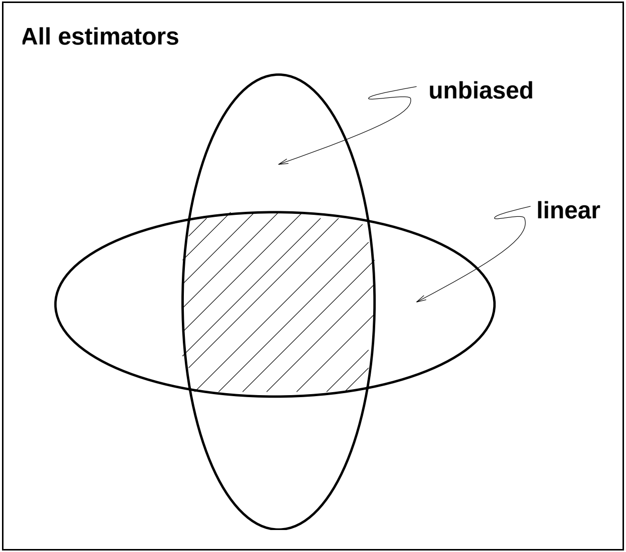
- Fails to hold when the assumptions are violated!

Gauss-Markov Theorem



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The population error term is independent of the explanatory variable, $u \perp\!\!\!\perp X$, and is normally distributed with mean zero and variance σ_u^2:

$$u \sim N(0, \sigma_u^2), \quad \textit{which implies} \quad Y|X \sim N(\beta_0 + \beta_1 X, \sigma_u^2)$$

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- Reminder: we don't need normality assumption in large samples

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Under Assumptions I–VI,

$$\widehat{\beta}_1 \sim N\left(\beta_1, Var[\widehat{\beta}_1 \mid X]\right)$$

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Given Assumptions I–VI, $\hat{\beta}_1$ is a linear combination of the i.i.d. normal random variables:

$$\hat{\beta}_1 = \beta_1 + \sum_{i=1}^n \frac{(x_i - \bar{x})}{SST_x} u_i \quad \text{where} \quad u_i \sim N(0, \sigma_u^2).$$

Any linear combination of independent normals is normal, and we can transform/standarize any normal random variable into a standard normal by subtracting off its mean and dividing by its standard deviation. □

Sampling distribution of OLS slope

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- All of this depends on Normal errors!

The t-Test for Single Population Parameters

- $SE[\hat{\beta}_1] = \frac{\sigma_u}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2}}$ involves the unknown population error variance σ_u^2
- Replace σ_u^2 with its unbiased estimator $\hat{\sigma}_u^2 = \frac{\sum_{i=1}^n \hat{u}_i^2}{n-2}$, and we obtain:

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Proof.

The logic is perfectly analogous to the t-value for the population mean — because we are estimating the denominator, we need a distribution that has fatter tails than $N(0,1)$ to take into account the additional uncertainty.

This time, $\hat{\sigma}_u^2$ contains two estimated parameters ($\hat{\beta}_0$ and $\hat{\beta}_1$) instead of one, hence the degrees of freedom $= n-2$. □

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Now let's briefly return to some of the large sample properties.

Large Sample Properties: Consistency

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Theorem (Consistency of OLS Estimator)

Given Assumptions I–IV, the OLS estimator $\widehat{\beta}_1$ is consistent for β_1 as $n \to \infty$:

$$\operatorname*{plim}_{n\to\infty} \widehat{\beta}_1 = \beta_1$$

- Technical note: We can slightly relax Assumption IV:

$$E[u|X] = 0 \quad \text{(any function of } X \text{ is uncorrelated with } u\text{)}$$

to its implication:

$$\text{Cov}[u, X] = 0 \quad (X \text{ is uncorrelated with } u)$$

for consistency to hold (but not unbiasedness).

Large Sample Properties: Consistency

Proof.

Similar to the unbiasedness proof:

$$\begin{aligned}
\hat{\beta}_1 &= \frac{\sum_{i=1}^n (x_i - \bar{x})y_i}{\sum_{i=1}^n (x_i - \bar{x})^2} = \beta_1 + \frac{\sum_i^n (x_i - \bar{x})u_i}{\sum_i^n (x_i - \bar{x})^2} \\
\text{plim } \widehat{\beta}_1 &= \text{plim } \beta_1 + \text{plim } \frac{\sum_i^n (x_i - \bar{x})u_i}{\sum_i^n (x_i - \bar{x})^2} \quad \text{(Wooldridge C.3 Property i)} \\
&= \beta_1 + \frac{\text{plim } \frac{1}{n}\sum_i^n (x_i - \bar{x})u_i}{\text{plim } \frac{1}{n}\sum_i^n (x_i - \bar{x})^2} \quad \text{(Wooldridge C.3 Property iii)} \\
&= \beta_1 + \frac{\text{Cov}[X, u]}{\text{Var}[X]} \quad \text{(by the law of large numbers)} \\
&= \beta_1 \quad (\text{Cov}[X, u] = 0 \text{ and } \text{Var}[X] > 0)
\end{aligned}$$

□

- OLS is inconsistent (and biased) unless $\text{Cov}[X, u] = 0$
- If $\text{Cov}[u, X] > 0$ then asymptotic bias is upward; if $\text{Cov}[u, X] < 0$ asymptotic bias is downwards

Large Sample Properties: Asymptotic Normality

- For statistical inference, we need to know the sampling distribution of $\hat{\beta}$ when $n \to \infty$.

Theorem (Asymptotic Normality of OLS Estimator)

Given Assumptions I–V, the OLS estimator $\widehat{\beta}_1$ is asymptotically normally distributed:

$$\frac{\hat{\beta}_1 - \beta_1}{\widehat{SE}[\hat{\beta}_1]} \overset{approx.}{\sim} N(0,1)$$

where

$$\widehat{SE}[\hat{\beta}_1] = \frac{\hat{\sigma}_u}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2}}$$

with the consistent estimator for the error variance:

$$\hat{\sigma}_u^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 \overset{p}{\to} \sigma_u^2$$

Large Sample Inference

Proof.

Proof is similar to the small-sample normality proof:

$$\hat{\beta}_1 = \beta_1 + \sum_{i=1}^{n} \frac{(x_i - \bar{x})}{SST_x} u_i$$

$$\sqrt{n}(\hat{\beta}_1 - \beta_1) = \frac{\sqrt{n} \cdot \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x}) u_i}{\frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2}$$

where the numerator converges in distribution to a normal random variable by CLT. Then, rearranging the terms, etc. gives you the right formula given in the theorem.

For a more formal and detailed proof, see Wooldridge Appendix 5A. □

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- We need homoskedasticity (Assumption V) for this result, but we do not need normality (Assumption VI).
- Result implies that asymptotically our usual standard errors, t-values, p-values, and CIs remain valid even without the normality assumption! We just proceed as in the small sample case where we assume normality.
- It turns out that, given Assumptions I–V, the OLS asymptotic variance is also the lowest in class (asymptotic Gauss-Markov).

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For 2 and 3, we need to know more than just the mean and the variance of the sampling distribution of $\hat{\beta}_1$. We need to know the full shape of the sampling distribution of our estimators $\hat{\beta}_0$ and $\hat{\beta}_1$.

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- Notice these are statements about the population parameters, not the OLS estimates.

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- Thus, under the null, we know the distribution of T and can use that to formulate a rejection region and calculate p-values.

Rejection region

- Choose a level of the test, α, and find rejection regions that correspond to that value under the null distribution:

$$\mathbb{P}(-t_{\alpha/2,n-2} < T < t_{\alpha/2,n-2}) = 1 - \alpha$$

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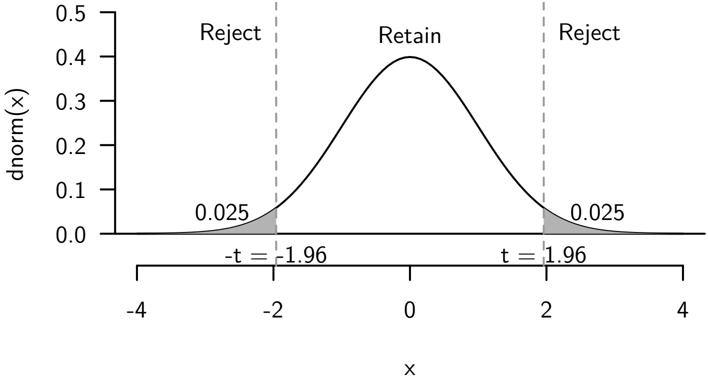
- This is exactly the same as with sample means and sample differences in means, except that the degrees of freedom on the t distribution have changed.



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- If the p-value is less than α we would reject the null at the α level.

Confidence intervals

- Very similar to the approach with sample means. By the sampling distribution of the OLS estimator, we know that we can find t-values such that:

$$\mathbb{P}\left(- t_{\alpha/2,n-2} \leq \frac{\widehat{\beta}_1 - \beta_1}{\widehat{SE}[\widehat{\beta}_1]} \leq t_{\alpha/2,n-2} \right) = 1 - \alpha$$

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CIs Simulation Example

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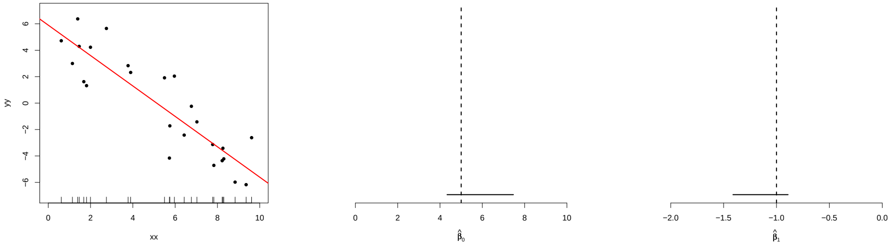

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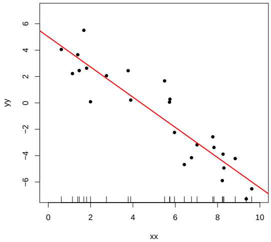

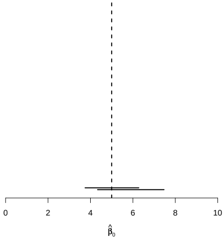

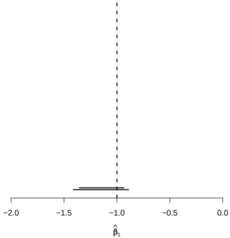

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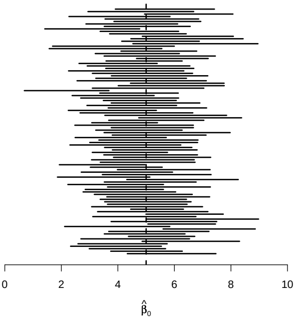

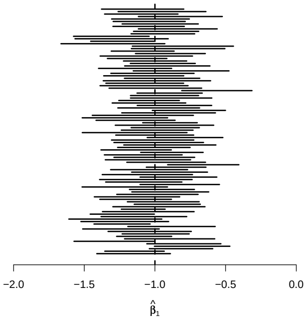

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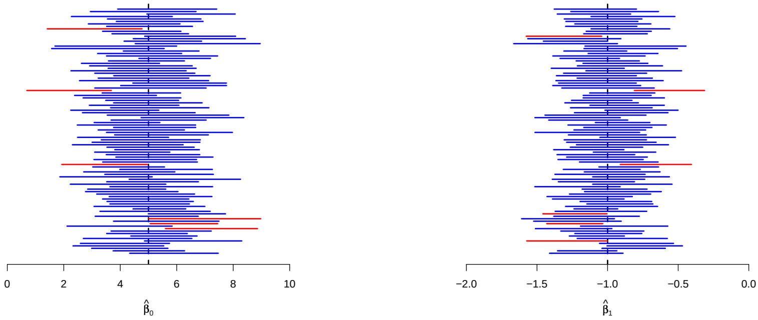

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- Once we have estimated our model, we have new prediction errors, which are just the sum of the squared residuals or SS_{res}:

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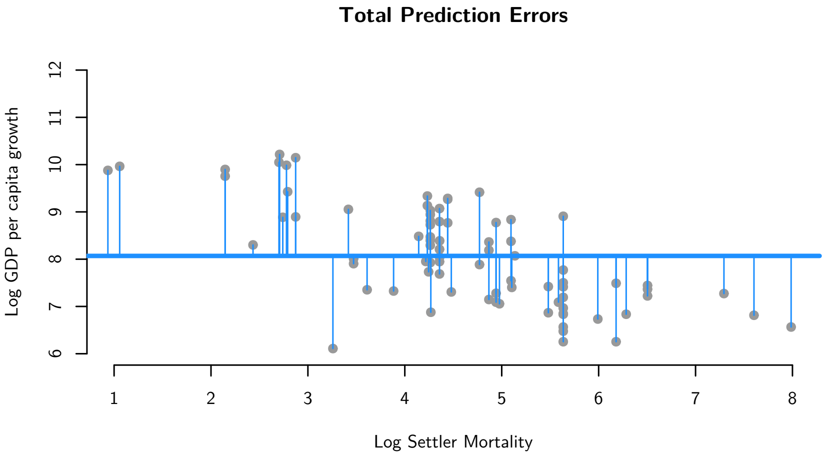
Sum of Squares



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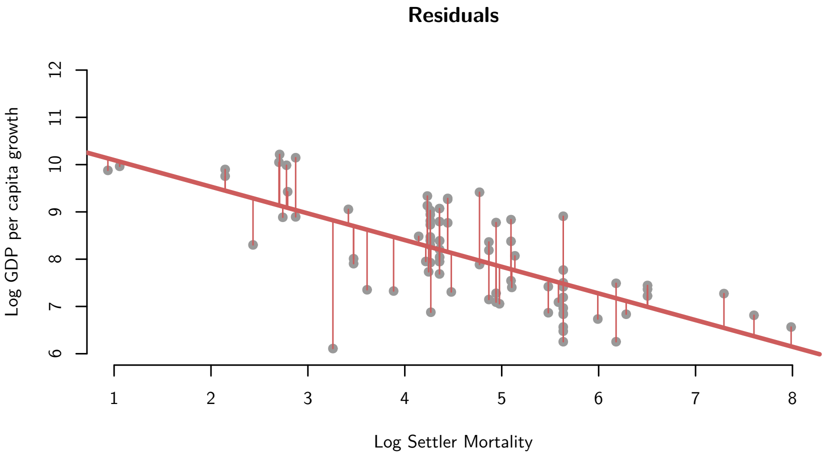

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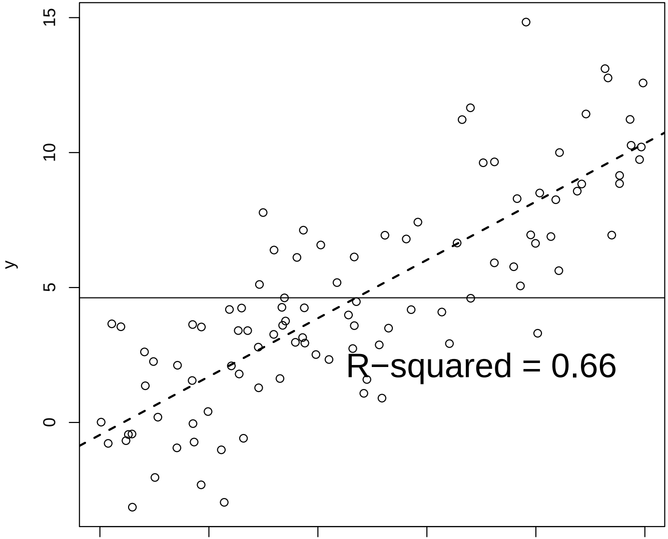

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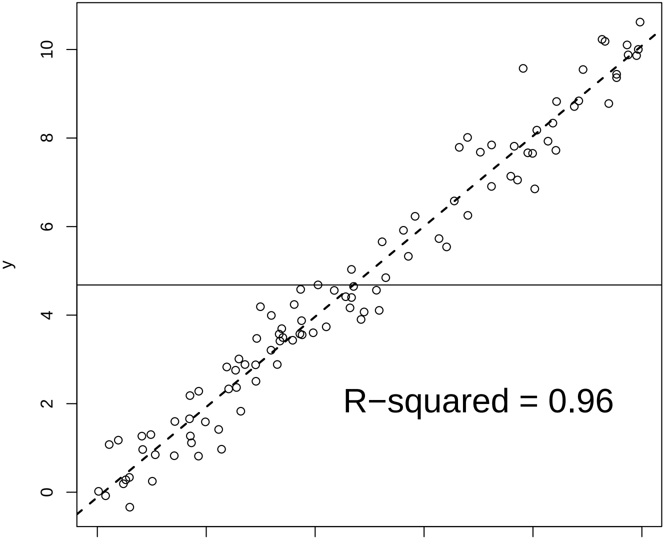

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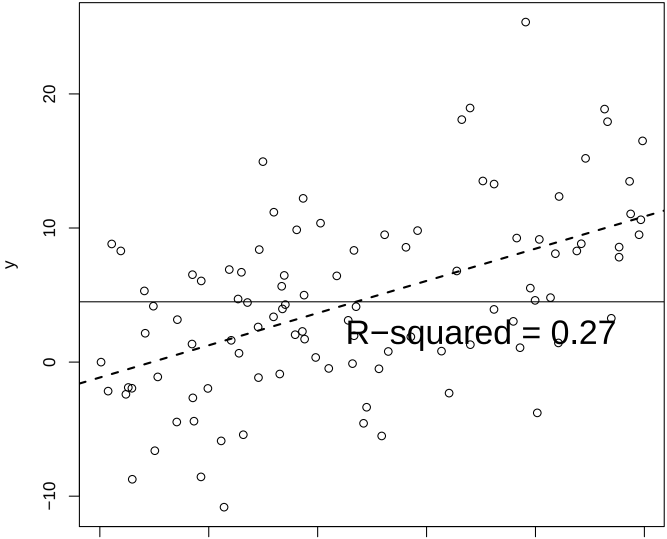

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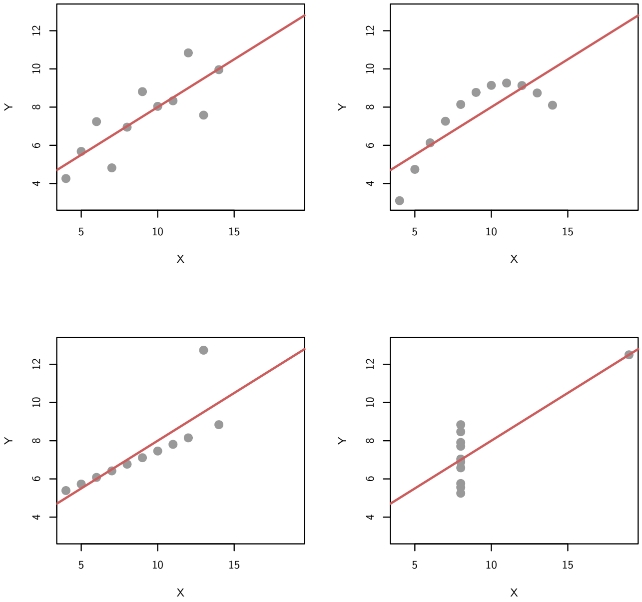
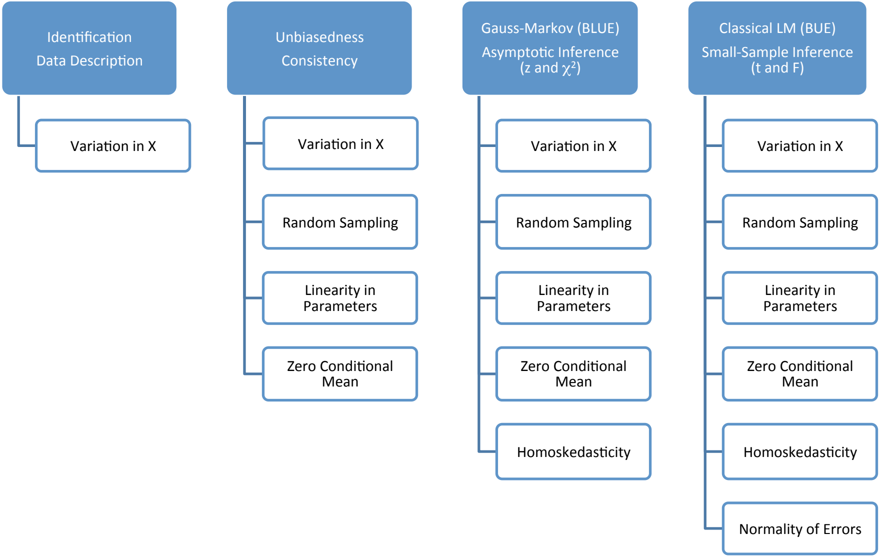


OLS Assumptions Summary



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   - We will come back to this in the last few weeks.

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- Note that Assumption I would make OLS the best, not just best linear, predictor, so it is certainly desired

State Legislators and African American Population

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Coefficients:
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(no effect size or direction)

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"A one percentage point increase in the African American population causes a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(unwarranted causal language)

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"A one percentage point increase in the African American population is associated with a 0.35 percentage point increase in the fraction of African American state legislators ($p < 0.001$)."

(hints at causality)

State Legislators and African American Population

Interpretations of increasing quality:

```
> summary(lm(beo ~ bpop, data = D))

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) -1.31489    0.32775  -4.012 0.000264 ***
bpop         0.35848    0.02519  14.232  < 2e-16 ***
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Residual standard error: 1.317 on 39 degrees of freedom
Multiple R-squared: 0.8385,Adjusted R-squared: 0.8344
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"In states where an additional .01 proportion of the population is African American, we observe on average .035 proportion more African American state legislators ($p < 0.001$)."

(p value doesn't help people with uncertainty)

State Legislators and African American Population

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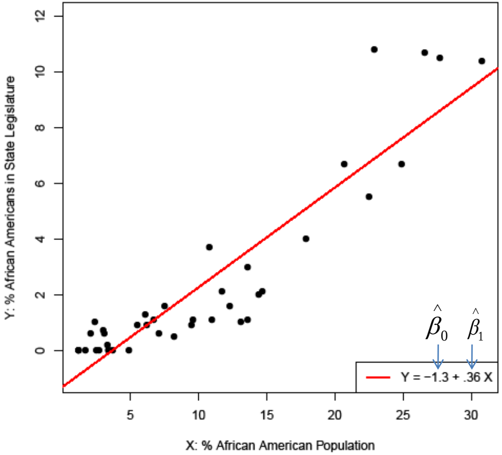
"In states where an additional .01 proportion of the population is African American, we observe on average .035 proportion more African American state legislators (between .03 and .04 with 95% confidence)."

(still not perfect, the best will be subject matter specific. is fairly clear it is non-causal, gives uncertainty.)

Graphical

Graphical presentations are often the most informative. We will talk more about them later in the semester.



Ground Rules: Interpretation of the Slope

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1. Give a short, but precise interpretation of the association using interpretable language and units
2. If the association has a causal interpretation explain why, otherwise do not imply a causal interpretation.
3. Provide a meaningful sense of uncertainty
4. Indicate the practical significance of the finding for your argument.

Next Week

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- OLS with two regressors

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- Dummy variables, interactions, polynomials
- Reading:
  - Optional Fox Chapters 5-7
  - For more on logs, Gelman and Hill (2007) pg 59-61 is nice

Fun with Non-Linearities

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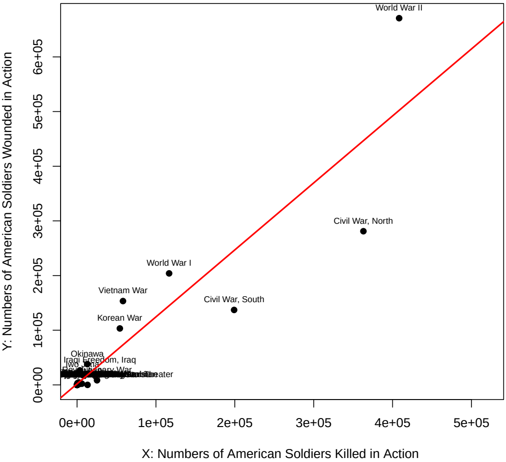
    - Note that these approximations work only for small increments
    - In particular, they do not work when X is a discrete random variable

Example from the American War Library



$\hat{\beta}_1 = 1.23 \longrightarrow$

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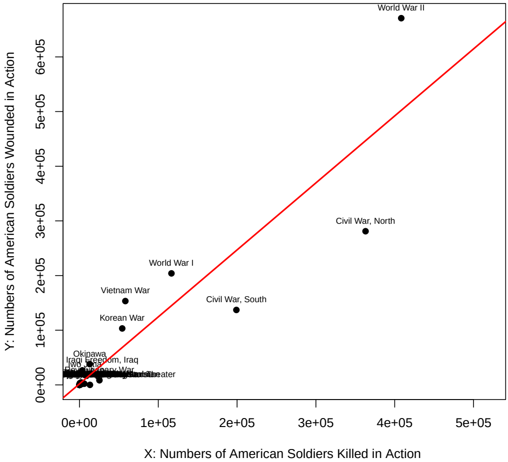
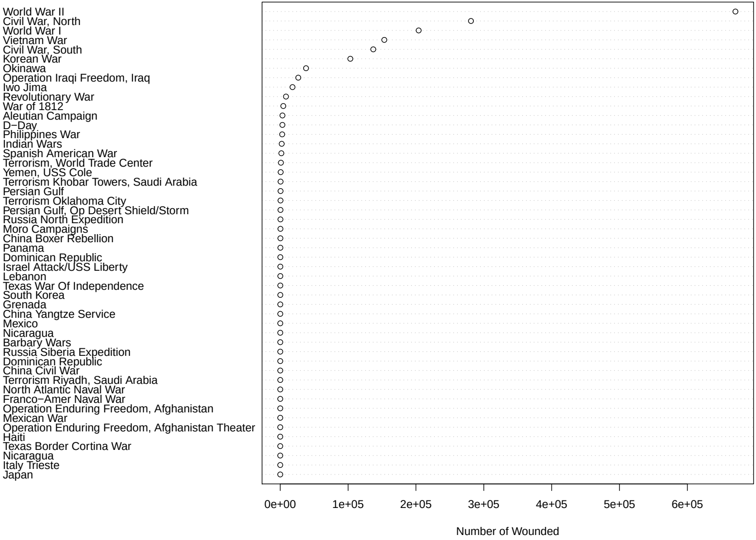
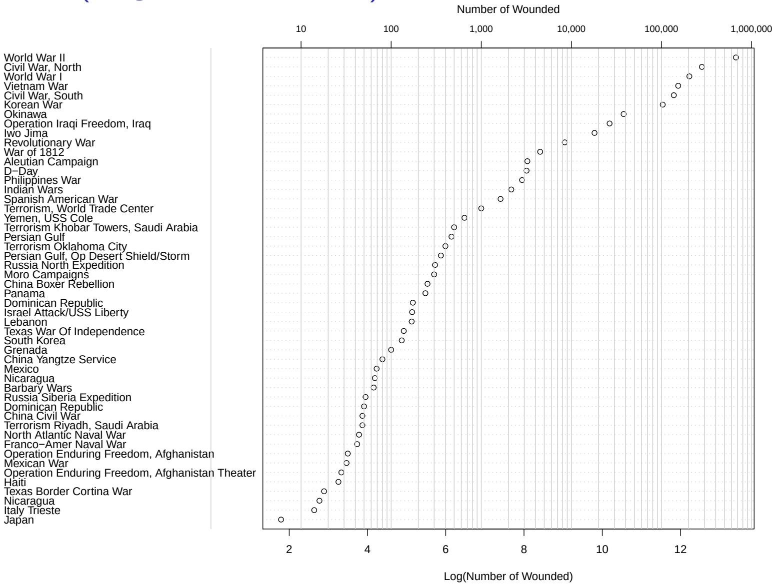


$\hat{\beta}_1 = 1.23 \longrightarrow$ One additional soldier killed predicts 1.23 additional soldiers wounded on average

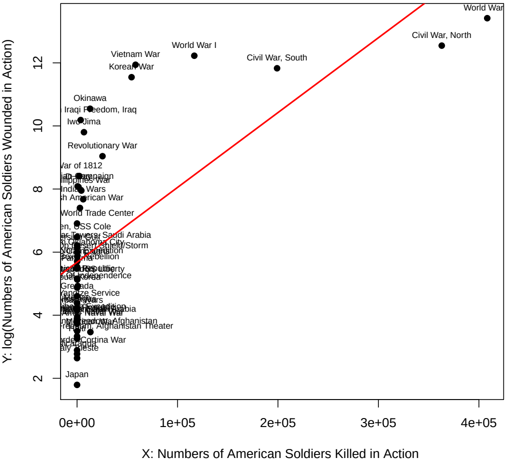
Wounded (Scale in Levels)



Wounded (Logarithmic Scale)

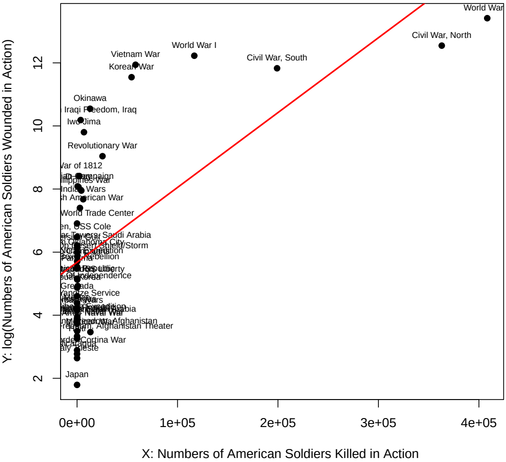


Regression: Log-Level



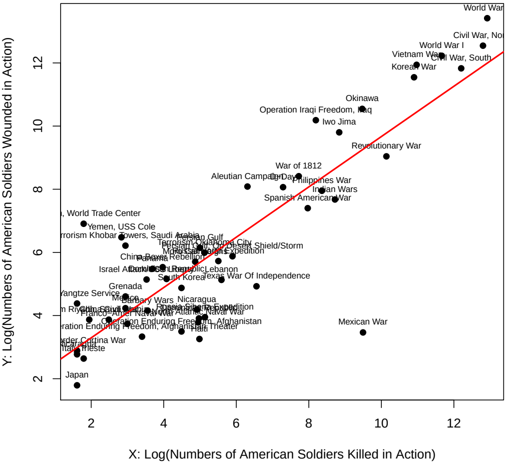
$\hat{\beta}_1 = 0.0000237 \longrightarrow$

Regression: Log-Level



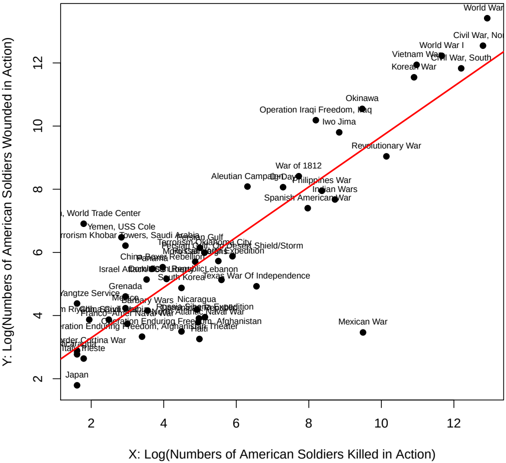
$\hat{\beta}_1 = 0.0000237 \longrightarrow$ One additional soldier killed predicts 0.0023 percent increase in the number of soldiers wounded on average

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$

Regression: Log-Log



$\hat{\beta}_1 = 0.797 \longrightarrow$ A percent increase in deaths predicts 0.797 percent increase in the wounded on average

Why r^2?

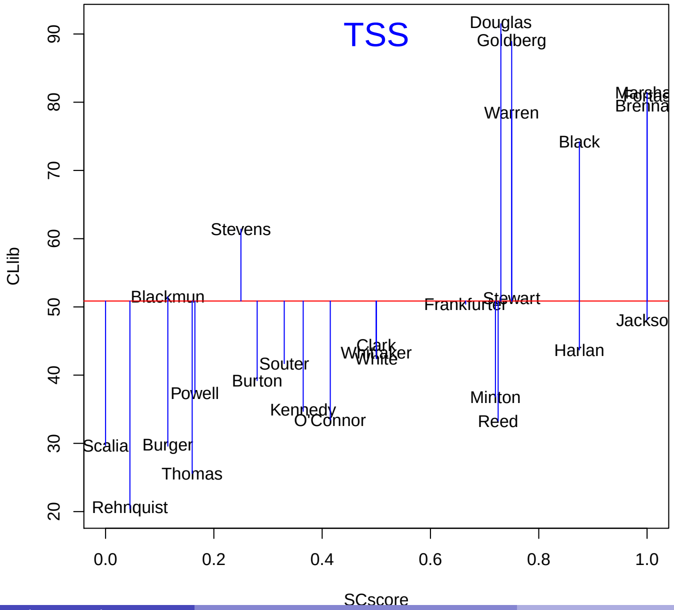
To calculate r^2, we need to think about the following two quantities:

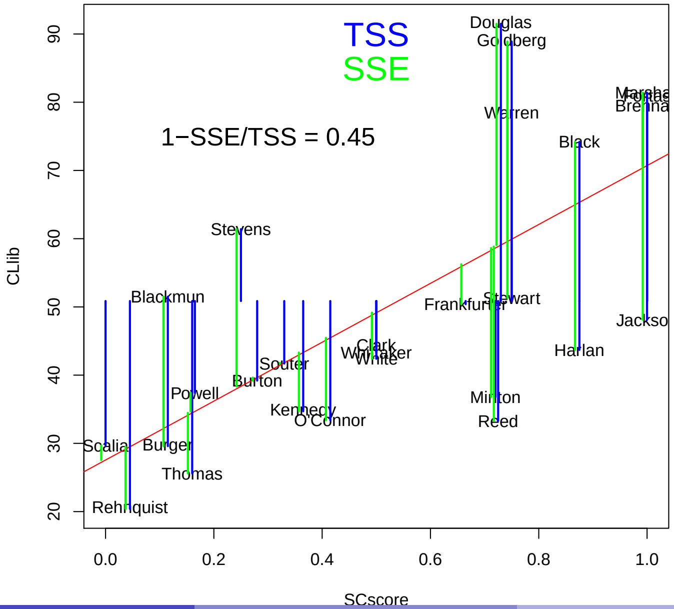
1. TSS: Total sum of squares
2. SSE: Sum of squared errors

$$TSS = \sum_{i=1}^{n}(y_i - \bar{y})^2.$$

$$SSE = \sum_{i=1}^{n} u_i^2.$$

$$r^2 = 1 - \frac{SSE}{TSS}.$$


TSS
CLlib
SCscore
0.0
0.2
0.4
0.6
0.8
1.0
20
30
40
50
60
70
80
90
Douglas
Goldberg
Warren
Black
Stevens
Blackmun
Frankfurter
Stewart
Jackso
Harlan
Clark
Whittaker
White
Souter
Burton
Powell
Kennedy
O'Connor
Minton
Reed
Scalia
Burger
Thomas
Rehnquist


TSS
SSE
1−SSE/TSS = 0.45
Douglas
Goldberg
Warren
Black
Stevens
Blackmun
Frankfurter
Stewart
Jackson
Harlan
Clark
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0.0
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20
30
40
50
60
70
80
90

Derivation

$$\sum_{i=1}^{n}(y_i - \bar{y})^2 = \sum_{i=1}^{n}\{\hat{u}_i + (\widehat{y}_i - \bar{y})\}^2$$

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Derivation

$$\begin{aligned}\sum_{i=1}^{n}(y_i - \bar{y})^2 &= \sum_{i=1}^{n}\{\hat{u}_i + (\widehat{y}_i - \bar{y})\}^2 \\ &= \sum_{i=1}^{n}\{\hat{u}_i^2 + 2\hat{u}_i(\widehat{y}_i - \bar{y}) + (\widehat{y}_i - \bar{y})^2\} \\ &= \sum_{i=1}^{n}\hat{u}_i^2 + 2\sum_{i=1}^{n}\hat{u}_i(\widehat{y}_i - \bar{y}) + \sum_{i=1}^{n}(\widehat{y}_i - \bar{y})^2\end{aligned}$$

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r^2 is a measure of how much of the variation in Y is accounted for by X.

References

Acemoglu, Daron, Simon Johnson, and James A. Robinson. "The colonial origins of comparative development: An empirical investigation." 2000.

Wooldridge, Jeffrey. 2000. *Introductory Econometrics.* New York: South-Western.