Week 8: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

Brandon Stewart[1]

Princeton

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[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller, Erin Hartman and Kevin Quinn.

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This "Week"
  - Monday (5):
    - unusual and influential data → robust estimation
  - Wednesday (7):
    - non-linearity → generalized additive models
  - Monday (12):
    - unusual errors → sandwich SEs
- Next Week
  - regression in social science
- Long Run
  - probability → inference → regression → causal inference

Questions?

Argument for Next Three Classes

Residuals are important. Look at them.

Review of the OLS assumptions

1. Linearity: $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (K+1)$ matrix with rank $K+1$
4. Zero conditional mean: $\mathbb{E}[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2\mathbf{I}_n$
6. Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2\mathbf{I}_n)$

- 1-4 give us unbiasedness/consistency
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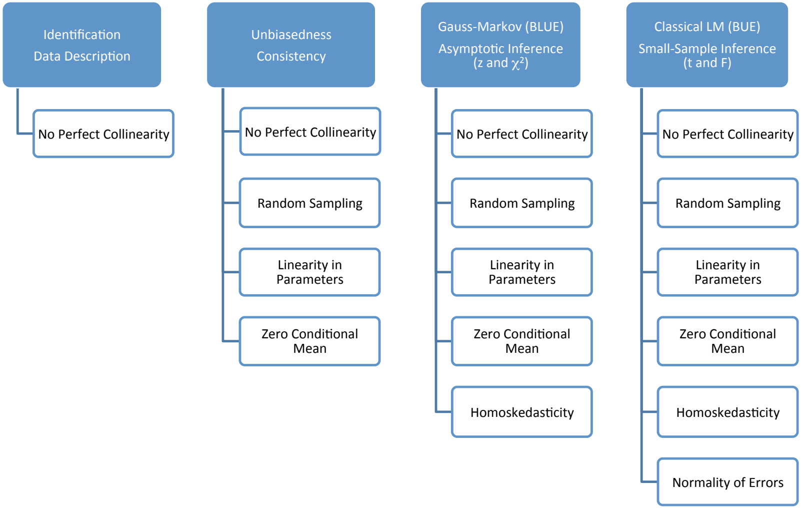
Let's talk about what's at stake in diagnostics under different views of what regression is doing.

Review of the OLS Assumptions



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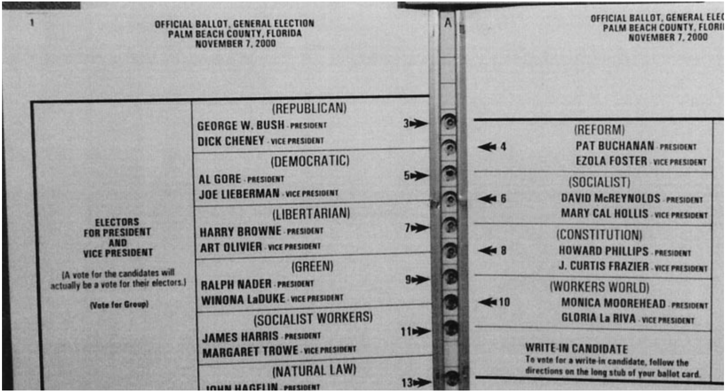
Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL's 25 electoral votes and the presidency.

FIGURE 1. The Palm Beach County Bufferfly Ballot



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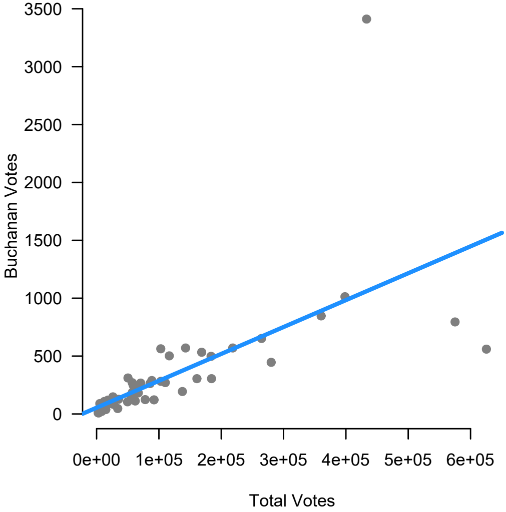

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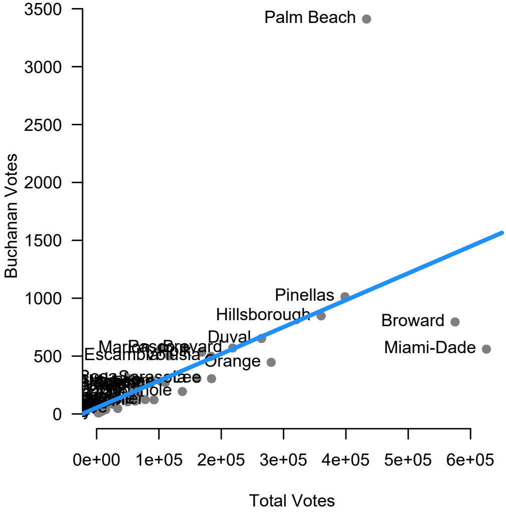

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- Fix $\mathbf{x}_i'$ and the distribution of errors should be Normal

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.

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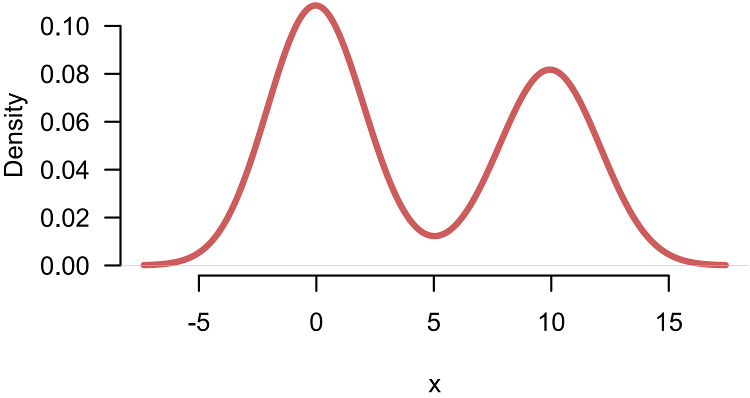
- The marginal distribution of y can be non-Normal even if the conditional distribution is Normal!
- The plausibility depends on the X chosen by the researcher.

Example: Is this a violation?



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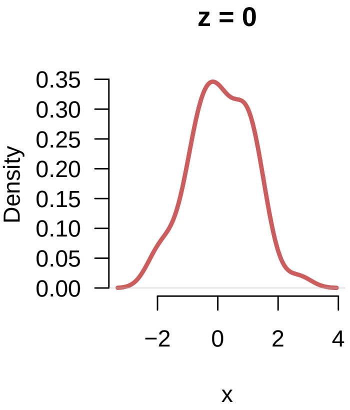

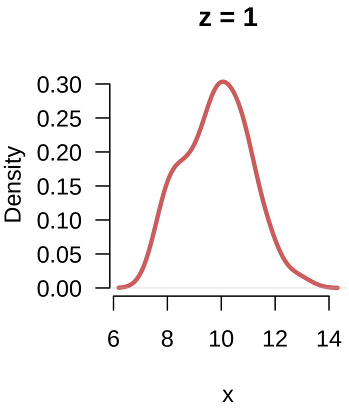

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Solution: Carefully investigate the residuals numerically and graphically.

To understand the relationship between residuals and errors, we need to derive the distribution of the residuals (which we will do over the next few slides).

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  - $\mathbf{H}$ is idempotent: $\mathbf{H}\mathbf{H} = \mathbf{H}$

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With the hat matrix, we are ready to relate the residuals to the errors.

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- Note that each residual is a function of all of the errors.

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The variance of the ith residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the ith diagonal element of the matrix $\mathbf{H}$ (called the hat value).

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What if we could transform the residuals to address the two issues above?

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where $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ and $\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$ is our usual estimate of the error variance.

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}_i'$ are not independent. This makes the distribution of $\hat{u}_i'$ nonstandard. If the distribution is non-standard, we can't easily check for violations.

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- Deviations from this t distribution of the residuals imply violation of Normality in the errors.

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- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

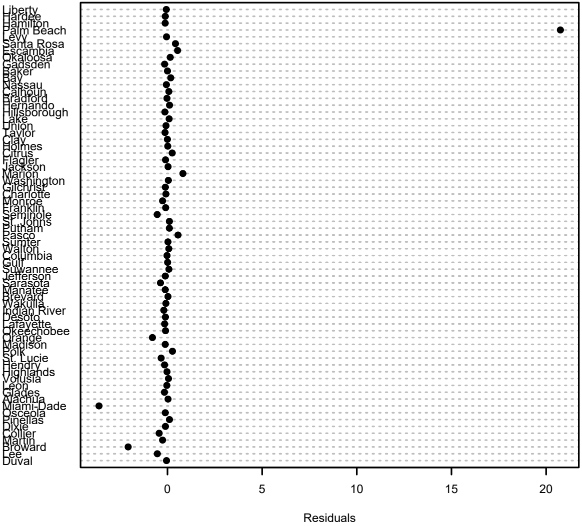
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min      1Q  Median      3Q     Max 
-947.05  -41.74  -19.47   20.20 2350.54 

Coefficients:
              Estimate Std. Error t value Pr(>|t|)    
(Intercept)  5.423e+01  4.914e+01   1.104    0.274    
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared: 0.4628,       Adjusted R-squared: 0.4545 
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

> residuals             <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals  <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals



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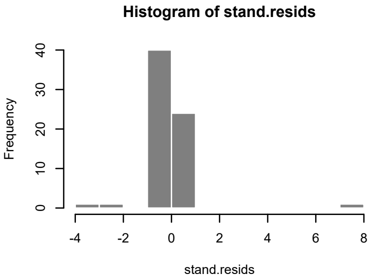

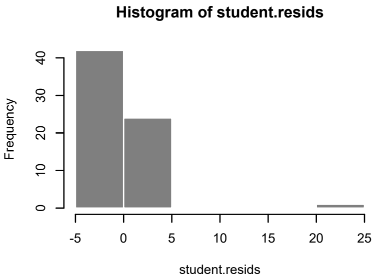

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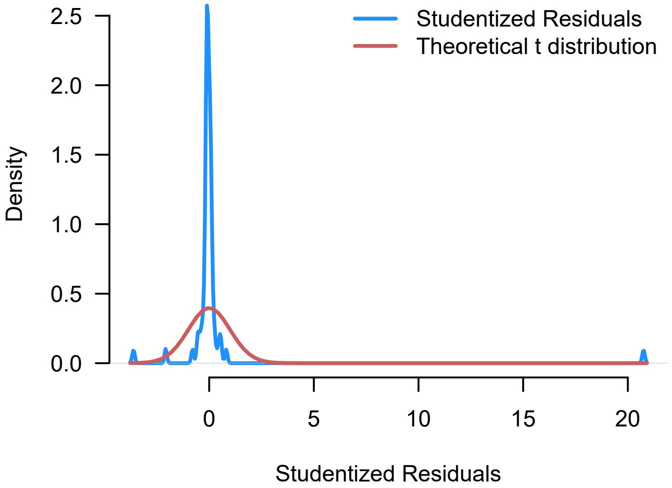

Quantile-Quantile plots

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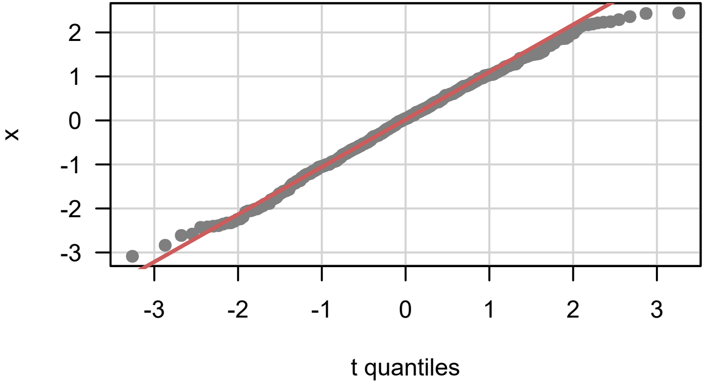
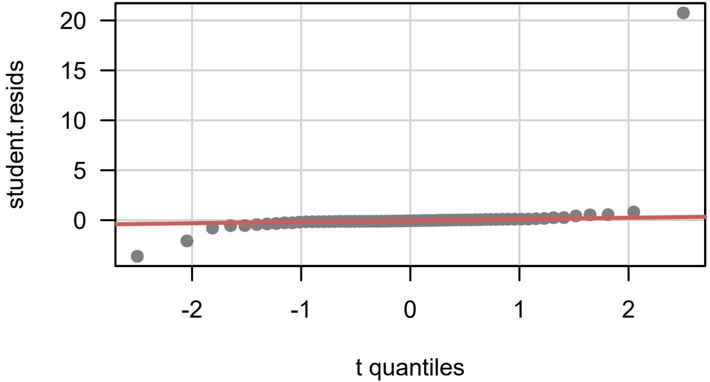
- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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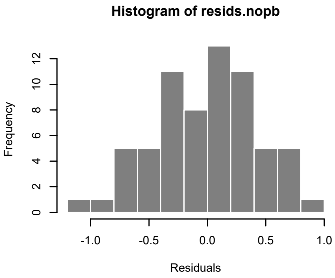
- Use estimators other than OLS that are robust to nonnormality (later this class)
- Consider other causes (next two classes)

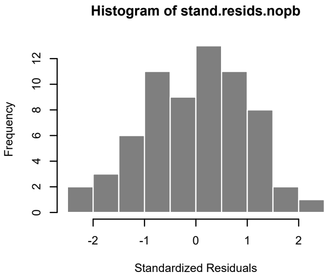
Buchanan revisited

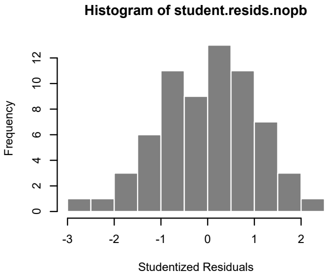
Let's delete Palm Beach and also use log transformations for both variables

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -2.48597    0.37889  -6.561 1.09e-08 ***
## log(edaytotal)  0.70311    0.03621  19.417  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4362 on 64 degrees of freedom
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526
## F-statistic:   377 on 1 and 64 DF,  p-value: < 2.2e-16
```

Buchanan revisited







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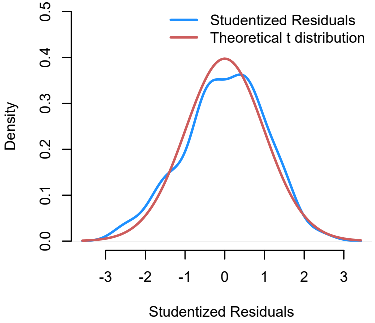

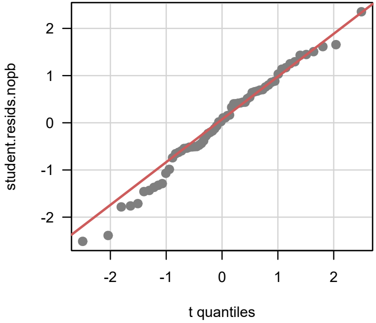

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- Jensen's inequality gives us information on this relation: $f(E[X]) \leq E[f(X)]$ for any convex function $f()$
- The results will in general be consistent which ensures that the bias decreases in sample size.

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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814	-.192	-.278	.137
	(4.7)	(2.0)	(2.4)	(2.9)
Norway Obs Excluded	.641	-.068	-.138	.054
	(4.8)	(0.9)	(1.5)	(1.3)

Creative curve fitting with Norway

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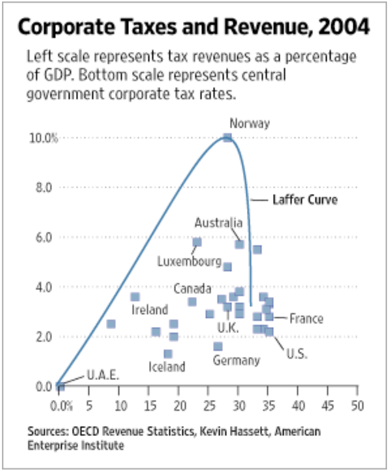

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Always Carefully Examine the Data First!!

1. Examine summary statistics: `summary(data)`
2. Scatterplot matrix for densities and bivariate relationships:
   E.g. `scatterplotMatrix(data)` from `car` library.
3. Further conditional plots for multivariate data:
   E.g. `ggplot2`

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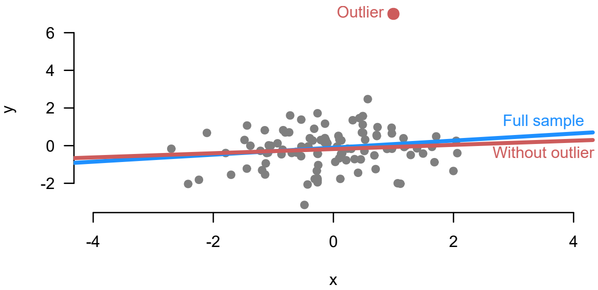
- Can be a violation of iid (not identically distributed)

Outlier definition



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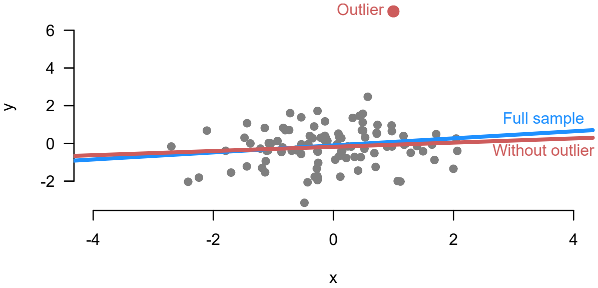

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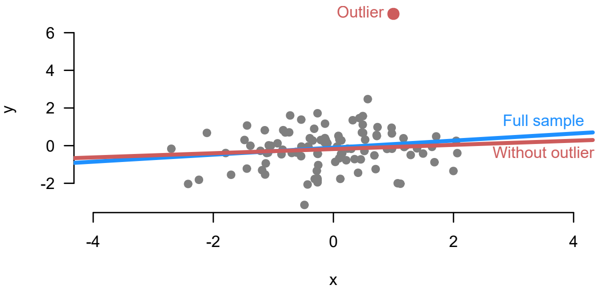

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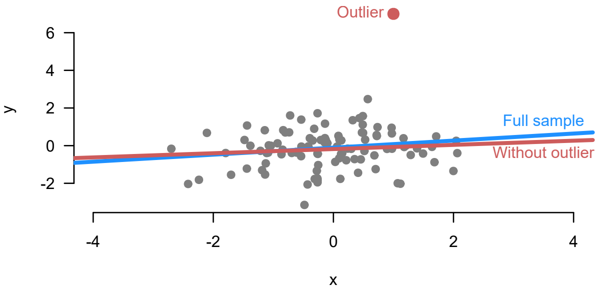

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- No bias if typical in the x's

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- $\widehat{\sigma} > \widehat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\widehat{u}_i' < \widehat{u}_i^*$

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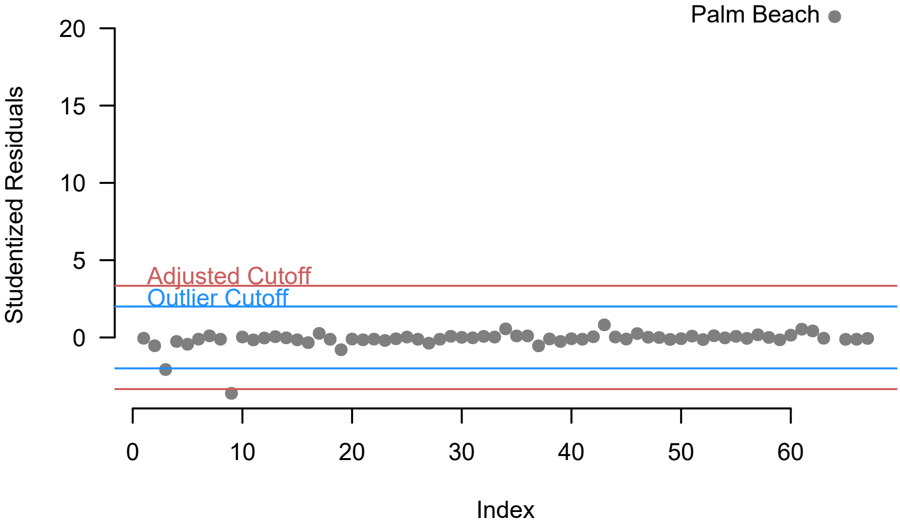
- People usually adjust cutoff for multiple testing

Buchanan outliers



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  - Use a method that is robust to outliers (robust regression)

A Cautionary Tale: The "Discovery" of the Ozone Hole

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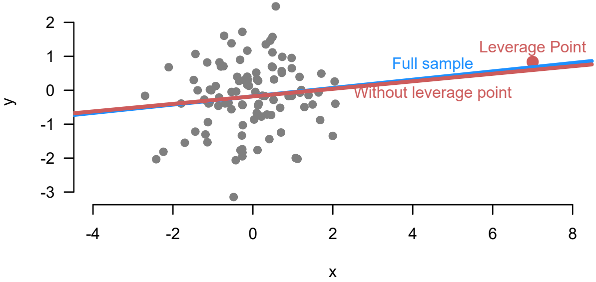
- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

Leverage point definition



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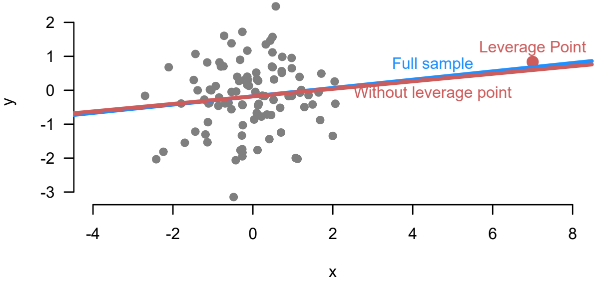

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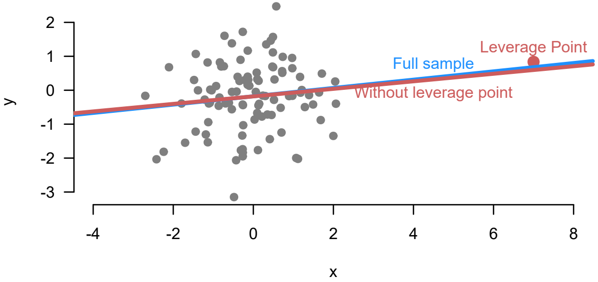

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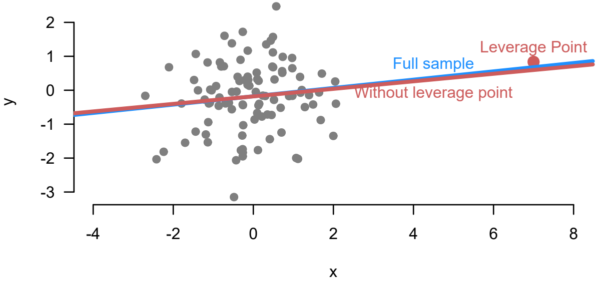

- Values that are extreme in the x direction
- That is, values far from the center of the covariate distribution
- Decrease SEs (more X variation)
- No bias if typical in y dimension

Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix **H**:

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}$$

H is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

$$\hat{y}_i = \mathbf{h}_i'\mathbf{y} = \begin{bmatrix} h_{i,1} & h_{i,2} & \cdots & h_{i,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^{n} h_{i,j} y_j$$

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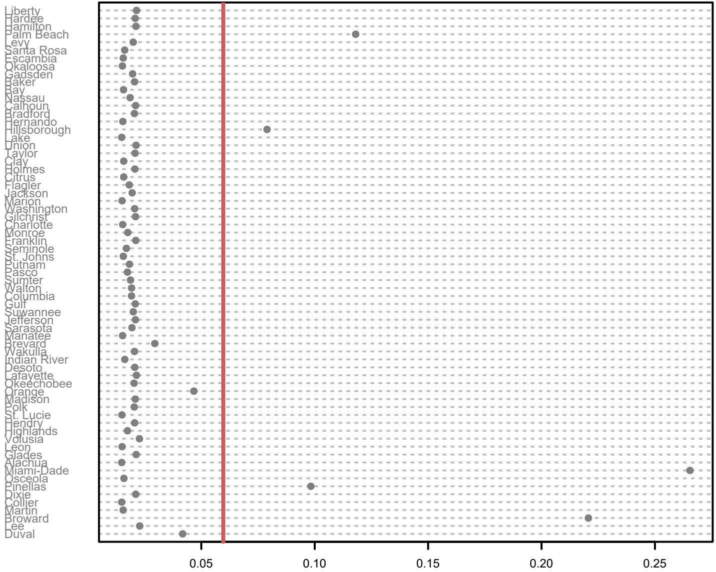
- Intuitively, the hat values measure how far a unit's vector of characteristics $\mathbf{x}_i$ is from the vector of means of $\mathbf{X}$
- Rule of thumb: examine hat values greater than $2(k+1)/n$

Buchanan hats



Influence points

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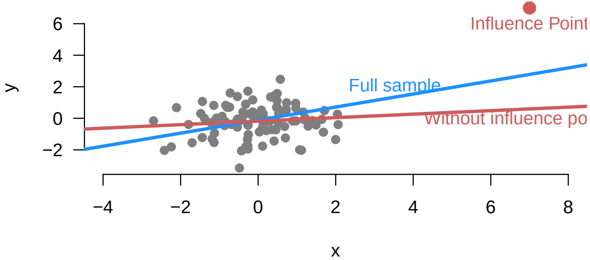

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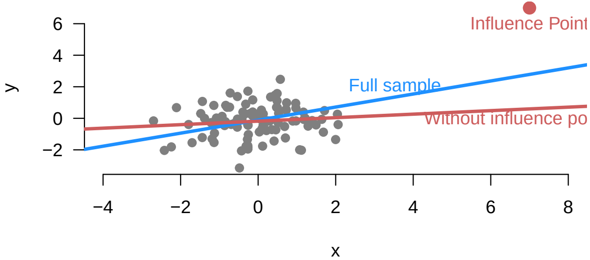

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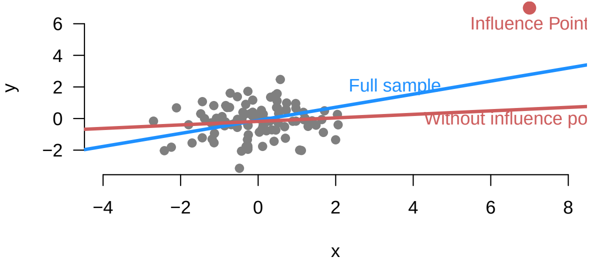

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- Causes the regression line to move toward it (bias?)

Detecting Influence Points/Bad Leverage Points

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  Influence on coefficients $=$ Leverage $\times$ Outlyingness

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- More formally: Measure the change that occurs in the slope estimates when an observation is removed from the data set. Let

$$D_{ij} = \hat{\beta}_j - \hat{\beta}_{j(-i)}, \quad i = 1, \ldots, n, \quad j = 0, \ldots, k$$

  where $\hat{\beta}_{j(-i)}$ is the estimate of the jth coefficient from the same regression once observation i has been removed from the data set.

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- D_{ij} is called the DFbeta, which measures the influence of observation i on the estimated coefficient for the jth explanatory variable.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

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- In R: `dfbetas(model)`

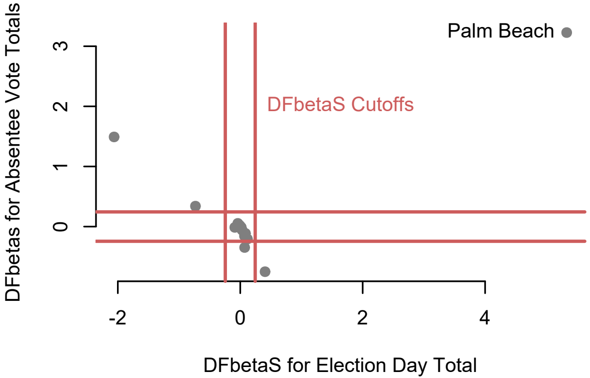
Buchanan influence

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal     1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
##   (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

```
##     (Intercept)     edaytotal  absnbuchanan
## 1  0.3454475146  0.4050504921 -0.7505222758
## 2 -0.0234266617 -0.0241000045 -0.0131672181
## 3  0.0650795039 -0.7319311820  0.3401669862
## 4 -0.0333980968  0.0133802934 -0.0087505576
## 5 -0.0397626659 -0.0073746223  0.0096551713
## 6 -0.0009277798  0.0001505476  0.0002210247
```

Buchanan influence



- Palm Beach county moves each of the coefficients by more than 3 standard errors!

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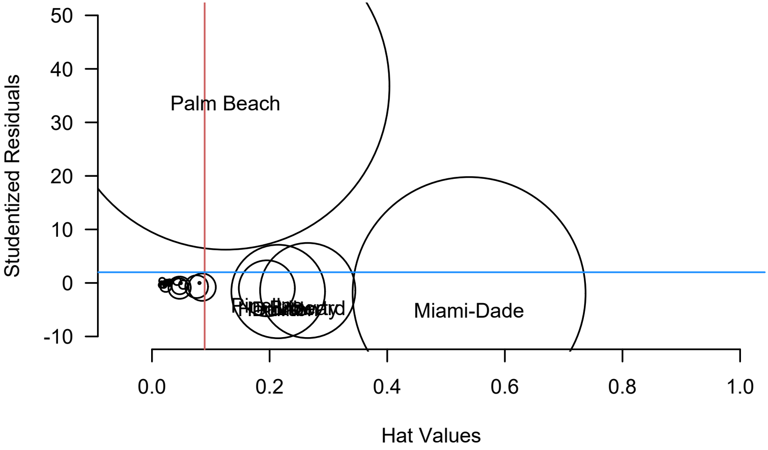
Influence Plot Buchanan



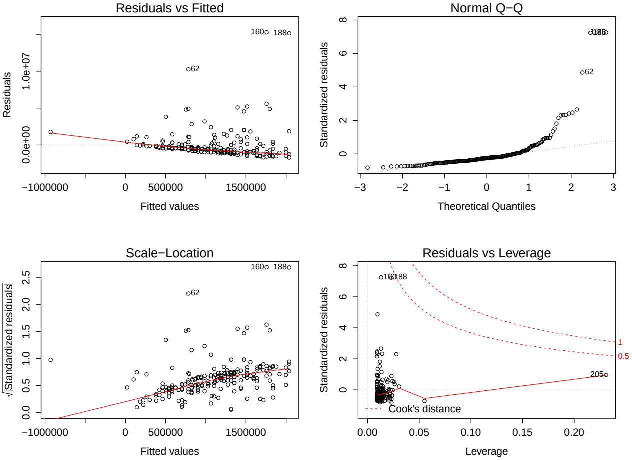
Code for Influence Plot

```
mod3 <- lm(edaybuchanan ~ edaytotal + absnbuchanan, data = flvote)
symbols(y = rstudent(mod3), x = hatvalues(mod3),
            circles = sqrt(cooks.distance(mod3)),
            ylab = "Studentized Residuals",
            xlab = "Hat Values", xlim = c(-0.05, 1),
            ylim = c(-10, 50), las = 1, bty = "n")
cutoffstud <- 2
cutoffhat <- 2 * (3)/nrow(flvote)
abline(v = cutoffhat, col = "indianred")
abline(h = cutoffstud, col = "dodgerblue")
filter <- rstudent(mod3) > cutoffstud | hatvalues(mod3) > cutoffhat
text(y = rstudent(mod3)[filter],
       x = hatvalues(mod3)[filter],
       flvote$county[filter], pos = 1)
```

A Quick Function for Standard Diagnostic Plots

R Code

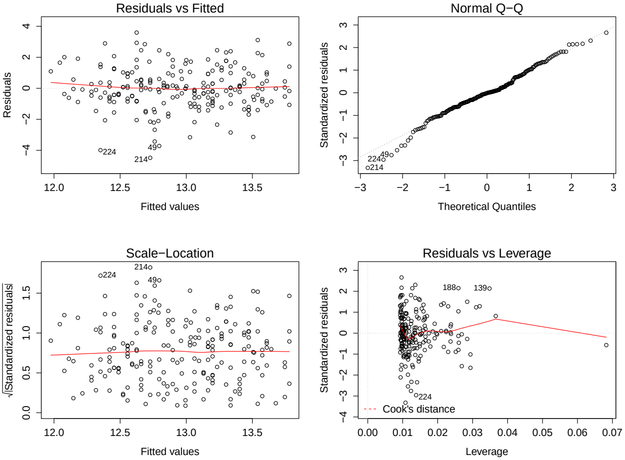
```
> par(mfrow=c(2,2))
> plot(mod1)
```



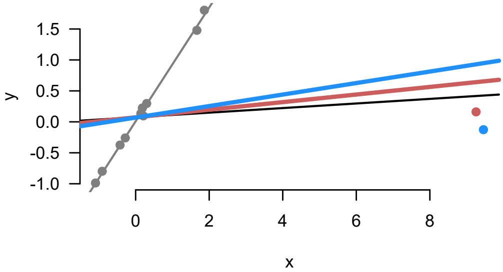
The Improved Model

R Code

```
> par(mfrow=c(2,2))
> plot(mod2)
```



Limitations of the standard tools



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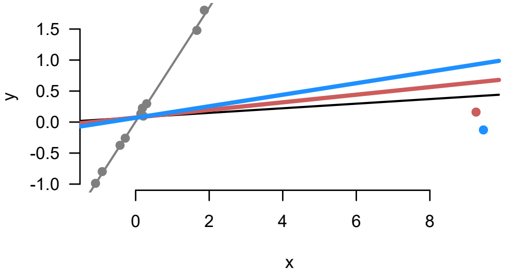

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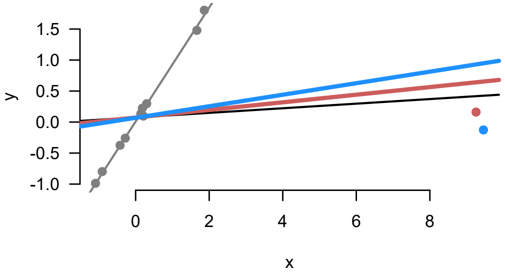

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point
- Neither of the "leave-one-out" approaches helps recover the line

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- The Linear point is an artificial restriction. It means the estimator has to be of the form $\hat{\beta} = \mathbf{W}y$ but why only use those?
- With normality assumption we get Best Unbiased Estimator (BUE) which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).

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"[Even without normally distributed errors] OLS coefficient estimators remain unbiased and efficient." - Berry (1993)

Quotes from Rainey and Baissa (2015) presentation

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> *"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators" - Wooldridge (2013)*

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"We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator"
- Gujarati (2004)

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This flies in the face of most conventional wisdom in textbooks.

> *"The Gauss-Markov theorem allows us to have considerable confidence in the least squares estimators." - Berry and Feldman (1993)*

Quotes from Rainey and Baissa (2015) presentation

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- We can measure sensitivity with the influence function which measures change in estimator based on corruption in one datapoint.

Influence Function

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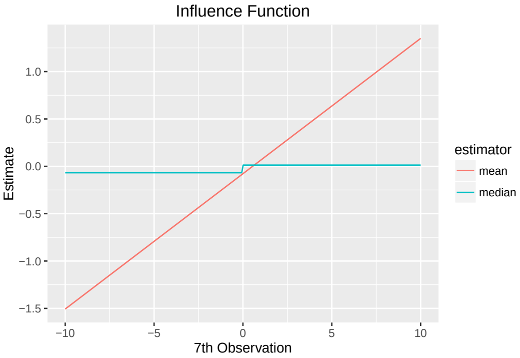

Example from Fox

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- The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.

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- Other objectives include the Huber objective and Tukey's biweight objective which have different properties.
- Calculating robust M estimators often requires an iterative procedure and a careful initialization.

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  - Least Trimmed Squares: choose $\hat{\beta}$ to minimize the sum of the p smallest elements of $\left\{(y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}_{\text{LTS}})^2\right\}_{i=1}^{n}$. High breakdown point and more efficient, still not as efficient as some.

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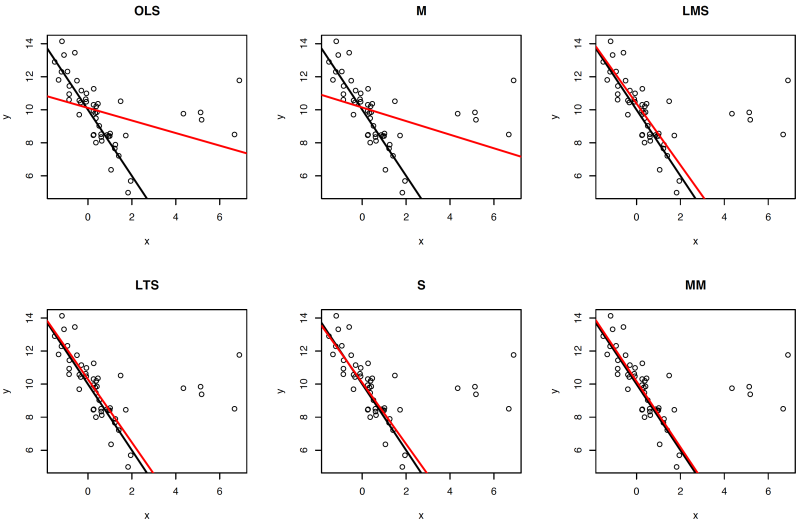
  - *MM*-estimator: with Huber's loss is what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for M-estimators but I would bootstrap it if possible as the asymptotics kick in slowly.

```
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 +  rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
```

Simulation Results



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- In most cases I personally would start with OLS, do diagnostics and then consider a robust alternative. If I don't have time for diagnostics, maybe robust is better from the outset.
- I highly recommend Baissa and Rainey (2016) "When BLUE is Not Best: Non-Normal Errors and the Linear Model" for more on this topic.
- The Fox textbook Chapter 19 is also quite good on this and points out to the key references

Concluding Thoughts for the Day

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- Easy to test some of these and hard to test others.
- Always check your data!
- Don't let regression be a magic black box for you- understand what is in your data that is leading to the findings.

Fun With Outliers

Prostitution and the sex discrepancy in reported number of sexual partners

Devon D. Brewer[*,†], John J. Potterat[‡], Sharon B. Garrett[*], Stephen Q. Muth[‡], John M. Roberts, Jr.[§], Danuta Kasprzyk[¶], Daniel E. Montano[¶], and William W. Darrow[‖]

[*]Alcohol and Drug Abuse Institute, University of Washington, 3937 15th Avenue NE, Seattle, WA 98105; [‡]El Paso County Department of Health and Environment, 301 South Union Boulevard, Colorado Springs, CO 80910; [§]Department of Sociology, University of New Mexico, Albuquerque, NM 87131; [¶]Centers for Public Health Research and Evaluation, Battelle Memorial Institute, 4000 NE 41st Street, P.O. Box 5395, Seattle, WA 98105-5395; [‖]Department of Public Health, Florida International University, 3000 NE 145th Street, ACI-394F, North Miami, FL 33181

Communicated by A. Kimball Romney, University of California, Irvine, CA, August 16, 2000 (received for review June 21, 2000)

One of the most reliable and perplexing findings from surveys of sexual behavior is that men report substantially more sexual partners than women do. We use data from national sex surveys and studies of prostitutes and their clients in the United States to examine sampling bias as an explanation for this disparity. We find that prostitute women are underrepresented in the national surveys. Once their undersampling and very high numbers of sexual partners are factored in, the discrepancy disappears. Prostitution's role in the discrepancy is not readily apparent because men are reluctant to acknowledge that their reported partners include prostitutes.

Our analyses of the GSS are based on data from 1988 to 1991 combined. Overall sample sizes are 5,907 for the GSS and 3,159 for the NHSLS.

Analysis and Results. Because of slight differences in the numbers of men and women in the population at large and differences in the proportions of men and women who are heterosexual, estimates must be obtained at the United States population level rather than simply by relying on the surveys' sample means for men's and women's numbers of partners. (Detailed calculations

Thanks to Matt Salganik for pointing me to this example

AMERICAN SOCIOLOGICAL ASSOCIATION

Does Diversity Pay? A Replication of Herring (2009)

American Sociological Review
2017, Vol. 82(4) 857–867

DOI: 10.1177/0003122417714422
journals.sagepub.com/home/asr
SAGE

Dragana Stojmenovska,[a] Thijs Bol,[a] and Thomas Leopold[a]

Abstract

In an influential article published in the *American Sociological Review* in 2009, Herring finds that diverse workforces are beneficial for business. His analysis supports seven out of eight hypotheses on the positive effects of gender and racial diversity on sales revenue, number of customers, perceived relative market share, and perceived relative profitability. This comment points out that Herring's analysis contains two errors. First, missing codes on the outcome variables are treated as substantive codes. Second, two control variables—company size and establishment size—are highly skewed, and this skew obscures their positive associations with the predictor and outcome variables. We replicate Herring's analysis correcting for both errors. The findings support only one of the original eight hypotheses, suggesting that diversity is nonconsequential, rather than beneficial, to business success.

In our correspondence with Herring, he did not offer a definitive explanation for these discrepancies, but indicated that he may have treated all codes other than "not applicable" (−999) as substantive codes. Given (1) the large difference between his sample size and the number of valid observations in the NOS, and (2) the large number of missing values due to reasons other than "not applicable"—in particular for sales revenue and number of customers—this coding error appears likely to account for much of the discrepancies. This means, for example, that 206 business organizations in which the sales revenue was unknown were treated as if they had sales of 88,888,888,888 US Dollars. Yet, even when we replicated this error (i.e., keeping all organizations with missing values other than −999 in our sample), we were unable to recover Herring's sample sizes, although the differences were smaller.

Appendix: Characterizing Estimator Robustness (formally)

Definition (Breakdown Point)

The breakdown point of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82)

Definition (Influence Function)

Let $F_p = (1-p)F + p\delta_{\mathbf{z}_0}$ where F is a probability measure, $\delta_{\mathbf{z}_0}$ is the point mass at $\mathbf{z}_0 \in \mathbb{R}^k$, and $p \in (0,1)$.
Let $T(\cdot)$ be a statistical functional. The influence function of T is

$$IF(\mathbf{z}_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}$$

The influence function is a function of $\mathbf{z}_0$ given T and F. It describes how T changes with small amounts of contamination at $\mathbf{z}_0$ (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).

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$$\frac{1}{n}\sum_{i=1}^{n}\rho\left(\frac{y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}_S}{s}\right) \geq K$$

where K is user-defined constant (typically set to 0.5) and $\rho : \mathbb{R} \rightarrow [0,1]$ is a function with the following properties (Davies, 1990, p. 1653):

1. $\rho(0) = 1$
2. $\rho(u) = \rho(-u), u \in \mathbb{R}$
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Good properties, but costly to compute (usually impossible to compute exactly).

References

- Wand, Jonathan N., Kenneth W. Shotts, Jasjeet S. Sekhon, Walter R. Mebane Jr, Michael C. Herron, and Henry E. Brady. "The butterfly did it: The aberrant vote for Buchanan in Palm Beach County, Florida." *American Political Science Review* (2001): 793-810.
- Lange, Peter, and Geoffrey Garrett. "The politics of growth: Strategic interaction and economic performance in the advanced industrial democracies, 19741980." *The Journal of Politics* 47, no. 03 (1985): 791-827.
- Jackman, Robert W. "The Politics of Economic Growth in the Industrial Democracies, 197480: Leftist Strength or North Sea Oil?." The *Journal of Politics* 49, no. 01 (1987): 242-256.

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This "Week"
  - Monday (5):
    - unusual and influential data → robust estimation
  - Wednesday (7):
    - non-linearity → generalized additive models
  - Monday (12):
    - unusual errors → sandwich SEs
- Next Week
  - regression in social science
- Long Run
  - probability → inference → regression → causal inference

Questions?

Residuals are still important. Look at them.

Nonlinearity

Linearity of the Conditional Expectation Function ($\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$) is a key assumption. Why?

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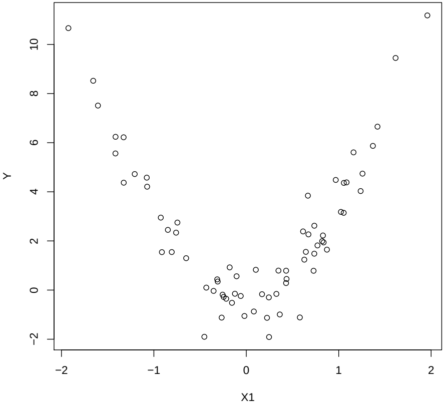

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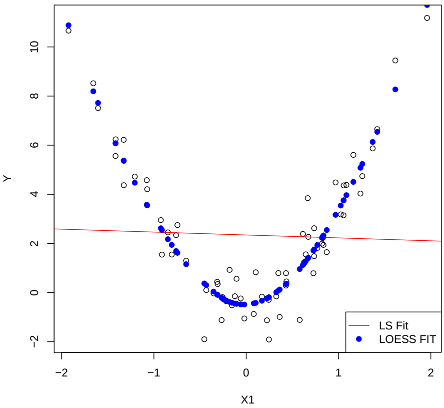

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- Usually we employ "linearity by default" but we should try to make sure this is appropriate: detect non-linearities and model them accurately

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  - Non-parametric multiple regression techniques (beyond the scope of this course)

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- OLS fit to this plot will have exactly $\widehat{\beta}_j$ and 0 intercept (drawing on the partialing out interpretation we discussed before)

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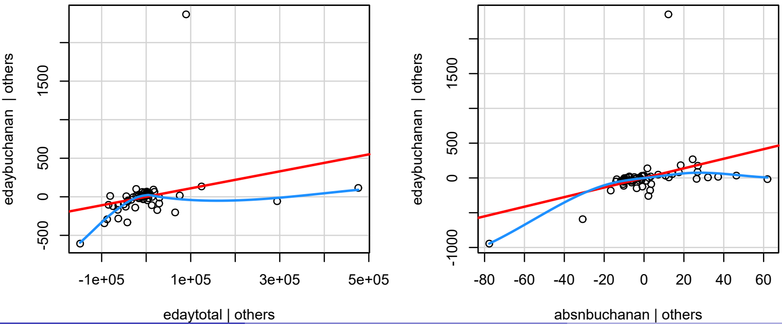
  3. Plot residuals from (1) against residuals from (2)
- In R: `avPlots(model)` from the `car` package
- OLS fit to this plot will have exactly $\widehat{\beta}_j$ and 0 intercept (drawing on the partialing out interpretation we discussed before)
- Use local smoother (loess) to detect any non-linearity

Buchanan AV plot

R Code

```
par(mfrow = c(1,2))
out <- avPlots(mod3, "edaytotal")
lines(loess.smooth(x = out$edaytotal[,1],
 y= out$edaytotal[,2]), col = "dodgerblue", lwd = 2)
out2 <- avPlots(mod3, "absnbuchanan")
lines(loess.smooth(x = out2$absnbuchanan[,1],
y= out2$absnbuchanan[,2]), col = "dodgerblue", lwd = 2)
```



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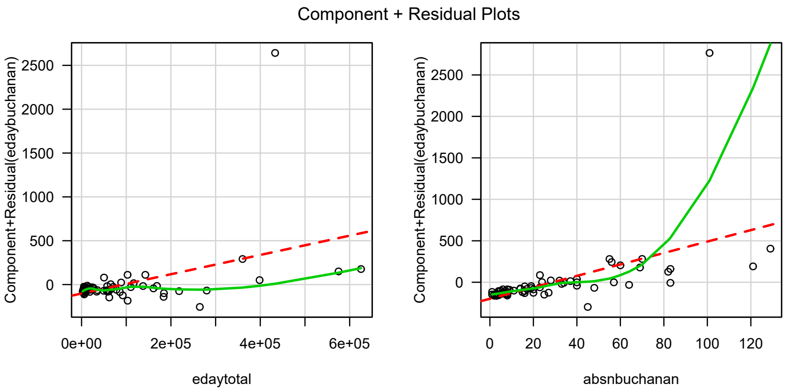
Buchanan CR plot

R Code

```
crPlots(mod3, las = 1)
```



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  - This suggests that linearizing the relationship between the Xs through transformations can be helpful
  - Experience suggests weak non-linearities among Xs do not invalidate CR plots

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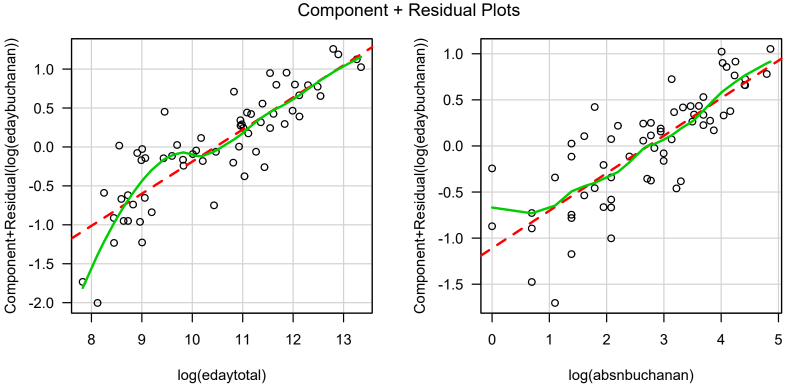
- Many more flexible, nonlinear regression models exist beyond the scope of this course.

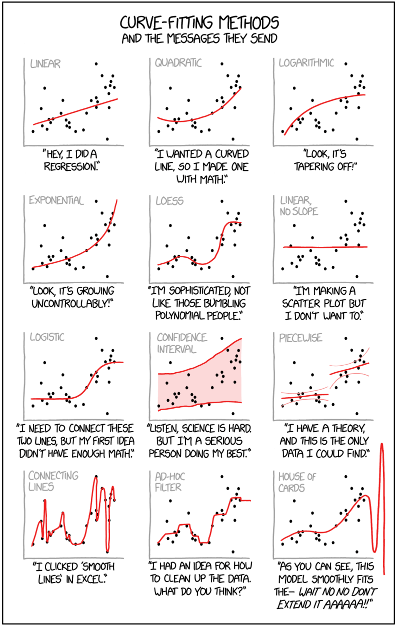
I will teach you some, but many options.

Transformed Buchanan regression

R Code

```
mod.nopb2 <- lm(log(edaybuchanan) ~ log(edaytotal) + log(absnbuchanan),
data = flvote, subset = county != "Palm Beach")
crPlots(mod.nopb2, las = 1)
```





Thanks XKCD for having a comic for everything!

Bias-Variance Tradeoff

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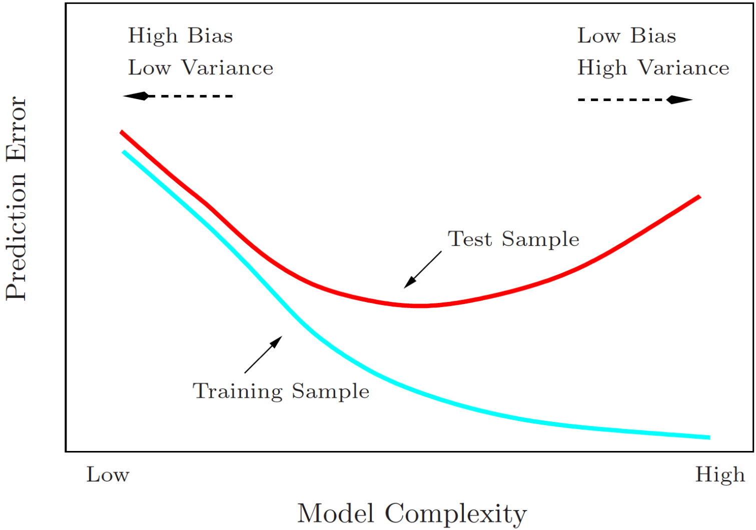
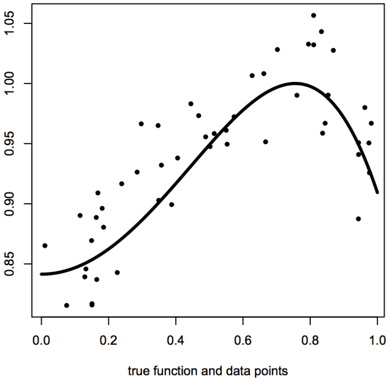


Example Synthetic Problem

$$y = \sin(1 + x^2) + \epsilon$$



This section adapted from slides by Radford Neal.

Linear Basis Function Models

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- In general the idea is to do a linear regression of y on $\phi_1(x), \phi_2(x), \ldots, \phi_{m-1}(x)$ where ϕ_j are basis functions.
- The model is now:

$$y = f(x, \beta) + \epsilon$$

$$f(x, \beta) = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$

Polynomial Basis Functions

We can look at OLS fits with polynomial basis functions of increasing order.

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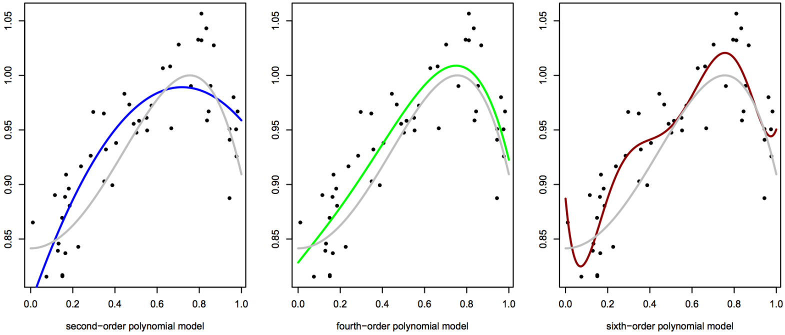

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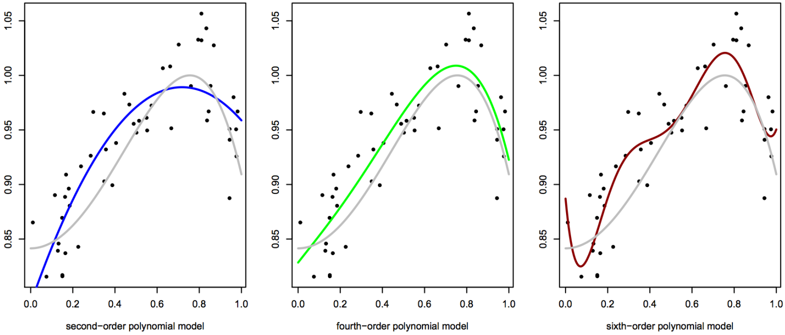

It appears that the last model is too complex and is overfitting a bit.

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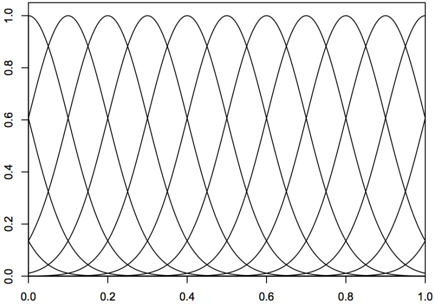
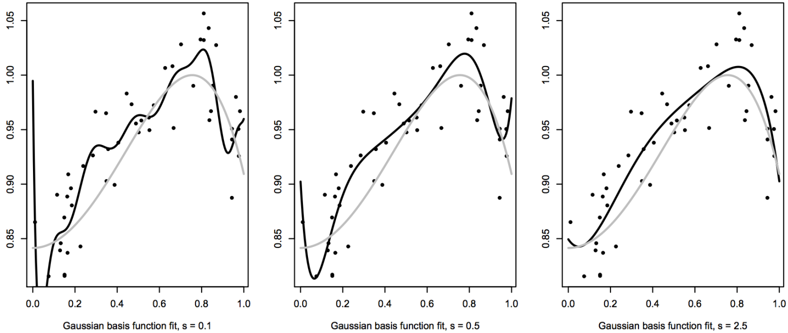


Gaussian basis functions, s = 0.1

Gaussian Basis Fits



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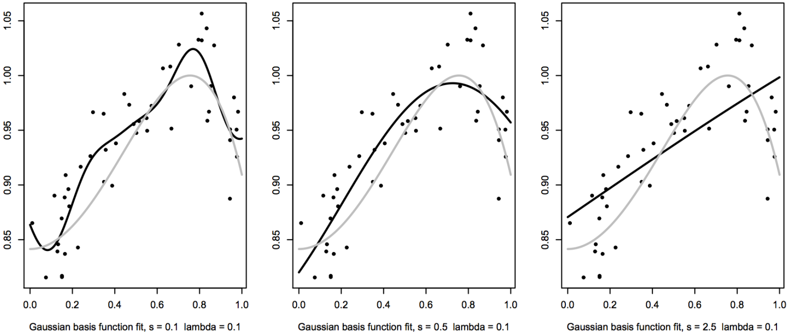
- The trick in general is how to set λ

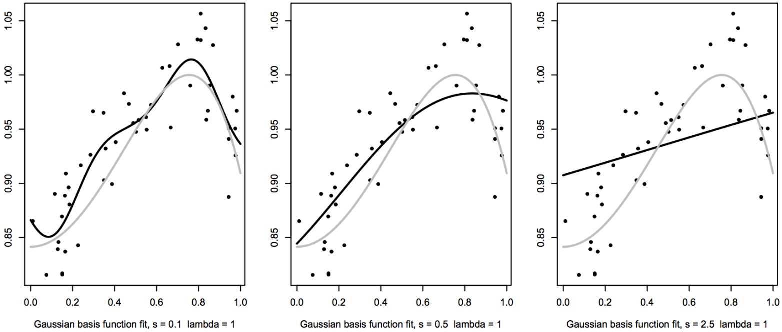
Results

Here are the results with $\lambda = 0.1$:



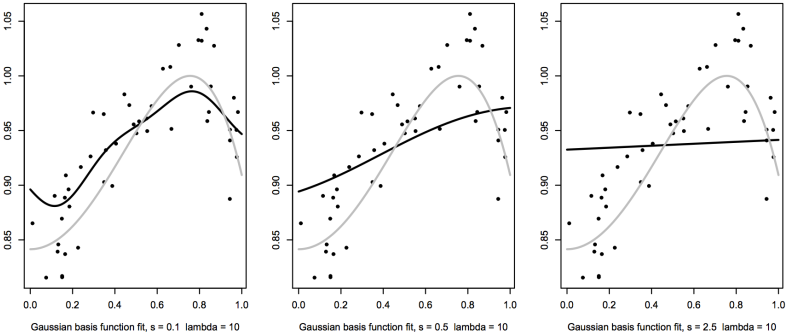
Results

Here are the results with $\lambda = 1$:



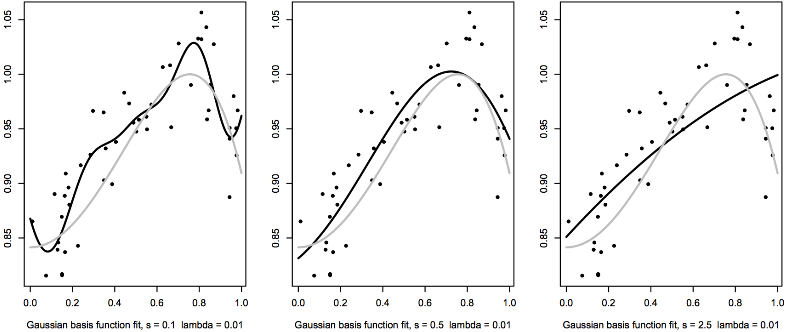
Results

Here are the results with $\lambda = 10$:



Results

Here are the results with $\lambda = 0.01$:



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- next up, Generalized Additive Models

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Recall the linear model,

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- Theory and estimation are somewhat involved, but they are easy to use:
  - `gam.out <- gam(y~s(x1)+s(x2)+x3)`
    `plot(gam.out)`
  - Multiple functions but I recommend `mgcv` package

Generalized Additive Models (GAM)

The GAM approach can be extended to allow interactions ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

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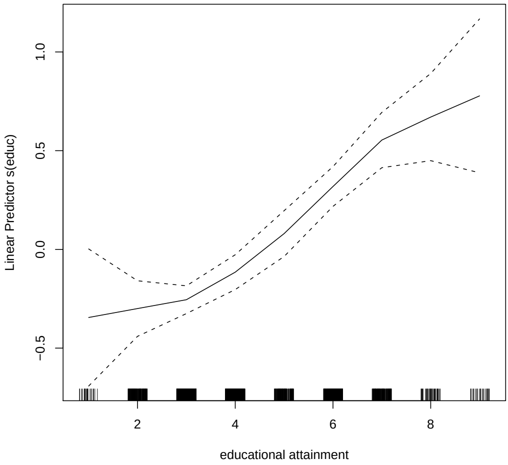
It can also be used for hybrid models where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

GAM Fit to Attitudes Toward Immigration



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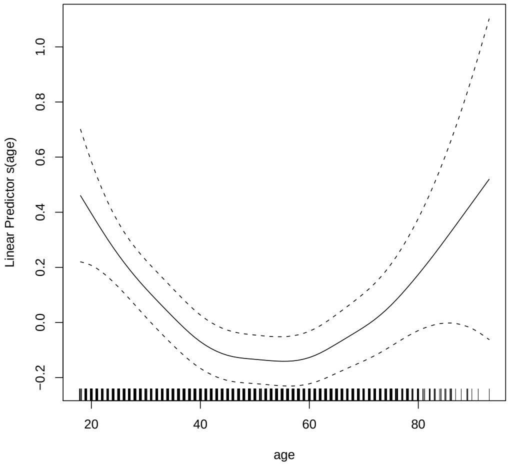

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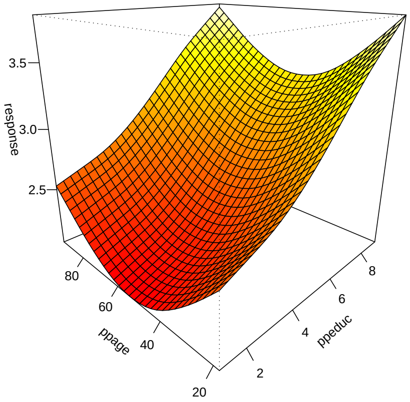

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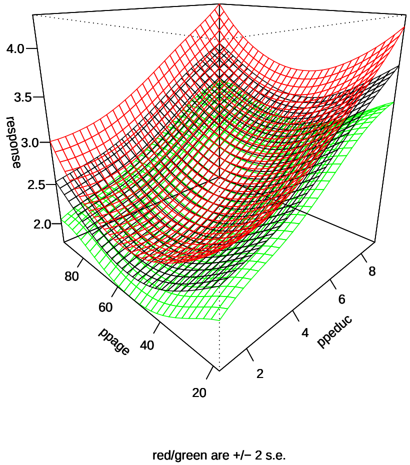

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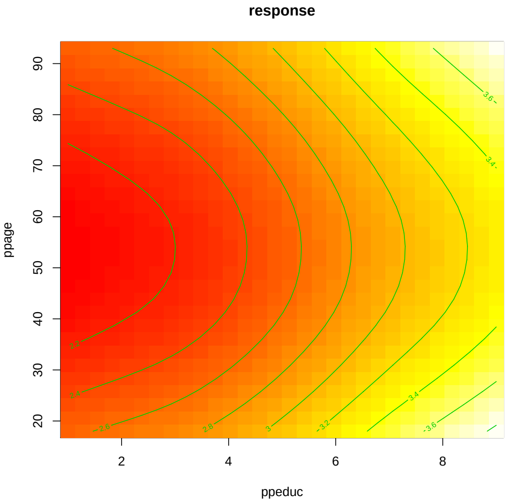
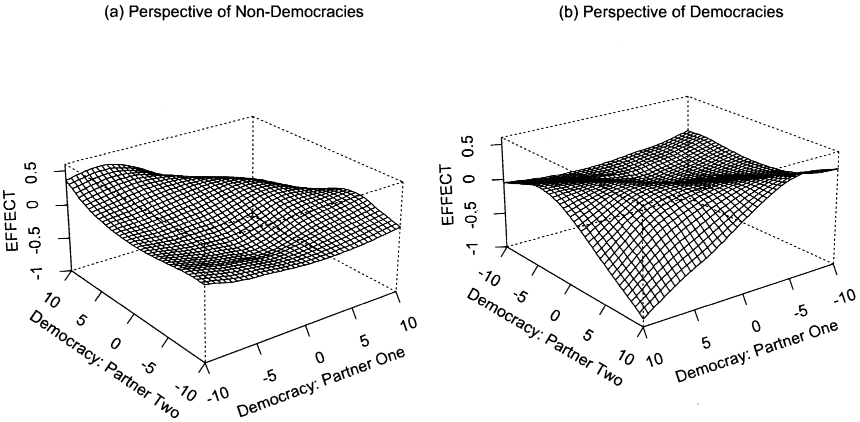


GAM Fit to Dyadic Democracy and Militarized Disputes



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- GAMs are a great way to model/detect non-linearity but transformations are often simpler
- However, be wary of the global properties of transformations and polynomials
- Non-linearity concerns are most relevant for continuous covariates with a large range (age)

Fun With Kernels

Hainmueller and Hazlett (2013). "Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach" *Political Analysis*.[2]

[2] I thank Chad Hazlett for sharing many of the slides that follow

Motivation: Misspecification Bias

Consider a data generating process such as:

```
> # Predictors
> GDP = runif(500)
> Polity  = .5*GDP^2 + .2*runif(200)
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Regressing `Stability` on `polity` and `GDP`:

```
> # OLS
> lm(Stability ~ Polity + GDP)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.3000     0.1039 -22.145  < 2e-16 ***
Polity       -3.1983     0.7613  -4.201 3.15e-05 ***
GDP           4.3443     0.4237  10.252  < 2e-16 ***
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> Polity  = .5*GDP^2 + .2*runif(200)
>
> # True Model
> Stability = log(GDP)+rnorm(500)
```

Regressing `Stability` on `polity` and `GDP`:

```
> # OLS
> lm(Stability ~ Polity + GDP)

            Estimate Std. Error t value Pr(>|t|)
(Intercept)  -2.3000     0.1039 -22.145  < 2e-16 ***
Polity       -3.1983     0.7613  -4.201 3.15e-05 ***
GDP           4.3443     0.4237  10.252  < 2e-16 ***
```

Entirely wrong conclusions!

Misspecification Bias

Try more flexible method that still reports marginal effects:

```
> krls(y=Stability,X=cbind(GDP,Polity))

Average Marginal Effects:
              Est Std. Error    t value     Pr(>|t|)
GDP     3.3855912  0.5217110  6.4893996 2.084441e-10
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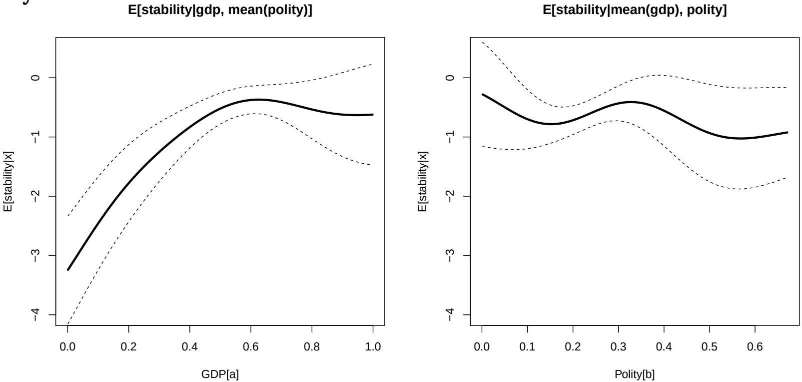

Kernel Basics

Kernel

For now, a kernel is a function $\mathbb{R}^{\mathbb{P}} \times \mathbb{R}^{\mathbb{P}} \rightarrow \mathbb{R}$

$$k(x_i, x_j) \rightarrow \mathbb{R}$$

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Some kernels are naturally interpretable as a distance metric, e.g. the Gaussian:

Gaussian Kernel

$$k(\cdot, \cdot) : \mathbb{R}^D \times \mathbb{R}^P \mapsto \mathbb{R}$$

$$k(x_j, x_i) = e^{-\frac{||x_j - x_i||^2}{\sigma^2}}$$

where $||X_j - X_i||$ is the Euclidean distance between X_j and X_i

Using the Kernel Trick for Regression

- A feature map, $\phi : \mathbb{R}^P \mapsto \mathbb{R}^{P'}$, such that: $k(X_i, X_j) = \langle \phi(X_i), \phi(X_j) \rangle$

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- Regularized (ridge) regression:

$$\underset{\theta \in \mathbb{R}^{P'}}{\text{argmin}} \sum_{i=1}^{N} (Y_i - \phi(X_i)^T \theta)^2 + \lambda \langle \theta, \theta \rangle$$

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- Solve the F.O.C.s:

$$R(\theta, \lambda) = \sum_{i=1}^{N} (Y_i - \phi(X_i)^\top \theta)^2 + \lambda \theta^\top \theta$$

$$\frac{\partial R(\theta, \lambda)}{\partial \theta} = -2 \sum_{i=1}^{N} \phi(X_i)(Y_i - \phi(X_i)^T \theta) + 2\lambda\theta = 0$$

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$$E[alt|lat, long] = \beta_0 + \beta_1 lat + \beta_2 long + \beta_3 lat \times long + \ldots$$

Similarity model:

$$E[alt|lat, long] = c_1(\text{similarity to obs1}) + \ldots + c_5(\text{similarity to obs5})$$

Intuition: Similarity

Think of this function space as built on similarity:

$$
\begin{aligned}
f(X^\star) &= \sum_{i=1}^{N} c_i k(X^\star, X_i) \\
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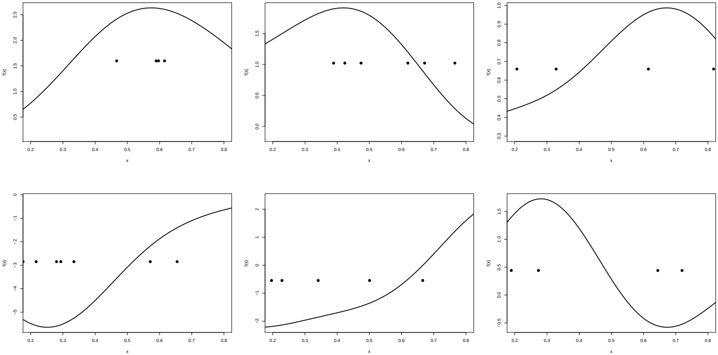
Some random functions from this space:



A real example: Harff 2003

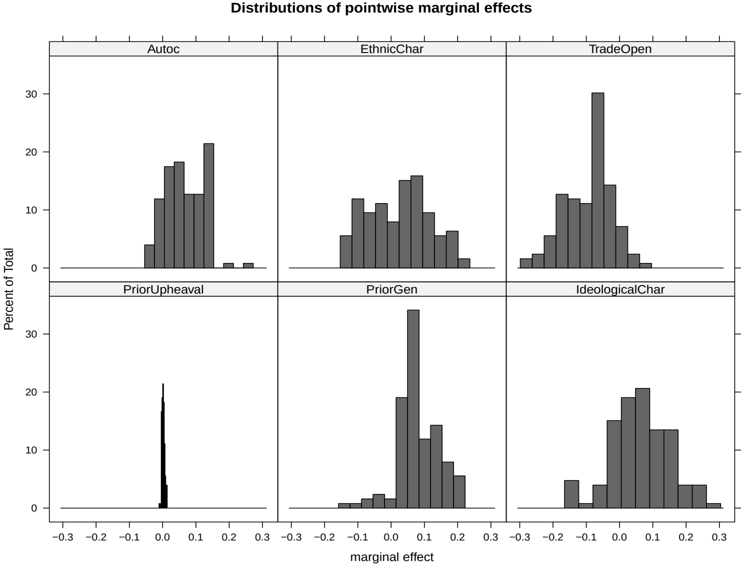
From `summary(krls(y,X))`

DV: Genocide onset

	β_{OLS}	$E[\frac{\hat{dy}}{dx_i}]$
Prior upheaval	0.009*	0.00
	(0.004)	0.00
Prior genocide	0.26*	0.19*
	(0.12)	(0.08)
Ideological char. elite	0.15*	0.13*
	(0.084)	(0.08)
Autocracy	0.16*	0.12*
	(0.077)	(0.07)
Ethnic char. elite	0.12	0.05
	(0.084)	(0.08)
log(trade openness)	-0.17*	-0.09*
	(0.057)	(0.03)

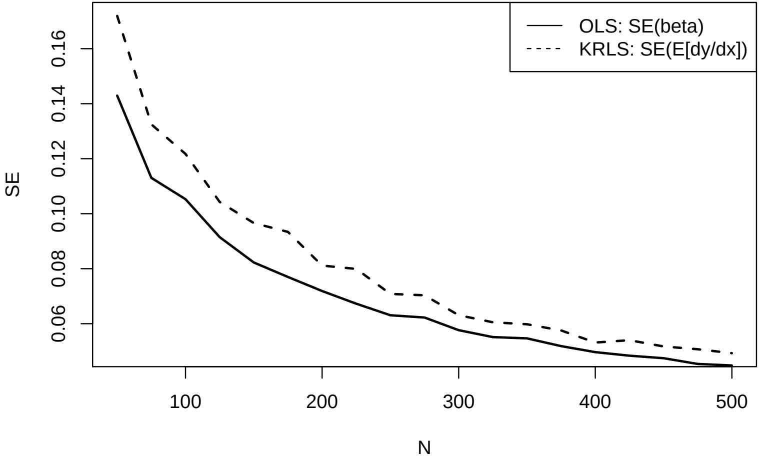
Behind the averages

`plot(krls(X,y))`

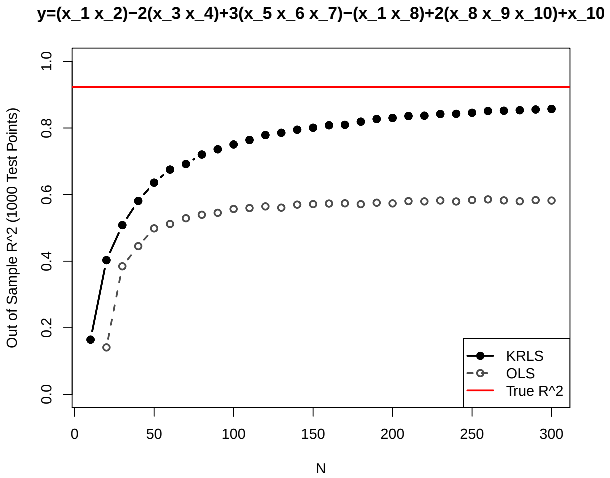


Efficiency Comparison

$$y = 2x + \epsilon,\ x \sim N(0,1), \epsilon \sim N(0,1)$$



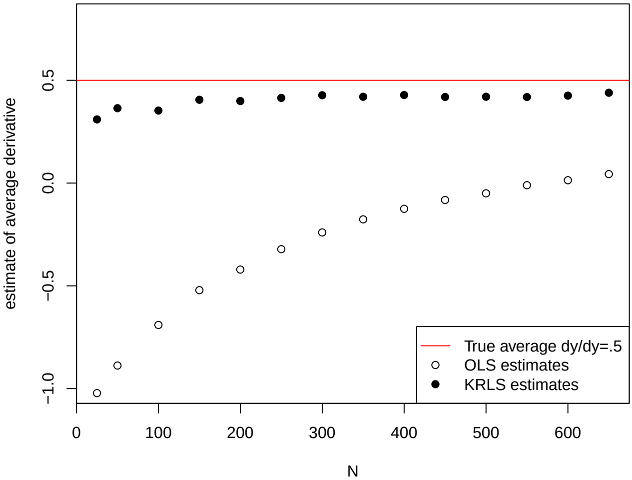
High-dimensional data with non-linearities



$y = (X_1X_2) - 2(X_3X_4) + 3(X_5X_6X_7) - (X_1X_8) + 2(X_8X_9X_{10}) + X_{10} + \epsilon$ where all X are i.i.d. *Bernoulli*(p) at varying p, $\varepsilon \sim N(0, .5)$. 1,000 test points.

Linear model with bad leverage points

- $y = .5x + \varepsilon$ where $\varepsilon \sim N(0, .3)$
- One bad point, $(y_i = -5, x_i = 5)$.



Interaction or non-linearity?

Truth: $y = 5x_1^2 + \varepsilon, \quad \rho(x_1, x_2) = .72$
$\varepsilon \sim (0, .44)$. $x_1 \sim Uniform(0, 2)$

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OLS Model: $y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 x_1 * x_2$

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Estimator	OLS	KRLS			
$\partial y / \partial x_{ij}$	Average	Average	1st Qu.	Median	3rd Qu.
const	-1.50				
	(0.34)				
x_1	7.51	9.22	5.22	9.38	14.03
	(0.40)	(0.52)	(0.82)	(0.85)	(0.79)
x_2	-1.28	0.02	-0.08	0.00	0.10
	(0.21)	(0.13)	(0.19)	(0.16)	(0.20)
$(x_1 \cdot x_2)$	1.24				
	(0.15)				
N	250				

Concluding Thoughts

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  - extremely powerful at detecting interactions

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- Strengths
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- Difficulties/Future Work
  - computation scales in number of datapoints ($O(N^3)$) which means it doesn't work for more than about 5000 datapoints
  - it may model deep interactions but it is still hard to summarize deep interactions

References

- Hainmueller and Hazlett (2013). "Kernel Regularized Least Squares: Reducing Misspecification Bias with a Flexible and Interpretable Machine Learning Approach" *Political Analysis*.

- Beck, N. and Jackman, S. 1998. Beyond Linearity by Default: Generalized Additive Models. *American Journal of Political Science*.

- Wood (2003). "Thin plate regression splines." Journal of the Royal Statistical Society: Series B.

- Hastie, T.J. and Tibshirani, R.J. 1990. *General Additive Models*.

- Hastie, Tibshirani, and Friedman (2009). *The Elements of Statistical Learning: Data Mining, Inference, and Prediction*. Springer.

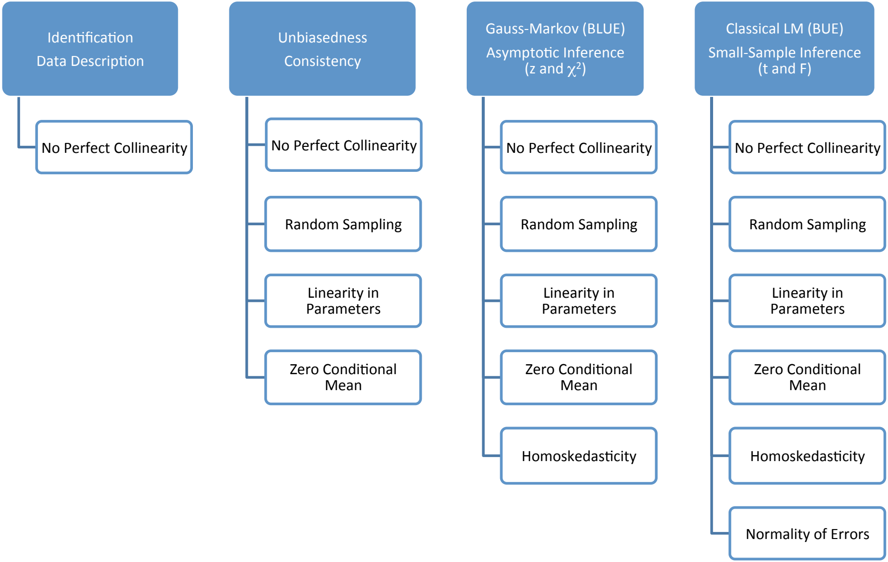
- Schölkopf and Smola (2002). *Learning with kernels: Support vector machines, regularization, optimization, and beyond*. Cambridge, MA: MIT Press.

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This "Week"
  - Monday (5):
    - ★ unusual and influential data → robust estimation
  - Wednesday (7):
    - ★ non-linearity → generalized additive models
  - Monday (12):
    - ★ unusual errors → sandwich SEs
- Next Week
  - regression in social science
- Long Run
  - probability → inference → regression → causal inference

Questions?

Review of the OLS Assumptions



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1. Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (K+1)$ matrix with rank $K+1$
4. Zero conditional mean: $\mathbb{E}[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
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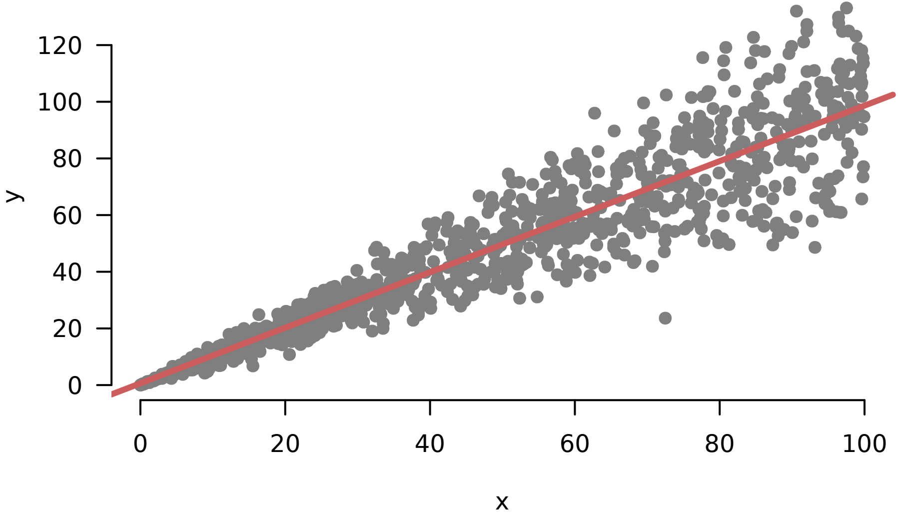
- 1-4 give us unbiasedness/consistency
- 1-5 are the Gauss-Markov, allow for large-sample inference
- 1-6 allow for small-sample inference

Today: How Do We Deal With This?



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Each is a violation of homoskedasticity, but each has its own diagnostics and corrections.

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Then we will discuss a contrarian view

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\end{aligned}$$

- Replace σ^2 with estimate $\widehat{\sigma}^2$ will give us our estimate of the covariance matrix

Non-constant Error Variance

- Homoskedastic:

$$V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

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- $\text{Cov}(u_i, u_j|\mathbf{X}) = 0$

Non-constant Error Variance

- Homoskedastic:

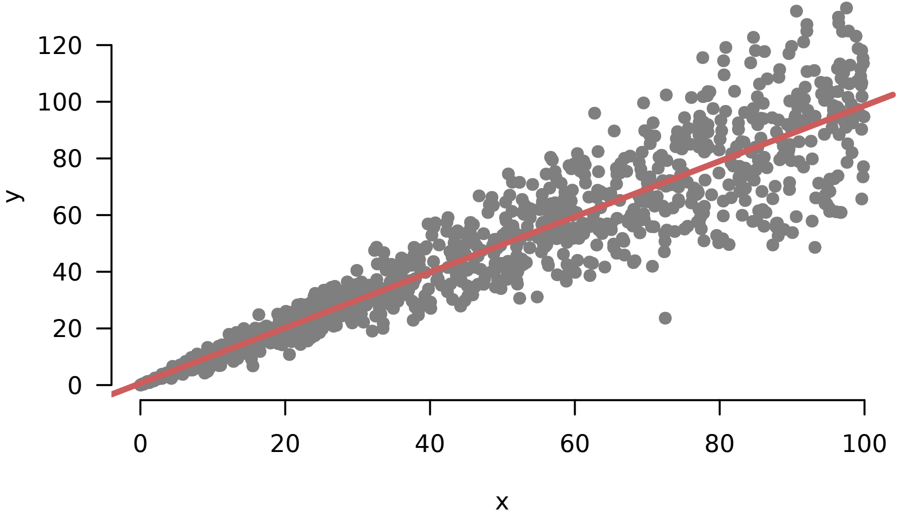
$$V[\mathbf{u}|\mathbf{X}] = \sigma^2\mathbf{I} = \begin{bmatrix} \sigma^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma^2 \end{bmatrix}$$

- Heteroskedastic:

$$V[\mathbf{u}|\mathbf{X}] \ = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

- Independent, not identical
- $\text{Cov}(u_i, u_j|\mathbf{X}) = 0$
- $\text{Var}(u_i|\mathbf{X}) = \sigma_i^2$

Classic Heteroskedasticity



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  - degree of the problem depends on how serious the heteroskedasticity is

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    - In R, `plot(mod, which = 3)`

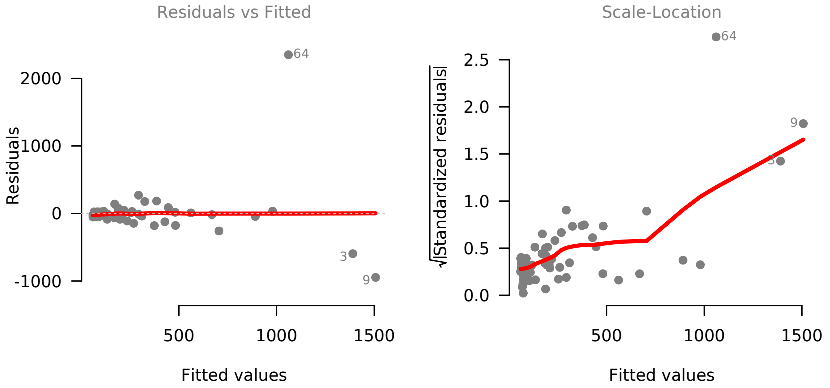
Example: Buchanan votes

```
flvote <- foreign::read.dta("flbuchan.dta")
mod <- lm(edaybuchanan ~ edaytotal, data = flvote)
summary(mod)
```

```
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 5.423e+01  4.914e+01   1.104    0.274
## edaytotal   2.323e-03  3.104e-04   7.483 2.42e-10 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 332.7 on 65 degrees of freedom
## Multiple R-squared:  0.4628, Adjusted R-squared:  0.4545
## F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10
```

Diagnostics

```
par(mfrow = c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 1, lwd = 3)
plot(mod, which = 3, lwd = 3)
```



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  1. Regression y_i on $\mathbf{x}_i'$ and store residuals, $\widehat{u}_i$
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  - In R, `bptest` in the `lmtest` package

Breush-Pagan Example

```
library(lmtest)
bptest(mod)

##
##  studentized Breusch-Pagan test
##
## data:  mod
## BP = 12.59, df = 1, p-value = 0.0003878
```

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4. Admit we have the wrong model and use a different approach

Appendix: Variance Stabilizing Transformations

If the variance for each error (σ_i^2) is proportional to some function of the mean ($\mathbf{x}_i\boldsymbol{\beta}$), then a variance stabilizing transformation may be appropriate.

Note: Transformations will affect the other regression assumptions, as well as interpretation of the regression coefficients.

Examples:

Transformation	Mean/Variance Relationship
$\sqrt{Y}$	$\sigma_i^2 \propto \mathbf{x}_i\boldsymbol{\beta}$
$\log Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^2$
$1/Y$	$\sigma_i^2 \propto (\mathbf{x}_i\boldsymbol{\beta})^4$

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- Under non-constant error variance:

$$\text{Var}[\mathbf{u}] = \mathbf{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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- When $\boldsymbol{\Sigma} \neq \sigma^2\mathbf{I}$, we are stuck with this expression:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\boldsymbol{\Sigma}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

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- Idea: If we can consistently estimate the components of $\mathbf{\Sigma}$, we could directly use this expression by replacing $\mathbf{\Sigma}$ with its estimate, $\hat{\mathbf{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form (but zero covariance):

$$V[\mathbf{u}] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma_1^2 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \sigma_n^2 \end{bmatrix}$$

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$$\widehat{V[\hat{\beta}|\mathbf{X}]} = (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}' \begin{bmatrix} \hat{\mathbf{u}}_1^2 & 0 & 0 & \dots & 0 \\ 0 & \hat{\mathbf{u}}_2^2 & 0 & \dots & 0 \\ & & & \vdots & \\ 0 & 0 & 0 & \dots & \hat{\mathbf{u}}_n^2 \end{bmatrix} \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}$$

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is a consistent estimator of $V[\hat{\beta}|\mathbf{X}]$ under any form of heteroskedasticity consistent with $V[\mathbf{u}]$ above.

The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors.

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3. Plug $\widehat{\boldsymbol{\Sigma}}$ into the sandwich formula to obtain the robust estimator of the variance-covariance matrix

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$$V[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\widehat{\boldsymbol{\Sigma}}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

- There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] = \frac{n}{n-k-1} \cdot \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\widehat{\boldsymbol{\Sigma}}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

Regular & Robust Standard Errors in Florida Example

R Code

```
> library(sandwich)
> library(lmtest)
> coeftest(mod1) # homoskedasticity
t test of coefficients:
              Estimate Std. Error t value  Pr(>|t|)
(Intercept)  5.4231e+01 4.9141e+01  1.1036    0.2738
TotalVotes00 2.3229e-03 3.1041e-04  7.4831 2.417e-10 ***

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC0")) # classic White
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.0612e+01  1.3353  0.18642
TotalVotes00 2.3229e-03 8.7047e-04  2.6685  0.00961 **

> coeftest(mod1,vcov = vcovHC(mod1, type = "HC1")) # small sample correction
t test of coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.4231e+01 4.1232e+01  1.3153  0.19304
TotalVotes00 2.3229e-03 8.8376e-04  2.6284  0.01069 *
```

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- Because it relies on consistency, it is a large sample result, best with large n
- For small n, performance might be poor (correction factors exist but are often insufficient)
- We can arrive at White's heteroskedasticity consistent standard errors using the plug-in principle and thus in some ways, these are the natural way of getting standard errors in the agnostic regression framework.

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Clustered Dependence: Intuition

- Think back to the Gerber, Green, and Larimer (2008) social pressure mailer example.
- Their design: randomly sample households and randomly assign them to different treatment conditions
- But the measurement of turnout is at the individual level
- Violation of iid/random sampling:
  - errors of individuals within the same household are correlated
  - $\rightsquigarrow$ violation of homoskedasticity
- Called clustering or clustered dependence

Clustered Dependence: notation

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- Ignoring clustering is "cheating": units not independent

Clustered Dependence: Example Model

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- $\rho \in (0, 1)$ is called the within-cluster correlation.
- What if we ignore this structure and just use ε_{ij} as the error?
- Variance of the composite error is σ^2:

$$\begin{aligned} \text{Var}[\varepsilon_{ij}] &= \text{Var}[v_j + u_{ij}] \\ &= \text{Var}[v_j] + \text{Var}[u_{ij}] \\ &= \rho\sigma^2 + (1-\rho)\sigma^2 = \sigma^2 \end{aligned}$$

Lack of Independence

- Covariance between two units i and s in the same cluster is $\rho\sigma^2$:

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- Zero covariance of two units i and s in different clusters j and k:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0$$

Example Covariance Matrix

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{bmatrix}'$$

$$\text{Var}[\varepsilon] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

Appendix: Example 6 Units, 2 Clusters

$$\boldsymbol{\varepsilon} = [\ \varepsilon_{1,1} \quad \varepsilon_{2,1} \quad \varepsilon_{3,1} \quad \varepsilon_{4,2} \quad \varepsilon_{5,2} \quad \varepsilon_{6,2}\ ]'$$

$$V[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \begin{bmatrix} V[\varepsilon_{1,1}] & Cov[\varepsilon_{2,1},\varepsilon_{1,1}] & Cov[\varepsilon_{3,1},\varepsilon_{1,1}] & . & . & . \\ Cov[\varepsilon_{1,1},\varepsilon_{2,1}] & V[\varepsilon_{2,1}] & Cov[\varepsilon_{3,1},\varepsilon_{2,1}] & . & . & . \\ Cov[\varepsilon_{1,1},\varepsilon_{3,1}] & Cov[\varepsilon_{2,1},\varepsilon_{3,1}] & V[\varepsilon_{3,1}] & . & . & . \\ Cov[\varepsilon_{1,1},\varepsilon_{4,2}] & Cov[\varepsilon_{2,1},\varepsilon_{4,2}] & Cov[\varepsilon_{3,1},\varepsilon_{4,2}] & V[\varepsilon_{4,2}] & . & . \\ Cov[\varepsilon_{1,1},\varepsilon_{5,2}] & Cov[\varepsilon_{2,1},\varepsilon_{5,2}] & Cov[\varepsilon_{3,1},\varepsilon_{5,2}] & Cov[\varepsilon_{4,2},\varepsilon_{5,2}] & V[\varepsilon_{5,2}] & . \\ Cov[\varepsilon_{1,1},\varepsilon_{6,2}] & Cov[\varepsilon_{2,1},\varepsilon_{6,2}] & Cov[\varepsilon_{3,1},\varepsilon_{6,2}] & Cov[\varepsilon_{4,2},\varepsilon_{6,2}] & Cov[\varepsilon_{5,2},\varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 \end{bmatrix}$$

which can be verified as follows:

- $V[\varepsilon_{ij}] = V[v_j + u_{ij}] = V[v_j] + V[u_{ij}] = \rho\sigma^2 + (1-\rho)\sigma^2 = \sigma^2$
- $Cov[\varepsilon_{ij},\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] - E[\varepsilon_{ij}]E[\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] = E[(v_j + u_{ij})(v_j + u_{lj})]$
  $= E[{v_j}^2] + E[v_j u_{ij}] + E[v_j u_{lj}] + E[u_{ij}u_{lj}]$
  $= E[{v_j}^2] + E[v_j]E[u_{ij}] + E[v_j]E[u_{lj}] + E[u_{ij}]E[u_{lj}]$
  $= E[{v_j}^2] = V[v_j] + (E[v_j])^2 = V[v_j] = \rho\sigma^2$
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  $= E[v_j]E[v_k] + E[v_j]E[u_{lk}] + E[v_k]E[u_{ij}] + E[u_{ij}]E[u_{lk}] = 0$

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The variance-covariance matrix of the error, $\mathbf{\Sigma}$, is block diagonal:

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Cluster-Robust SEs

- First, let's write the within-cluster regressions like so:

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- Under this clustered dependence, we can write this as:

$$\text{Var}[\hat{\beta}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\sum_{j=1}^{m}\mathbf{X}_j'\Sigma_j\mathbf{X}_j\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

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- There are multiple implementations in `R` including `multiwayvcov:cluster.vcov` and `sandwich::vcovCL`

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Cluster-Robust Standard Errors

- CRSE do not change our estimates $\widehat{\beta}$, cannot fix bias
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- There are numerous alternative clustered standard error variants out there.

A Contrarian View of Robust Standard Errors

King, Gary and Margaret E. Roberts. "How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It" *Political Analysis* (2015) 23: 159-179.[3]

[3] I thank Gary and Molly for the slides that follow.

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RSEs: Two Possibilities

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RSEs and SEs are the same

- Consistent with a correctly specified model
- RSEs are not useful, as a "fix"

Their Alternative Procedure

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- What they are not:

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Robust standard errors:

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1. Do RSEs and SEs differ?
2. If they do:

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1. Do RSEs and SEs differ?
2. If they do:
    - Use model diagnostics

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   - Evaluate misspecification
   - Respecify the model
3. Keeping going, until they don't differ.

For RSEs to help: Everything has to be Juuuussttt Right

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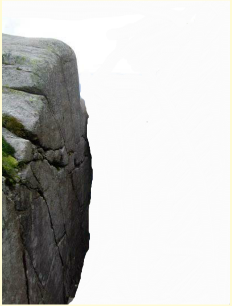
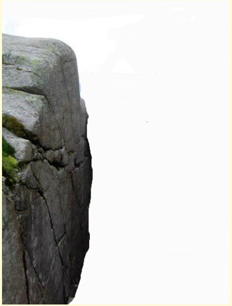


For RSEs to help: Everything has to be Juuuussttt Right

Model Correct



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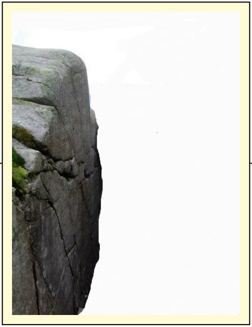
Model Correct

RSEs same as SEs



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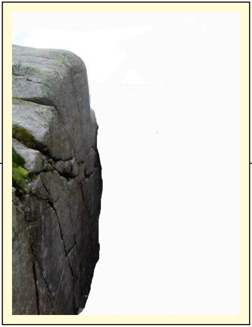
Model Correct

RSEs same as SEs

Point estimates correct



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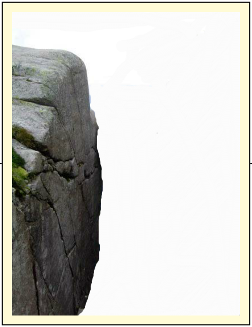

RSEs same as SEs

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Awesome!

For RSEs to help: Everything has to be Juuuussttt Right

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Model Misspecified

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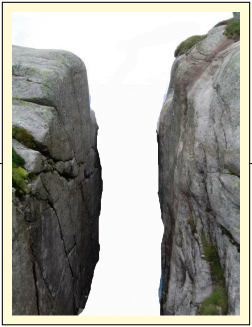

Model Misspecified

RSEs differ from SEs

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Model Correct

RSEs same as SEs

Point estimates correct

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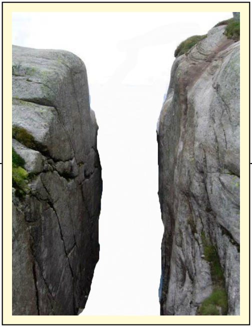

Model Misspecified

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Model Correct

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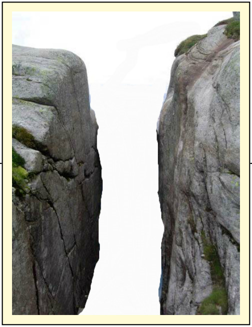

Model Misspecified

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Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Goldilocks Region

Model Correct

RSEs same as SEs

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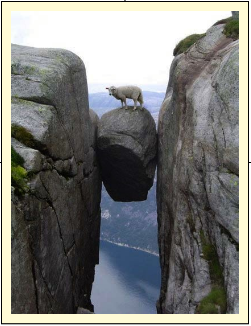
For RSEs to help: Everything has to be Juuuussttt Right

Goldilocks Region



Model Correct	Model Misspecified
RSEs same as SEs	RSEs differ from SEs
Point estimates correct	Point estimates biased
Awesome!	Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

Biased just enough to make RSEs useful,

Goldilocks Region

Model Correct

RSEs same as SEs

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Model Misspecified

RSEs differ from SEs

Point estimates biased

Respecify!

For RSEs to help: Everything has to be Juuuussttt Right

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Goldilocks Region

but not so much as to bias everything else

Model Correct

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Model Misspecified

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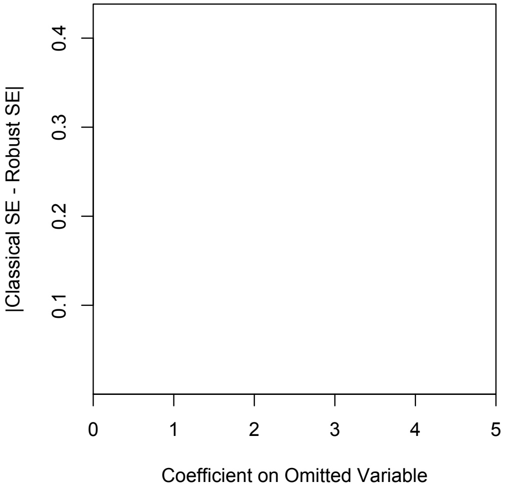
  - We can't check: whether model implications are realistic.
- Parts of the model are wrong; why do we think the rest are right?

Difference Between SE and RSE Exposes Misspecification



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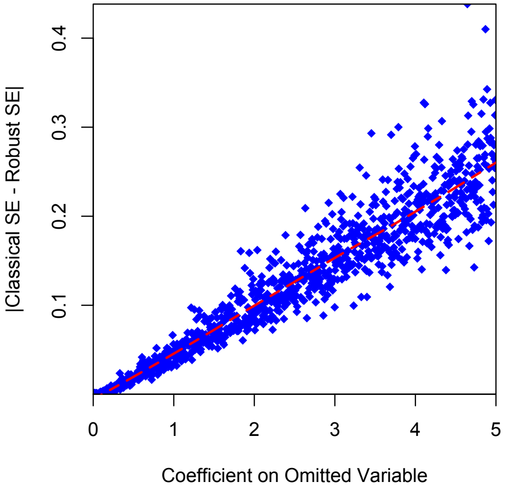

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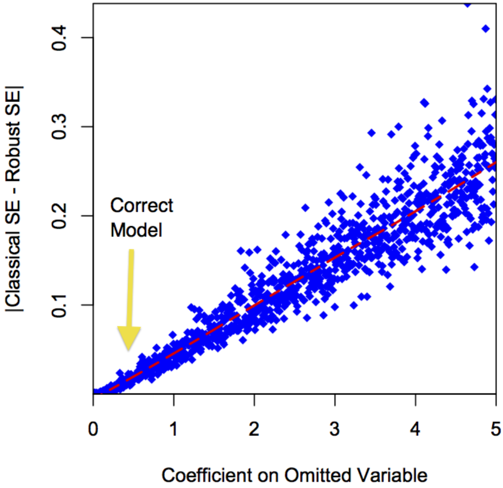

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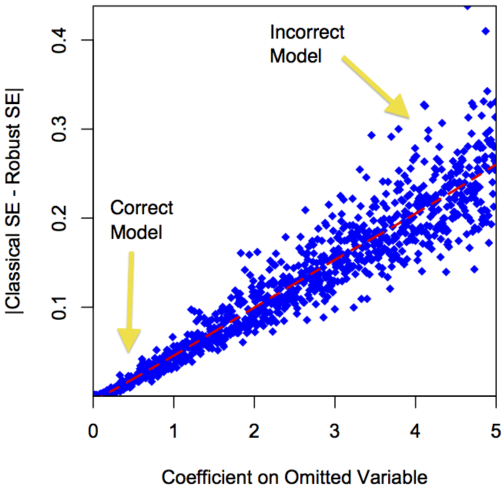

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  - $\Rightarrow$ indicates model misspecification

Problem: Highly Skewed Dependent Variable

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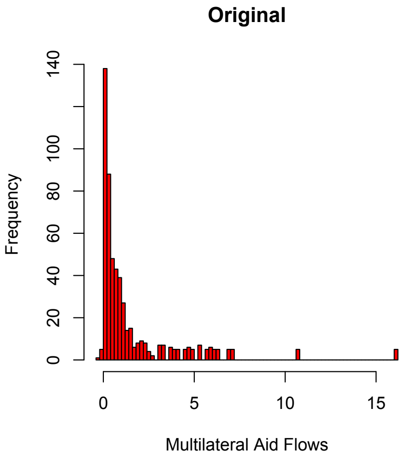

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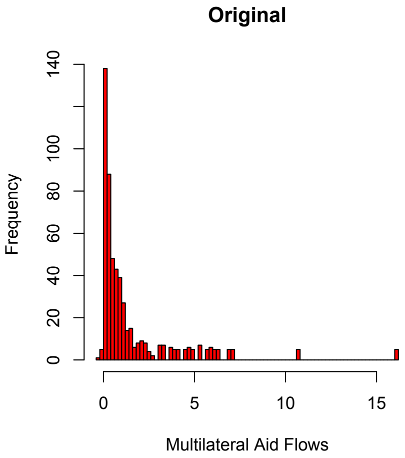

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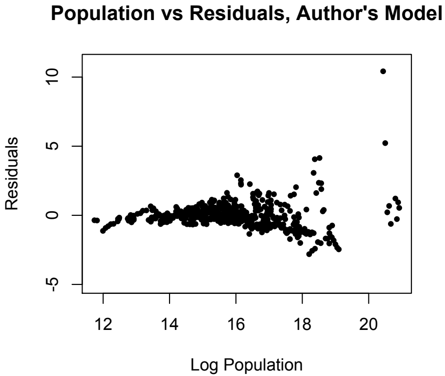

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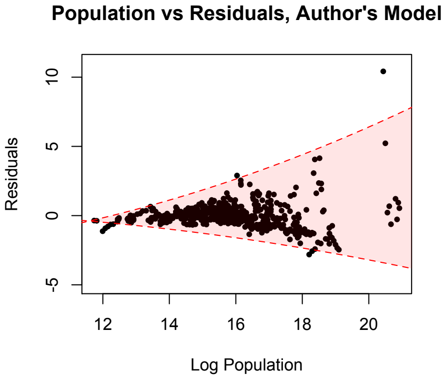

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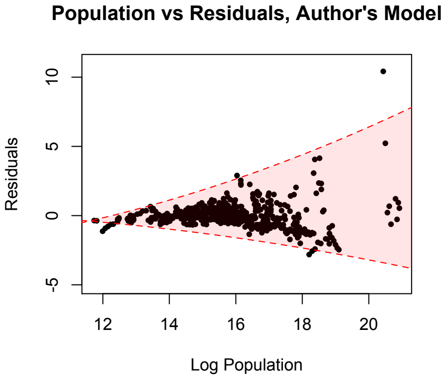
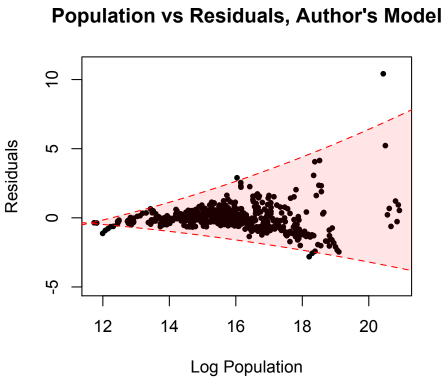
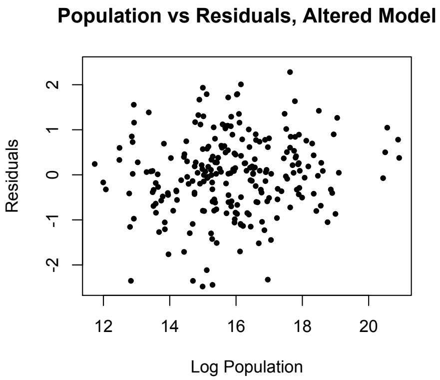


Textbook case of heteroskedasticity

Diagnostics: Reveal Misspecification





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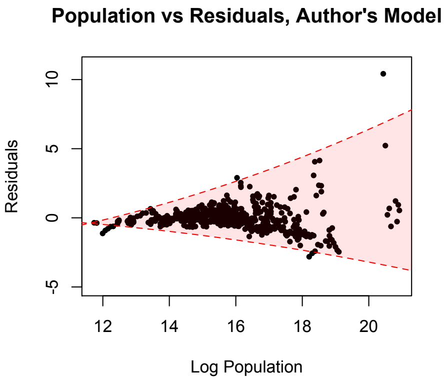

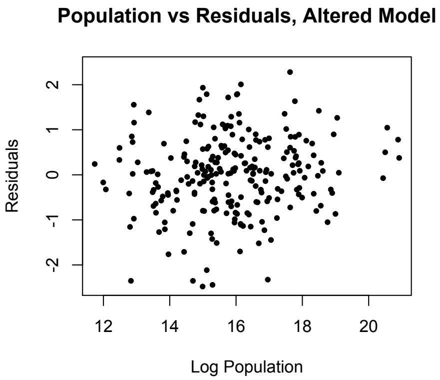

Textbook case of heteroskedasticity

Textbook case of homoskedasticity

After Fix: Different Conclusion

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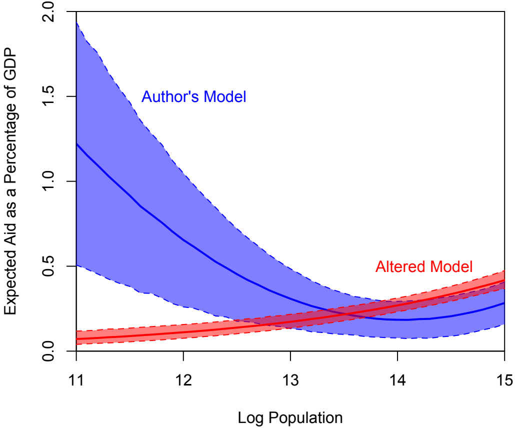

Concluding Contrarian Thoughts

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Their advice:

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Their Examples:

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- Robust SEs indicate fundamental modelling problems

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- Fixing these problems

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Their Examples:

- Robust SEs indicate fundamental modelling problems
- Easily identified with diagnostics
- Fixing these problems $\Rightarrow$ hugely different substantive conclusions

Concluding Thoughts on Diagnostics

Residuals are important. Look at them.

Next 'Week'

- Regression in the Social Sciences and An Introduction to Causal Inference
- Reading:
  - Healy *Data Visualization: A practical introduction* `http://socviz.co/` Chapter 1: Look at Data
  - Morgan and Winship Chapter 1: Causality and Empirical Research in the Social Sciences
  - Morgan and Winship Chapter 13.1: Objections to Adoption of the Counterfactual Approach
  - Angrist and Pishke Chapters 1-2
  - Hernan and Robins (2016) Chapter 1: A definition of a causal effect `https://www.hsph.harvard.edu/miguel-hernan/causal-inference-book/`
- As a side note: if you want to read the argument against the contrarian response: Aronow (2016) "A Note on 'How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It.'" It is an interesting piece- feel free to come talk to me about this debate!

Fun With Neighbors

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- We talked about error dependence induced by time and by cluster.
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Zhukov, Yuri M. and Brandon M. Stewart. "Choosing Your Neighbors: Networks of Diffusion in International Relations" *International Studies Quarterly* 2013; 57: 271-287.

Our Main Questions

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3. How do we make these assumptions?

How Do We Generally Choose Neighbors?

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3. Atheoretical choices, rarely justified
4. Different types of neighbors tell different stories

Visualization of Connections: Contiguity

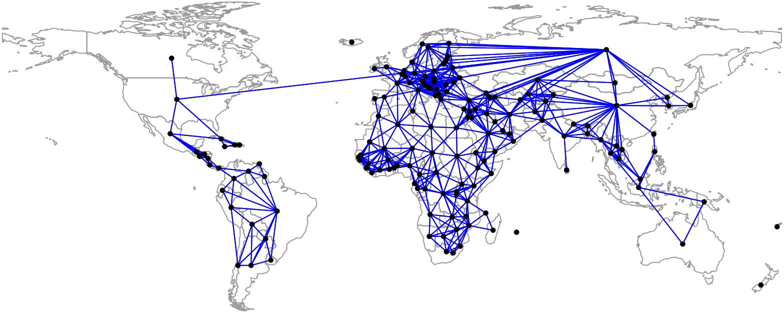
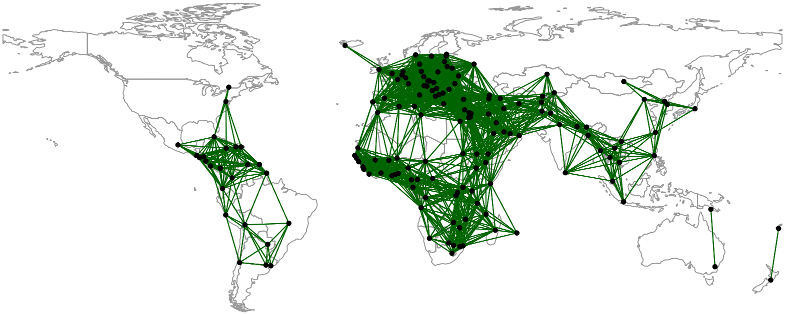


Figure: Contiguity neighbors with 500 km snap distance

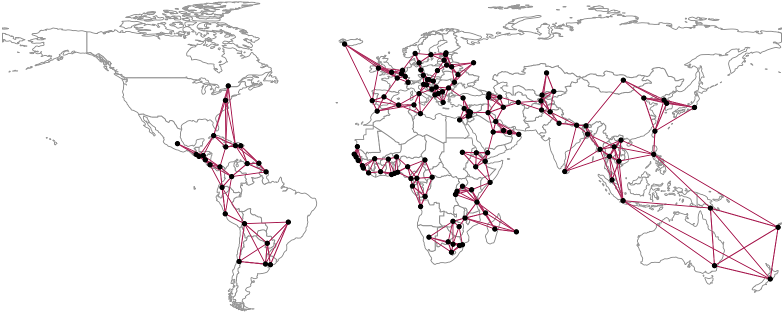
Visualization of Connections: Minimum Distance

Figure: Minimum distance neighbors (capital cities)



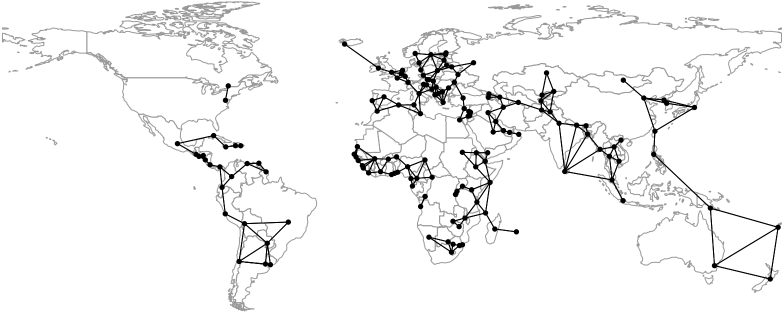
Visualization of Connections: K-Nearest Neighbors

Figure: $k = 4$ Nearest Neighbors (capital cities)



Visualization of Connections: Graph-based Neighbors

Figure: Sphere of Influence Neighbors (capital cities)



Application: Democratic Diffusion

Gleditsch and Ward (2006)
Changes of political regime modeled as a first-order Markov chain process with the transition matrix

$$\mathbf{K} = \begin{bmatrix} Pr(y_{i,t} = 0 | y_{i,t-1} = 0) & Pr(y_{i,t} = 1 | y_{i,t-1} = 0) \\ Pr(y_{i,t} = 0 | y_{i,t-1} = 1) & Pr(y_{i,t} = 1 | y_{i,t-1} = 1) \end{bmatrix}$$

where $y_{i,t} = 1$ if an (A)utocratic regime exists in country i at time t, and $y_{i,t} = 0$ if the regime is (D)emocratic.
... in other words:

$$\mathbf{K} = \begin{bmatrix} Pr(D \rightarrow D) & Pr(D \rightarrow A) \\ Pr(A \rightarrow D) & Pr(A \rightarrow A) \end{bmatrix}$$

Equilibrium Effects of Democratic Transition

If a regime transition takes place in country i, what is the change in predicted probability of a regime transition in country j (country i's neighbor)?

$$\text{QI} = Pr(y_{j,t}|y_{i,t} = y_{i,t-1}) - Pr(y_{j,t}|y_{i,t} \neq y_{i,t-1})$$

where $y_{i,t} = 0$ if country i is a democracy at time t and $y_{i,t} = 1$ if it is an autocracy. All other covariates are held constant.

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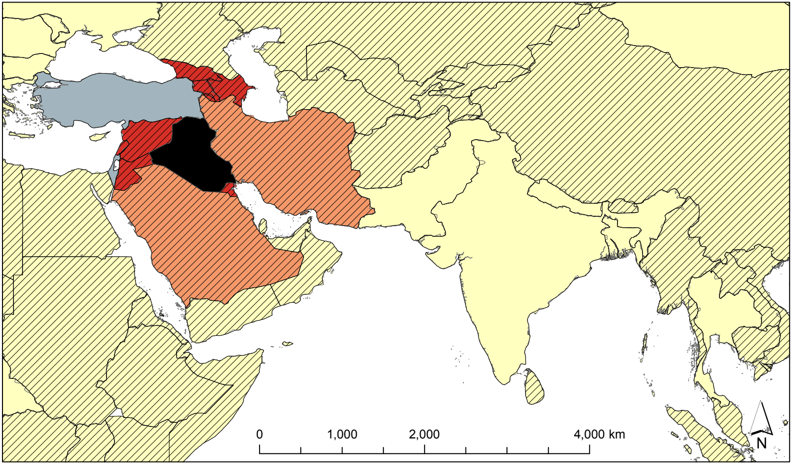
Illustrative cases

- Iraq transitions from autocracy to democracy.
- Russia transitions from democracy to autocracy.

Iraq's democratization and regional regime stability



Contiguity + 500 km

Iraq transitions from autocracy to democracy (1998 data)

Monte Carlo simulation (1,000 runs)

Iraq

Regime Type

Democracy

Autocracy

Change in Transition Probability

-0.05- -0.025

-0.025 - -0.001

0

0.001 - 0.025

0.025 - 0.05

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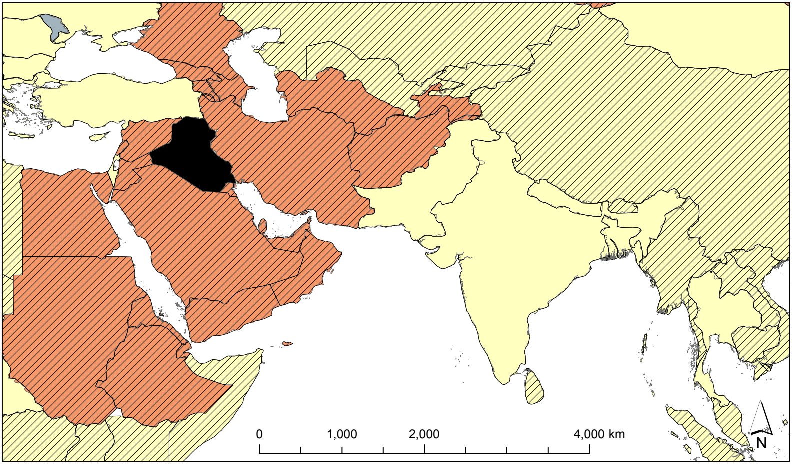

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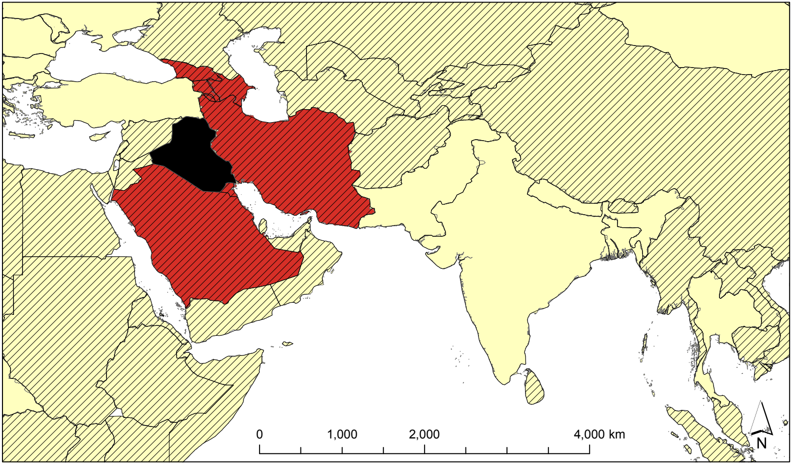

k = 4 Nearest Neighbors

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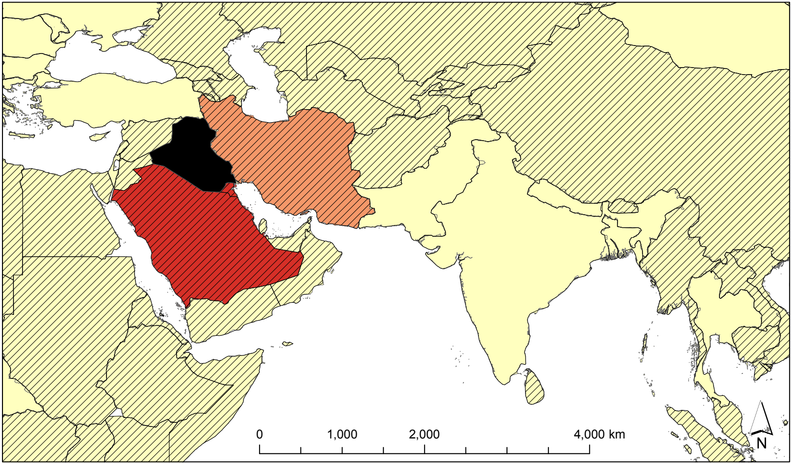

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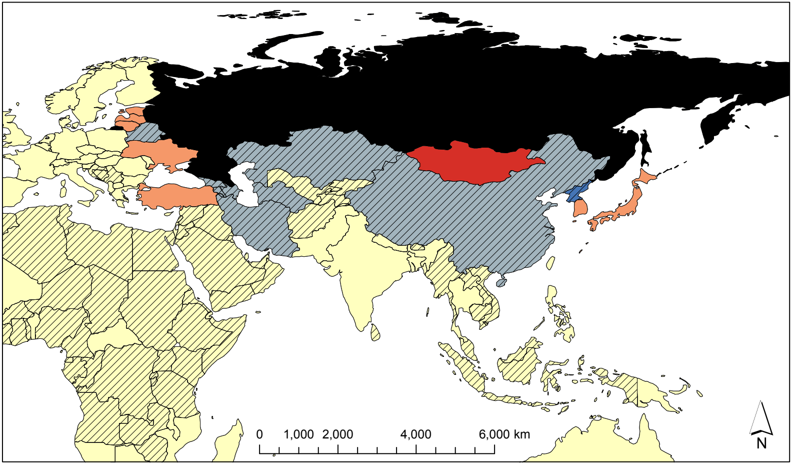

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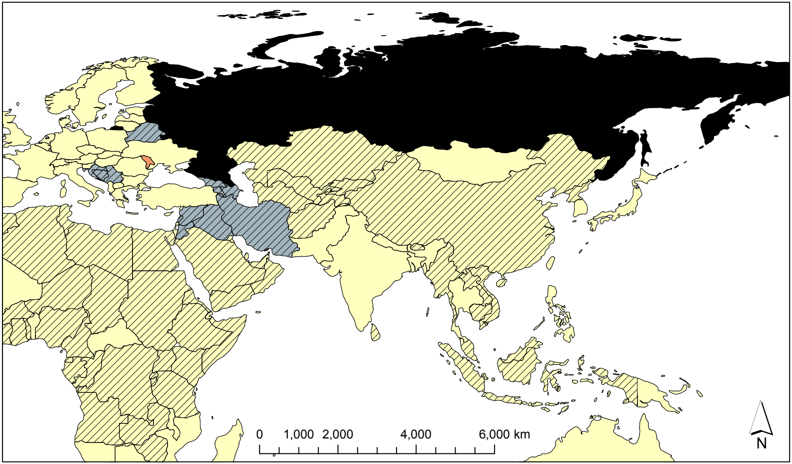

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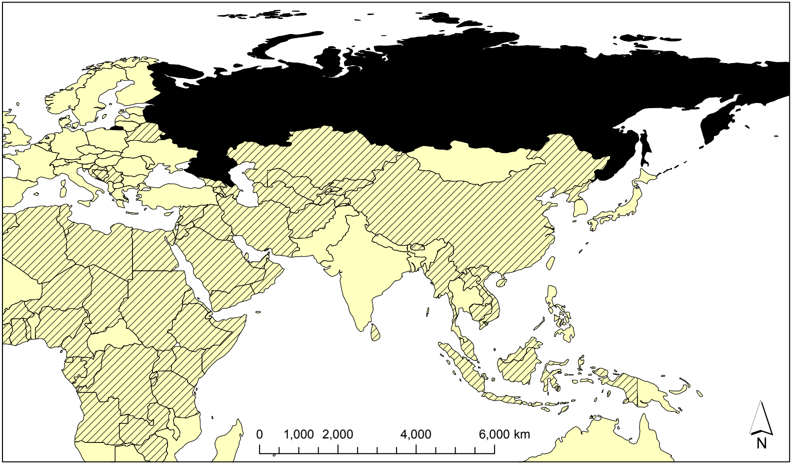

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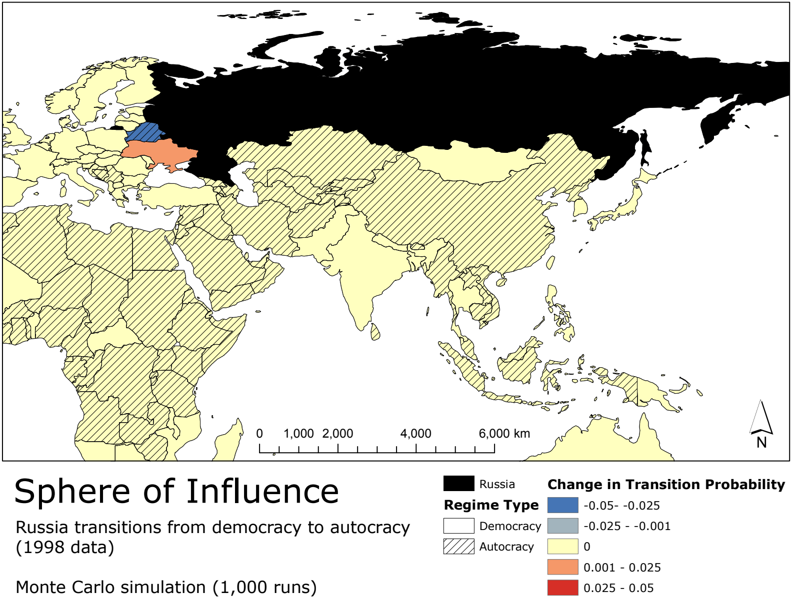

Sphere of Influence

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Appendix: Weighted Least Squares

- Suppose that the heteroskedasticity is known up to a multiplicative constant:

$$\text{Var}[u_i|\mathbf{X}] = a_i\sigma^2$$

  where $a_i = a_i(\mathbf{x}_i')$ is a positive and known function of $\mathbf{x}_i'$
- WLS: multiply y_i by $1/\sqrt{a_i}$:

$$y_i/\sqrt{a_i} = \beta_0/\sqrt{a_i} + \beta_1 x_{i1}/\sqrt{a_i} + \cdots + \beta_k x_{ik}/\sqrt{a_i} + u_i/\sqrt{a_i}$$

Appendix: Weighted Least Squares Intuition

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- Rescales errors to $u_i/\sqrt{a_i}$, which maintains zero mean error
- But makes the error variance constant again:

$$\text{Var}\left[\frac{1}{\sqrt{a_i}}u_i|\mathbf{X}\right] = \frac{1}{a_i}\text{Var}\left[u_i|\mathbf{X}\right]$$

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- If you know a_i, then you can use this approach to makes the model homoskedastic and, thus, BLUE again
- When do we know a_i?

Appendix: Weighted Least Squares procedure

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- Define the weighting matrix:

$$\mathbf{W} = \begin{bmatrix} 1/\sqrt{a_1} & 0 & 0 & 0 \\ 0 & 1/\sqrt{a_2} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1/\sqrt{a_n} \end{bmatrix}$$

- Run the following regression:

$$\begin{aligned} \mathbf{Wy} &= \mathbf{WX}\beta + \mathbf{Wu} \\ \mathbf{y}^* &= \mathbf{X}^*\beta + \mathbf{u}^* \end{aligned}$$

- Run regression of $\mathbf{y}^* = \mathbf{Wy}$ on $\mathbf{X}^* = \mathbf{WX}$ and all Gauss-Markov assumptions are satisfied
- Plugging into the usual formula for $\widehat{\beta}$:

$$\widehat{\beta}_W = (\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{X})^{-1}\mathbf{X}'\mathbf{W}'\mathbf{W}\mathbf{y}$$

Appendix: WLS Example

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- With the Buchanan data, maybe we think that the variance is proportional to the total number of ballots cast:

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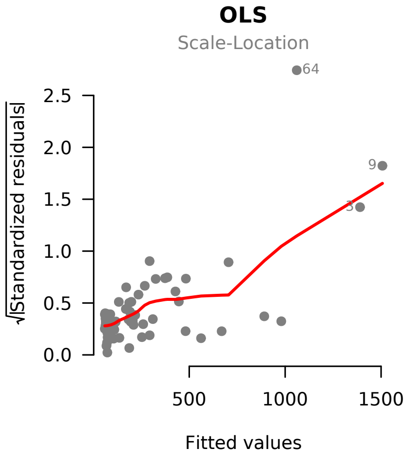
```
mod.wls <- lm(edaybuchanan ~ edaytotal, weights = 1/edaytotal,
                     data = flvote)
summary(mod.wls)

##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) 2.707e+01  8.507e+00   3.182  0.00225 **
## edaytotal   2.628e-03  2.502e-04  10.503 1.22e-15 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.5645 on 65 degrees of freedom
## Multiple R-squared:  0.6292, Adjusted R-squared:  0.6235
## F-statistic: 110.3 on 1 and 65 DF,  p-value: 1.22e-15
```

Appendix: Comparing WLS to OLS

```
par(mfrow=c(1,2), pch = 19, las = 1, col = "grey50", bty = "n")
plot(mod, which = 3, main = "OLS", lwd = 2)
plot(mod.wls, which = 3, main = "WLS", lwd = 2)
```



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- Sometimes we deal with data that is measured over time, $t = 1, \ldots, T$
- Examples: a country over several years or a person over weeks/months
- Often have serially correlated: errors in one time period are correlated with errors in other time periods
- Many different ways for this to happen, but we often assume a very limited type of dependence called AR(1).

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- Typically assume stationarity meaning that $V[u_t]$ and $Cov[u_t, u_{t+h}]$ are independent of t
- Generalizes to higher order serial correlation (e.g. an AR(2) model is given by $u_t = \rho u_{t-1} + \delta u_{t-2} + e_t$).

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The Error Structure for the AR(1) Model

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$$V[\mathbf{u}] = \mathbf{\Sigma} = \frac{\sigma^2}{(1-\rho^2)} \left[\begin{array}{ccccc} 1 & \rho & \rho^2 & \dots & \rho^{T-1} \\ \rho & 1 & \rho & \dots & \rho^{T-2} \\ \rho^2 & \rho & 1 & \dots & \rho^{T-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho^{T-1} & \rho^{T-2} & \rho^{T-3} & \dots & 1 \end{array} \right]$$

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ρ is usually positive, which implies that we underestimate the variance if we ignore serial correlation.

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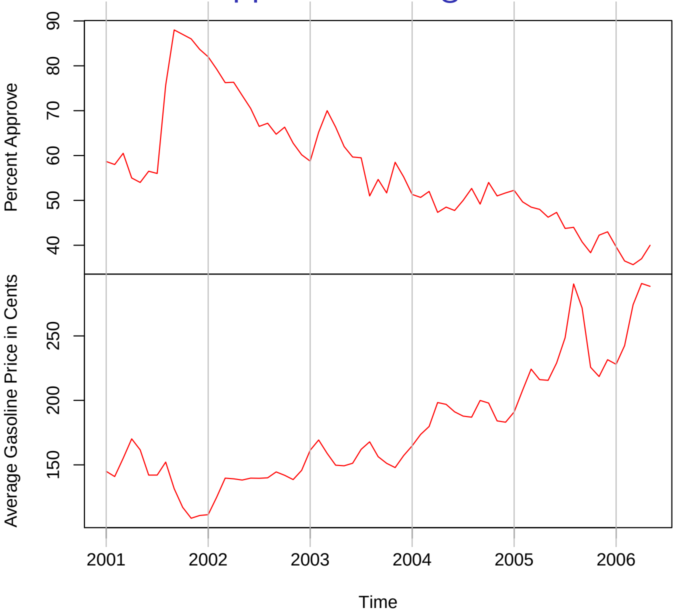
Monthly Presidential Approval Ratings and Gas Prices



Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(Zelig)
> data(approval)
> mod1 <- lm(approve ~ avg.price, data=approval)
> coeftest(mod1)

t test of coefficients:

              Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***
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Tests for Serial Correlation: Durbin-Watson

Recall our AR(1) model is:

$$y_t = \beta_0 + \beta_1 x_t + u_t$$

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- If $DW \approx 2$ then $\widehat{\rho} \approx 0$ (Note that $0 \leq DW \leq 4$)
- If $DW < 1$ we have serious positive serial correlation
- If $DW > 3$ we have serious negative serial correlation

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> library(lmtest)
> dwtest(approve ~ avg.price, data=approval)

        Durbin-Watson test

data:  approve ~ avg.price
DW = 0.4863, p-value = 1.326e-14
alternative hypothesis: true autocorrelation is greater than 0
```

The test suggests strong positive serial correlation. Standard errors are severely downward biased.

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  - HAC are consistent estimator for $V[\hat{\beta}]$ in the presence of heteroskedasticity and or autocorrelation
  - The `sandwich` package in R implements a variety of HAC estimators
  - A common option is `NeweyWest`

Monthly Presidential Approval Ratings and Gas Prices

R Code

```
> mod1 <- lm(approve~avg.price,data=approval)
> coeftest(mod1) # homoskedastic errors
t test of coefficients:

             Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076   3.567277  28.165 < 2.2e-16 ***
avg.price    -0.243885   0.019465 -12.529 < 2.2e-16 ***

> coeftest(mod1, vcov = NeweyWest) # HAC errors
t test of coefficients:

             Estimate Std. Error t value  Pr(>|t|)
(Intercept) 100.472076  14.499337  6.9294 2.652e-09 ***
avg.price    -0.243885   0.071733 -3.3999  0.001174 **
```

Once we correct for autocorrelation, standard errors increase dramatically.

Appendix: Derivation of Error Structure for the AR(1) Model

We have

$$V[u_t] = V[\rho u_{t-1} + e_t] = \rho^2 V[u_{t-1}] + \sigma^2$$

with stationarity, $V[u_t] = V[u_{t-1}]$, and so

$$V[u_t](1-\rho^2) = \sigma^2 \Rightarrow V[u_t] = \frac{\sigma^2}{(1-\rho^2)}$$

also

$$Cov[u_t, u_{t-1}] = E[u_t u_{t-1}] = E[(\rho u_{t-1} + e_t)e_{t-1}] = \rho V[e_{t-1}] = \rho \frac{\sigma^2}{(1-\rho^2)}$$

or generally

$$Cov[u_t, u_{t-h}] = \rho^h \frac{\sigma^2}{(1-\rho^2)}$$

References

- Angrist, Joshua D., and Jrn-Steffen Pischke. *Mostly harmless econometrics: An empiricist's companion*. Princeton university press, 2008.
- Breusch, Trevor S., and Adrian R. Pagan. "A simple test for heteroscedasticity and random coefficient variation." *Econometrica: Journal of the Econometric Society* (1979): 1287-1294.
- Durbin, James, and Geoffrey S. Watson. "Testing for serial correlation in least squares regression. II." *Biometrika* (1951): 159-177.
- Freedman, David A. "On the so-called Huber sandwich estimator and robust standard errors." *The American Statistician* 60.4 (2006).
- King, Gary and Margaret E. Roberts. "How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It" *Political Analysis* (2015) 23: 159-179.
- Newey, Whitney K., and Kenneth D. West. "A simple, positive semi-definite, heteroskedasticity and autocorrelationconsistent covariance matrix." (1986).
- Salganik MJ, Levy KEC (2015) Wiki Surveys: Open and Quantifiable Social Data Collection. *PLoS ONE* 10(5): e0123483.
- Wand, Jonathan N., Kenneth W. Shotts, Jasjeet S. Sekhon, Walter R. Mebane Jr, Michael C. Herron, and Henry E. Brady. "The butterfly did it: The aberrant vote for Buchanan in Palm Beach County, Florida." *American Political Science Review* (2001): 793-810.