Week 6: Linear Regression with Two Regressors

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Princeton

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Erin Hartman.

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- Next Week
  - multiple regression
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Questions?

Why Do We Want More Than One Predictor?

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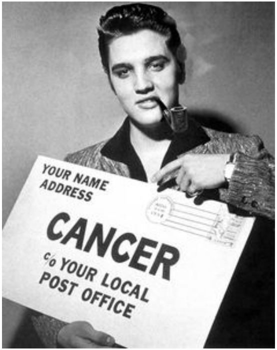
- Model interactive effects (e.g. $Y = \beta_0 + \beta_1 X + \beta_2 X_2 + \beta_3 X_1 X_2$)

Example 1: Cigarette Smokers and Pipe Smokers



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- What about the conditional relationship within departments?

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- Marginal relationships (admissions and gender) $\neq$ conditional relationship given third variable (department)

Sex Bias in Graduate Admissions: Data from Berkeley

Measuring bias is harder than is usually assumed, and the evidence is sometimes contrary to expectation.

P. J. Bickel, E. A. Hammel, J. W. O'Connell

Determining whether discrimination because of sex or ethnic identity is being practiced against persons seeking passage from one social status or locus to another is an important problem in our society today. It is legally important and morally important. It is also often quite difficult. This article is an exploration of some of the issues of measurement and assessment involved in one example of the general problem, by means of which we hope to shed some light on the difficulties. We deceision to admit or to deny admission. The question we wish to pursue is whether the decision to admit or to deny was influenced by the sex of the applicant. We cannot know with any certainty the influences on the evaluators in the Graduate Admissions Office, or on the faculty reviewing committees, or on any other administrative personnel participating in the chain of actions that led to a decision on an individual application. We can, however, say that if the admissions decision and the sex by using a familiar statistic, chi-square. As already noted, we are aware of the pitfalls ahead in this naive approach, but we intend to stumble into every one of them for didactic reasons.

We must first make clear two assumptions that underlie consideration of the data in this contingency table approach. Assumption 1 is that in any given discipline male and female applicants do not differ in respect of their intelligence, skill, qualifications, promise, or other attribute deemed legitimately pertinent to their acceptance as students. It is precisely this assumption that makes the study of "sex bias" meaningful, for if we did not hold it any differences in acceptance of applicants by sex could be attributed to differences in their qualifications, promise as scholars, and so on. Theoretically one could test the assumption, for example, by examining presumably unbiased estimators of academic qualification such as Graduate Record Examination scores, undergraduate grade point averages, and so on. There are, however, enormous practical difficulties in this. We therefore predicate our discussion on the validity of assumption 1.

Bickel, Peter J., Eugene A. Hammel, and J. William O'Connell. "Sex bias in graduate admissions: Data from Berkeley." *Science* 187, no. 4175 (1975): 398-404.

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- This general pattern repeats in many debates, often because of the limits of data collection.

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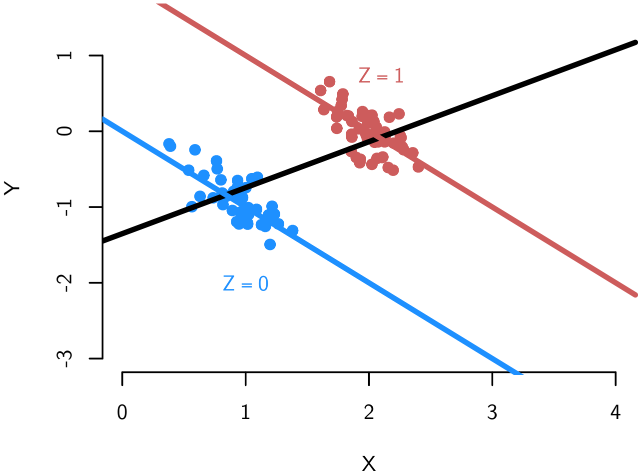
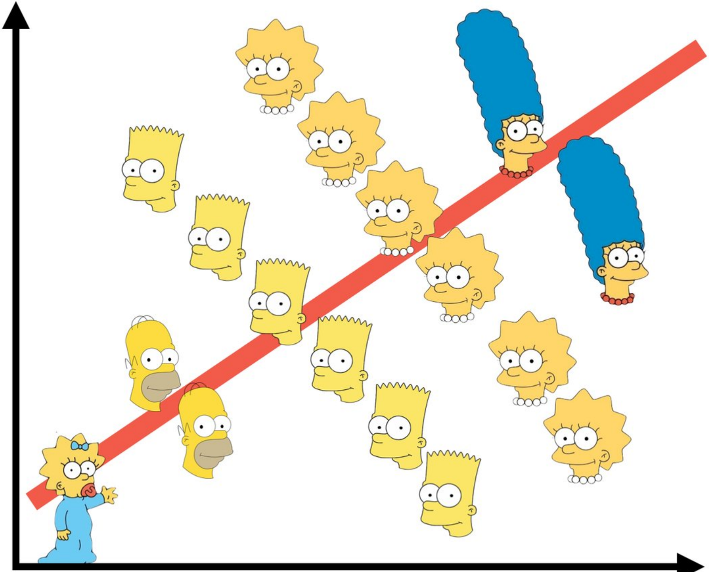


Core idea: a relationship in one direction between Y_i and X_i but the opposite relationship within strata defined by Z_i.

Simpson's paradox



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Instance of a more general problem called the ecological inference fallacy

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- β's are the population parameters we want to estimate

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1. Two Examples
2. Adding a Binary Variable
3. Adding a Continuous Covariate
4. Once More With Feeling
5. OLS Mechanics and Partialing Out
6. Fun With Red and Blue
7. Omitted Variables
8. Multicollinearity
9. Dummy Variables
10. Interaction Terms
11. Polynomials
12. Conclusion
13. Fun With Interactions

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  - X_1: Country income, measured as log(GDP per capita in \$1000s)
  - X_2: Ethnic heterogeneity (continuous) or British colonial heritage (binary)
- With one predictor we ask: Does income (X_1) predict or explain the level of democracy (Y)?
- With two predictors we ask questions like: Does income (X_1) predict or explain the level of democracy (Y), once we "control" for ethnic heterogeneity or British colonial heritage (X_2)?

Regression with Two Explanatory Variables

Example: data from Fish (2002) "Islam and Authoritarianism." *World Politics.* 55: 4-37. Data from 157 countries.

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- The rest of this lecture is designed to explain what is meant by "controlling for another variable" with linear regression.

Simple Regression of Democracy on Income

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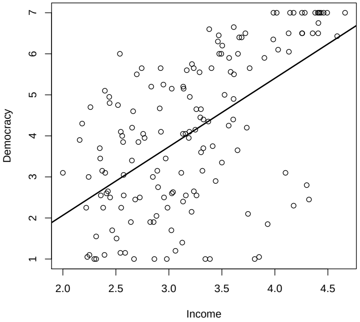
- Let's look at the bivariate regression of Democracy on Income:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_1$$

$$\widehat{Demo} = -1.26 + 1.6\, Log(GDP)$$



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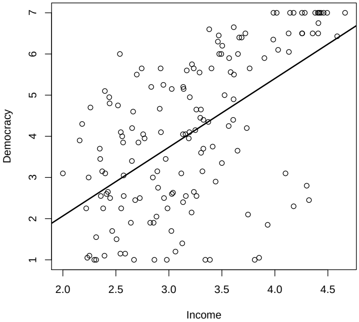

Interpretation:

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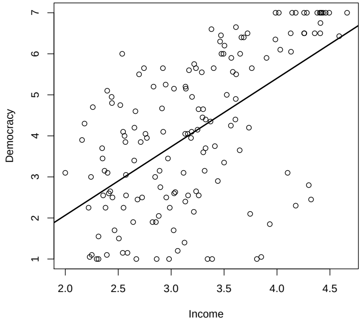
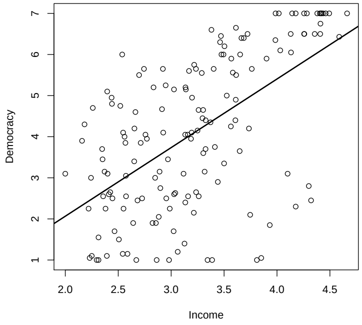


Interpretation: A one percent increase in GDP increases our prediction of democracy by .016.

Simple Regression of Democracy on Income

- But we can use more information in our prediction equation.



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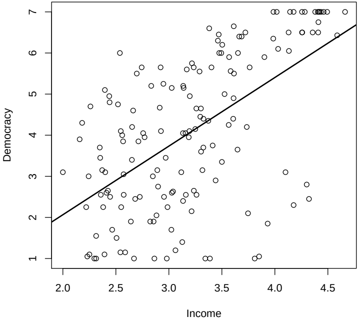

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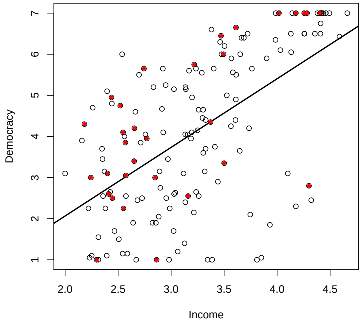

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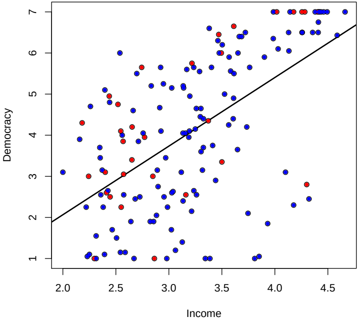
- For example, some countries were originally British colonies and others were not:
    - Former British colonies tend to have higher levels of democracy
    - Non-colony countries tend to have lower levels of democracy



Adding a Covariate

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How do we do this? We can generalize the prediction equation:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{1i} + \widehat{\beta}_2 x_{2i}$$

This implies that we want to predict y using the information we have about x_1 and x_2, and we are assuming a linear functional form.

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In words:

$$\widehat{Democracy} = \widehat{\beta}_0 + \widehat{\beta}_1\, Log(GDP) + \widehat{\beta}_2\, Colony$$

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What does this mean?

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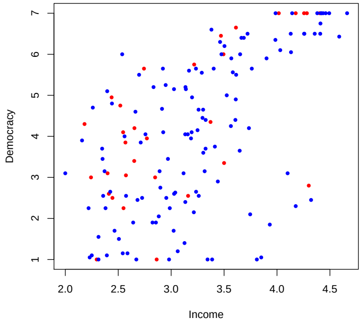
What does this mean? We are fitting two lines with the same slope but different intercepts.

Regression of Democracy on Income

From R, we obtain estimates $\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$:

```
Coefficients:
             Estimate
(Intercept)   -1.5060
GDP90LGN       1.7059
BRITCOL        0.5881
```



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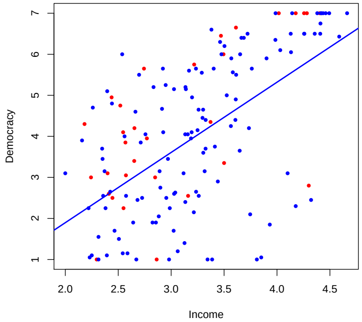

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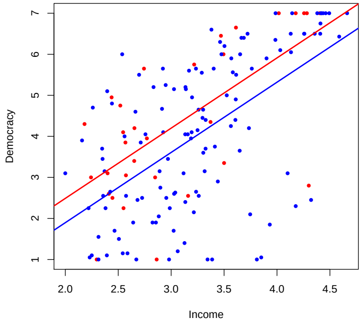

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$$\widehat{y} = \widehat{\beta}_0 + \widehat{\beta}_1 x_1$$
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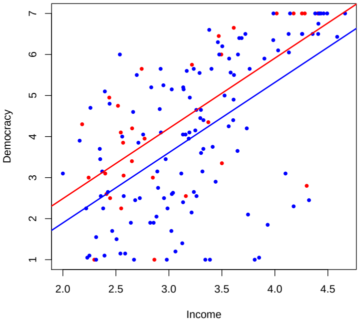
- Former British colonies:

$$\widehat{y} = (\widehat{\beta}_0 + \widehat{\beta}_2) + \widehat{\beta}_1 x_1$$
$$\widehat{y} = -.92 + 1.7\, x_1$$

Regression of Democracy on Income

Our prediction equation is:
$\widehat{y} = -1.5 + 1.7\, x_1 + .58\, x_2$

Where do these quantities appear on the graph?



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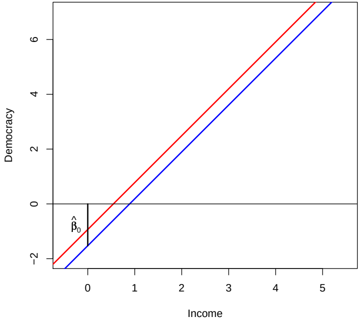

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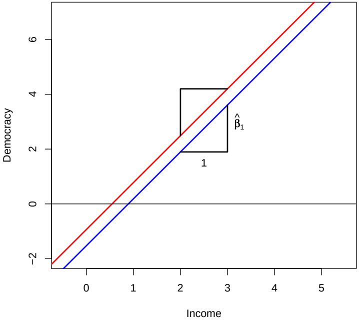

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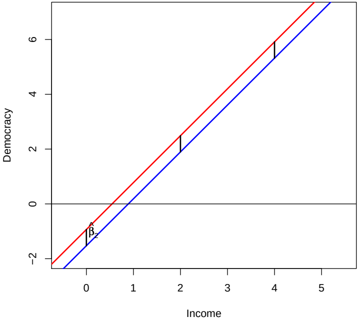
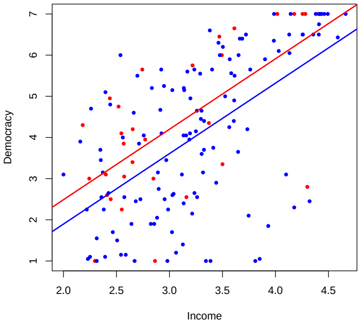


Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.



Fitting a regression plane

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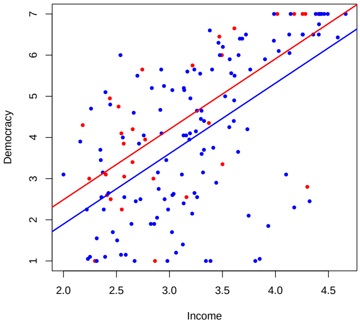
- This is easy to represent graphically in two dimensions because we can use colors to distinguish the two groups in the data.



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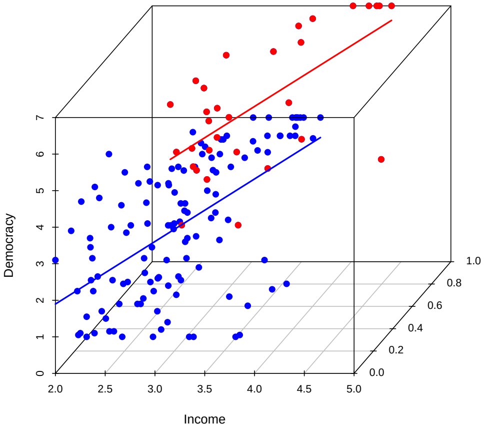
- These observations are actually located in a three-dimensional space.



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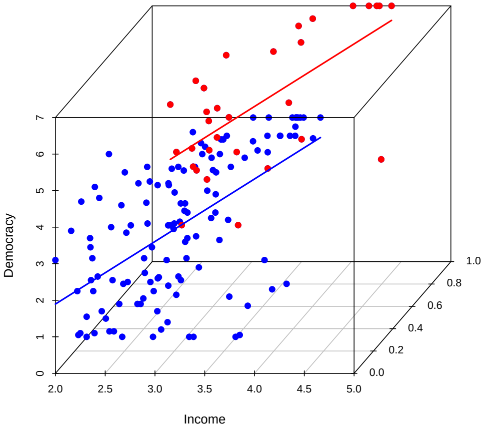

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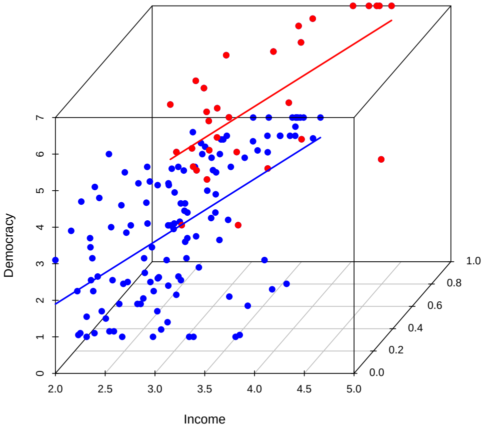
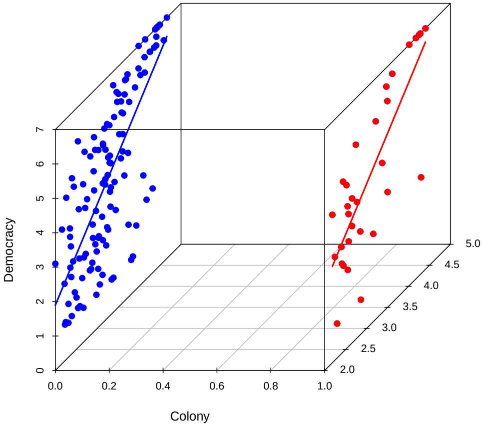
- We can try to represent this using a 3D scatterplot.
- In this view, we are looking at the data from the Income side; the two regression lines are drawn in the appropriate locations.



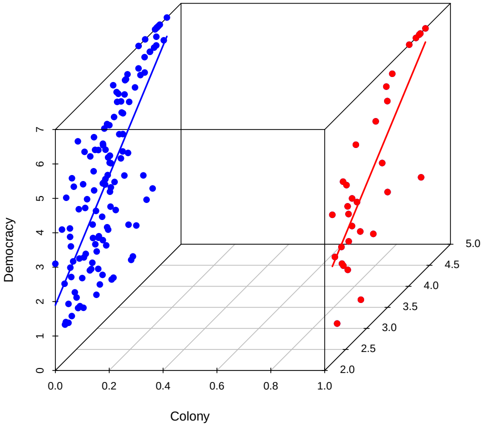
Regression of Democracy on Income

- We can also look at the 3D scatterplot from the British colony side.



Regression of Democracy on Income

- We can also look at the 3D scatterplot from the British colony side.
- While the British colonial status variable is either 0 or 1, there is nothing in the prediction equation that requires this to be the case.



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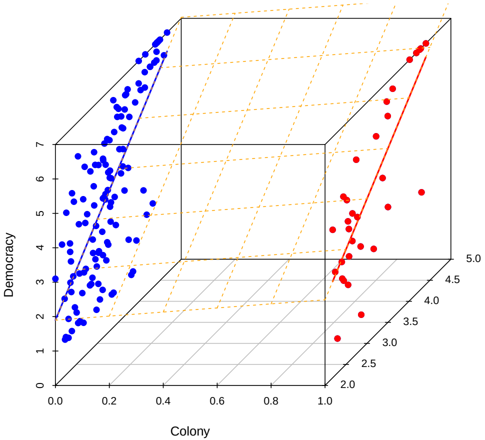
- We can also look at the 3D scatterplot from the British colony side.
- While the British colonial status variable is either 0 or 1, there is nothing in the prediction equation that requires this to be the case.
- In fact, the prediction equation defines a regression plane that connects the lines when $x_2 = 0$ and $x_2 = 1$.



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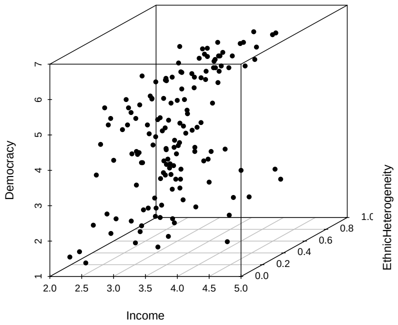
- For example, we might want to use:
  - X_1 Income and X_2 Ethnic Heterogeneity
  - Y Democracy

$$\widehat{\text{Democracy}} = \hat{\beta}_0 + \hat{\beta}_1 \text{Income} + \hat{\beta}_2 \text{Ethnic Heterogeneity}$$

Regression of Democracy on Income

- We can plot the points in a 3D scatterplot.



Regression of Democracy on Income

- We can plot the points in a 3D scatterplot.
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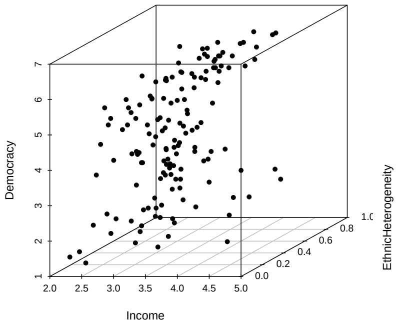
  - $\widehat{\beta}_1 = 1.6$ for Income
  - $\widehat{\beta}_2 = -.6$ for Ethnic Heterogeneity

  How does this look graphically?



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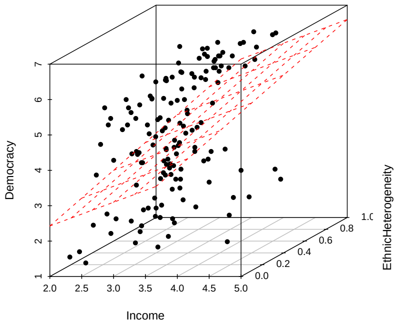

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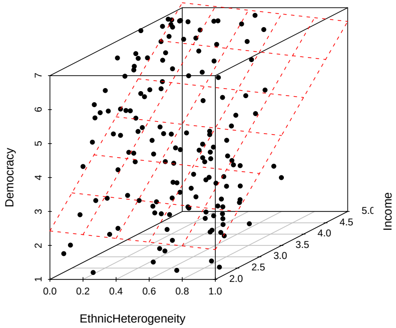

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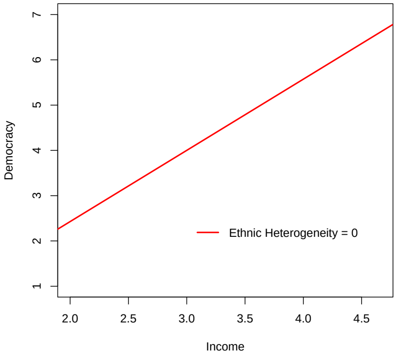

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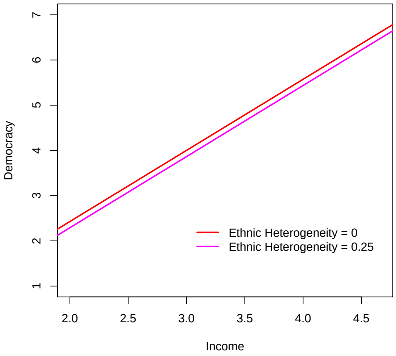

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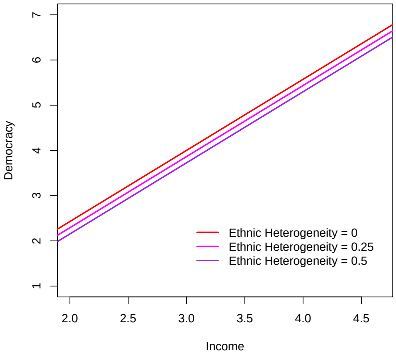

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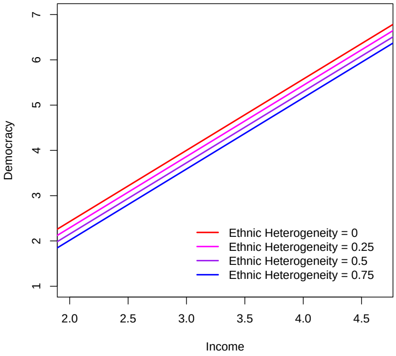

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- Predicted democracy is
  - $-.71 + 1.6 \cdot 3.5 - .6 \cdot .06 = 4.8$ for Chile

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- What is the predicted difference in democracy between
  - Chile with $X_1 = 3.5$ and $X_2 = .06$?
  - China with $X_1 = 2.5$ and $X_2 = .5$?
- Predicted democracy is
  - $-.71 + 1.6 \cdot 3.5 - .6 \cdot .06 = 4.8$ for Chile
  - $-.71 + 1.6 \cdot 2.5 - .6 \cdot 0.5 = 3$ for China.

More Complex Predictions

- We can also use the coefficient estimates for more complex predictions that involve changing multiple variables simultaneously.
- Consider our results for the regression of democracy on X_1 income and X_2 ethnic heterogeneity:
  - $\widehat{\beta}_0 = -.71$
  - $\widehat{\beta}_1 = 1.6$
  - $\widehat{\beta}_2 = -.6$
- What is the predicted difference in democracy between
  - Chile with $X_1 = 3.5$ and $X_2 = .06$?
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- Predicted democracy is
  - $-.71 + 1.6 \cdot 3.5 - .6 \cdot .06 = 4.8$ for Chile
  - $-.71 + 1.6 \cdot 2.5 - .6 \cdot 0.5 = 3$ for China.

  Predicted difference is thus: 1.8 or $(3.5 - 2.5)\widehat{\beta}_1 + (.06 - .5)\widehat{\beta}_2$

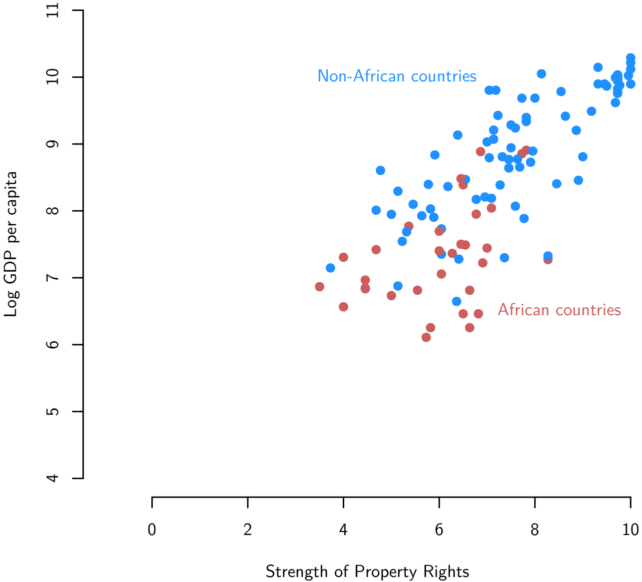
AJR Example

The Colonial Origins of Comparative Development:
An Empirical Investigation

By DARON ACEMOGLU, SIMON JOHNSON, AND JAMES A. ROBINSON*

http://www.jstor.org/stable/2677930

AJR Example


Non-African countries
African countries
Log GDP per capita
Strength of Property Rights
4
5
6
7
8
9
10
11
0
2
4
6
8
10

Basics

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- Ye olde model:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i$$

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- New model:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

AJR model

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  5.65556    0.31344  18.043  < 2e-16 ***
## avexpr       0.42416    0.03971  10.681  < 2e-16 ***
## africa      -0.87844    0.14707  -5.973 3.03e-08 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6253 on 108 degrees of freedom
##   (52 observations deleted due to missingness)
## Multiple R-squared:  0.7078, Adjusted R-squared:  0.7024
## F-statistic: 130.8 on 2 and 108 DF,  p-value: < 2.2e-16
```

Two lines in one regression

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$$
\begin{aligned}
\widehat{Y}_i &= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i \\
&= \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 \times 0 \\
&= \widehat{\beta}_0 + \widehat{\beta}_1 X_i
\end{aligned}
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- When $Z_i = 1$:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

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- Two different intercepts, same slope

Example interpretation of the coefficients

- Let's review what we've seen so far:

	Intercept for X_i	Slope for X_i
Non-African country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
African country ($Z_i = 1$)	$\widehat{\beta}_0 + \widehat{\beta}_2$	$\widehat{\beta}_1$

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- In this example, we have:

$$\widehat{Y}_i = 5.656 + 0.424 \times X_i - 0.878 \times Z_i$$

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  - $\widehat{\beta}_0$: average log income for non-African country ($Z_i = 0$) with property rights measured at 0 is 5.656

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Non-African country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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  - $\widehat{\beta}_0$: average log income for non-African country ($Z_i = 0$) with property rights measured at 0 is 5.656
  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)

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	Intercept for X_i	Slope for X_i
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- In this example, we have:

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  - $\widehat{\beta}_1$: A one-unit increase in property rights is associated with a 0.424 increase in average log incomes for two African countries (or for two non-African countries)
  - $\widehat{\beta}_2$: there is a -0.878 average difference in log income per capita between African and non-African counties **conditional on** property rights

General interpretation of the coefficients

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

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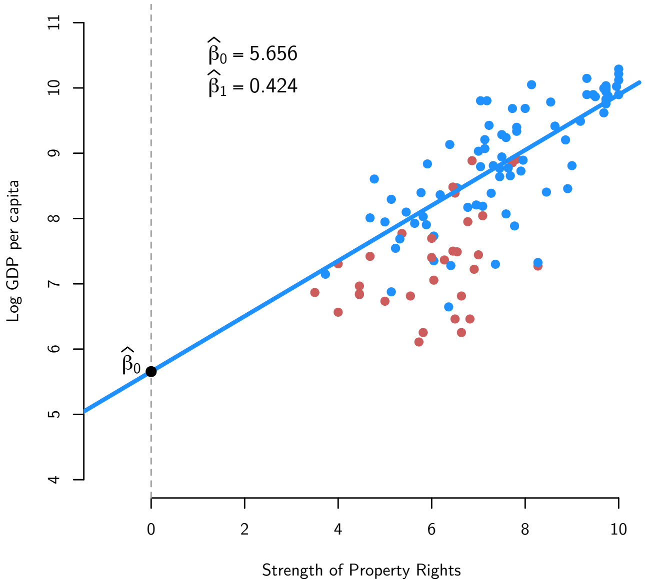
- $\widehat{\beta}_1$: A one-unit change in X_i produces a $\widehat{\beta}_1$-unit change in our prediction of Y_i conditional on Z_i
- $\widehat{\beta}_2$: average difference in Y_i between $Z_i = 1$ group and $Z_i = 0$ group conditional on X_i

Adding a binary variable, visually



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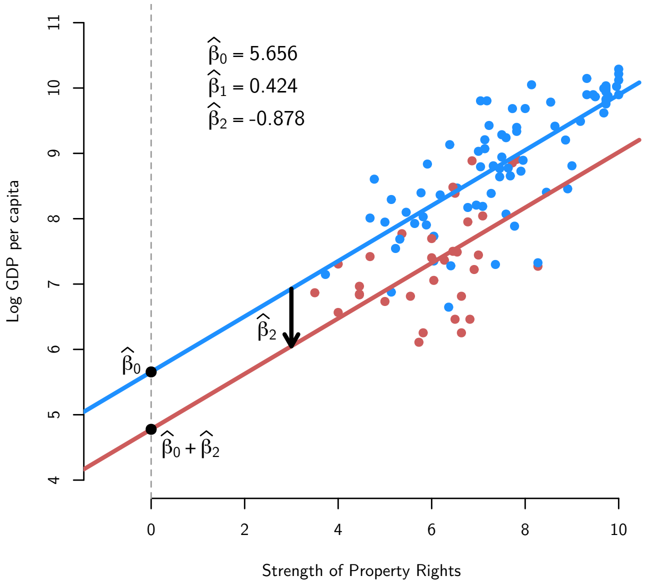

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- New model:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

AJR model, revisited

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  6.80627    0.75184   9.053 1.27e-12 ***
## avexpr       0.40568    0.06397   6.342 3.94e-08 ***
## meantemp    -0.06025    0.01940  -3.105  0.00296 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.6435 on 57 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.6155, Adjusted R-squared:  0.602
## F-statistic: 45.62 on 2 and 57 DF,  p-value: 1.481e-12
```

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^{\circ}\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^{\circ}\text{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
$Z_i = 21\,^{\circ}\text{C}$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 21$	$\widehat{\beta}_1$

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- In this example we have:

$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

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- In this example we have:

$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

- $\widehat{\beta}_0$: average log income for a country with property rights measured at 0 and a mean temperature of 0 is 6.806

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^\circ\mathrm{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

- $\widehat{\beta}_0$: average log income for a country with property rights measured at 0 and a mean temperature of 0 is 6.806
- $\widehat{\beta}_1$: A one-unit change in property rights is associated with a 0.406 change in average log incomes conditional on a country's mean temperature

Interpretation with a continuous Z

	Intercept for X_i	Slope for X_i
$Z_i = 0\,^\circ\text{C}$	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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$$\widehat{Y}_i = 6.806 + 0.406 \times X_i + -0.06 \times Z_i$$

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- $\widehat{\beta}_1$: A one-unit change in property rights is associated with a 0.406 change in average log incomes conditional on a country's mean temperature
- $\widehat{\beta}_2$: A one-degree increase in mean temperature is associated with a -0.06 change in average log incomes conditional on strength of property rights

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i for a fixed value of Z_i.

General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

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- The coefficient $\widehat{\beta}_2$ measures how the predicted outcome varies in Z_i for a fixed value of X_i.

Fitted values and residuals

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- Fitted values for $i = 1, \dots, n$:

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Fitted values and residuals

- Where do we get our hats? $\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2$
- To answer this, we first need to redefine some terms from simple linear regression.
- Fitted values for $i = 1, \ldots, n$:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

- Residuals for $i = 1, \ldots, n$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i$$

Least squares is still least squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?

Least squares is still least squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?
- Minimize the sum of the squared residuals, just like before:

$$(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2) = \arg\min_{b_0, b_1, b_2} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i - b_2 Z_i)^2$$

Least squares is still least squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?
- Minimize the sum of the squared residuals, just like before:

$$(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2) = \underset{b_0, b_1, b_2}{\arg\min} \sum_{i=1}^{n} (Y_i - b_0 - b_1 X_i - b_2 Z_i)^2$$

- The calculus is the same as last week, with 3 partial derivatives instead of 2

Least squares is still least squares

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- The calculus is the same as last week, with 3 partial derivatives instead of 2
- Let's start with a simple recipe and then rigorously show that it holds

OLS estimator recipe using two steps

- “Partialling out” OLS recipe:

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- Estimate of $\widehat{\beta}_1$ will be the same as running:
$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

Regression property rights on mean temperature

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  9.95678    0.82015  12.140  < 2e-16 ***
## meantemp    -0.14900    0.03469  -4.295 6.73e-05 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 1.321 on 58 degrees of freedom
##   (103 observations deleted due to missingness)
## Multiple R-squared:  0.2413, Adjusted R-squared:  0.2282
## F-statistic: 18.45 on 1 and 58 DF,  p-value: 6.733e-05
```

Regression of log income on the residuals

```
## (Intercept)  avexpr.res
##   8.0542783   0.4056757

## (Intercept)      avexpr    meantemp
##  6.80627375  0.40567575 -0.06024937
```

Residual/partial regression plot

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Useful for plotting the conditional relationship between property rights and income given temperature:

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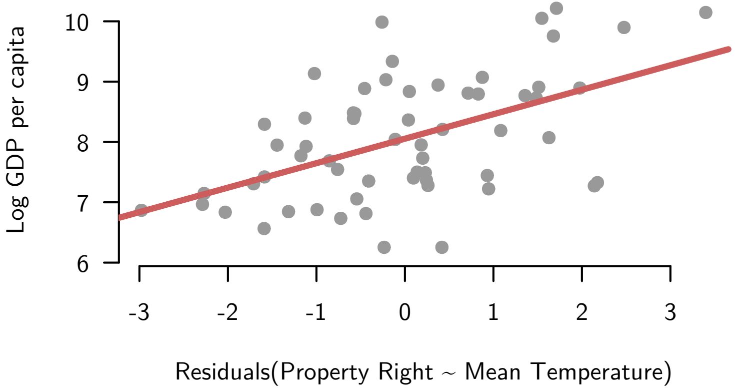

Deriving the Linear Least Squares Estimator

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- We use the same principle for picking $(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2)$ for regression with two regressors (x_i and z_i):

$$
\begin{aligned}
(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2) &= \underset{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} \widehat{u}_i^2 = \underset{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 \\
&= \underset{\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2}{\operatorname{argmin}} \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - x_i\tilde{\beta}_1 - z_i\tilde{\beta}_2)^2
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$$

- (The same works more generally for k regressors, but this is done more easily with matrices as we will see next week)

Deriving the Linear Least Squares Estimator

We want to minimize the following quantitity with respect to $(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)$:

$$S(\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2) = \sum_{i=1}^{n} (y_i - \tilde{\beta}_0 - \tilde{\beta}_1 x_i - \tilde{\beta}_2 z_i)^2$$

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1. Take the partial derivatives of S with respect to $\tilde{\beta}_0, \tilde{\beta}_1$ and $\tilde{\beta}_2$.

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Plan is conceptually the same as before

1. Take the partial derivatives of S with respect to $\tilde{\beta}_0, \tilde{\beta}_1$ and $\tilde{\beta}_2$.
2. Set each of the partial derivatives to 0 to obtain the first order conditions.
3. Substitute $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ for $\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2$ and solve for $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ to obtain the OLS estimator.

First Order Conditions

Setting the partial derivatives equal to zero leads to a system of 3 linear equations in 3 unknowns: $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$

$$\frac{\partial S}{\partial \tilde{\beta}_0} = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0$$

$$\frac{\partial S}{\partial \tilde{\beta}_1} = \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0$$

$$\frac{\partial S}{\partial \tilde{\beta}_2} = \sum_{i=1}^{n} z_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0$$

When will this linear system have a unique solution?

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\end{aligned}$$

When will this linear system have a unique solution?

- More observations than predictors (i.e. $n > 2$)
- x and z are linearly independent, i.e.,
  - neither x nor z is a constant
  - x is not a linear function of z (or vice versa)
- Wooldridge calls this assumption no perfect collinearity

The OLS Estimator

The OLS estimator for $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ can be written as

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} - \hat{\beta}_2 \bar{z} \\
\hat{\beta}_1 &= \frac{Cov(x,y)Var(z) - Cov(z,y)Cov(x,z)}{Var(x)Var(z) - Cov(x,z)^2} \\
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For $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ to be well-defined we need:

$$Var(x)Var(z) \neq Cov(x,z)^2$$

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Condition fails if:

1. If x or z is a constant ($\Rightarrow Var(x)Var(z) = Cov(x,z) = 0$)
2. One explanatory variable is an exact linear function of another ($\Rightarrow \; Cor(x,z) = 1 \; \Rightarrow \; Var(x)Var(z) = Cov(x,z)^2$)

"Partialling Out" Interpretation of the OLS Estimator

Assume $Y = \beta_0 + \beta_1 X + \beta_2 Z + u$. Another way to write the OLS estimator is:

$$\hat{\beta}_1 = \frac{\sum_i^n \hat{r}_{xz,i}\, y_i}{\sum_i^n \hat{r}_{xz,i}^2}$$

where $\hat{r}_{xz,i}$ are the residuals from the regression of X on Z:

$$X = \lambda + \delta Z + r_{xz}$$

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In other words, both of these regressions yield identical estimates $\hat{\beta}_1$:

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- That is, same as the simple regresson of Y on X alone.

Origin of the Partial Out Recipe

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- Residuals $\hat{r}_{xz}$ are the part of X that is uncorrelated with Z. Put differently, $\hat{r}_{xz}$ is X, after the effect of Z on X has been partialled out or netted out.
- Can use same equation with k explanatory variables; $\hat{r}_{xz}$ will then come from a regression of X on all the other explanatory variables.

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2. Random/iid sample
3. No perfect collinearity
4. Zero conditional mean error
$$\mathbb{E}[u_i | X_i, Z_i] = 0$$

New assumption

Assumption 3: No perfect collinearity

(1) No explanatory variable is constant in the sample and (2) there are no exactly linear relationships among the explanatory variables.

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- What's the correlation between Z_i and X_i? 1!

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- No! Because while Z_i is a deterministic function of X_i, it is not a linear function of X_i.

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```
##
## Coefficients: (1 not defined because of singularities)
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.71638    0.08991  96.941  < 2e-16 ***
## africa      -1.36119    0.16306  -8.348 4.87e-14 ***
## nonafrica         NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9125 on 146 degrees of freedom
##   (15 observations deleted due to missingness)
## Multiple R-squared:  0.3231, Adjusted R-squared:  0.3184
## F-statistic: 69.68 on 1 and 146 DF,  p-value: 4.87e-14
```

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```
## (Intercept)    meantemp  meantemp.f
##  10.8454999  -0.1206948          NA
```

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6. Normal conditional errors
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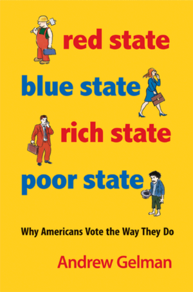
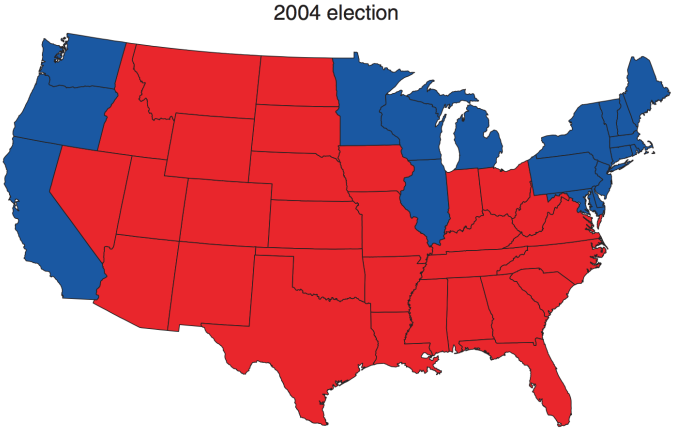
- $\rightsquigarrow$ small adjustments to the critical values and the t-values for our hypothesis tests and confidence intervals.

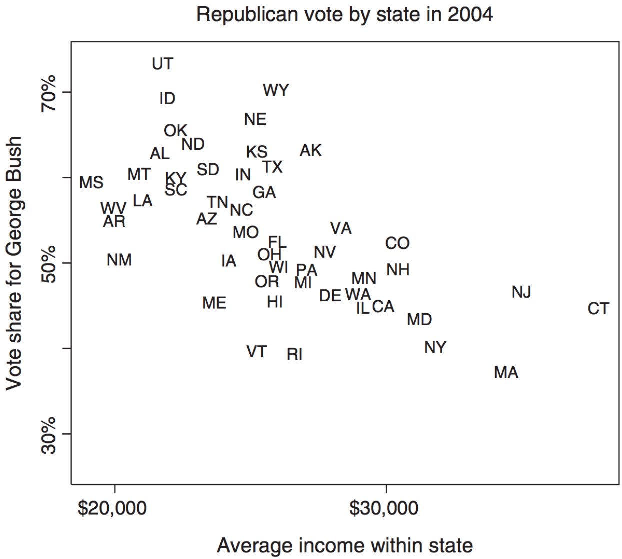
Red State Blue State



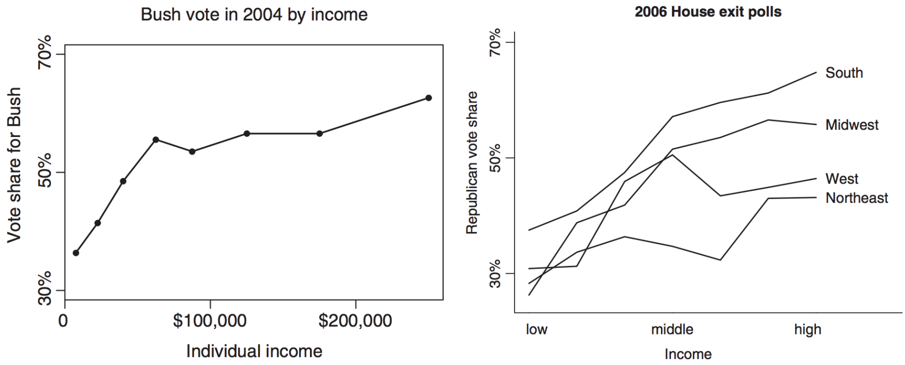
Red and Blue States



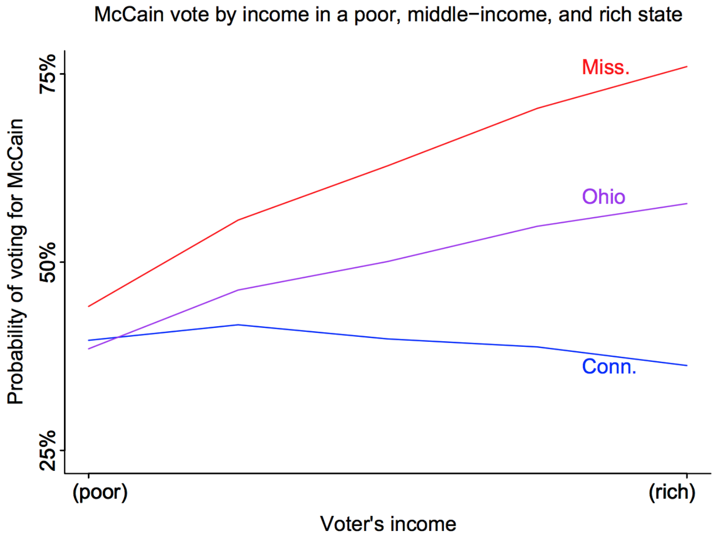
Rich States are More Democratic



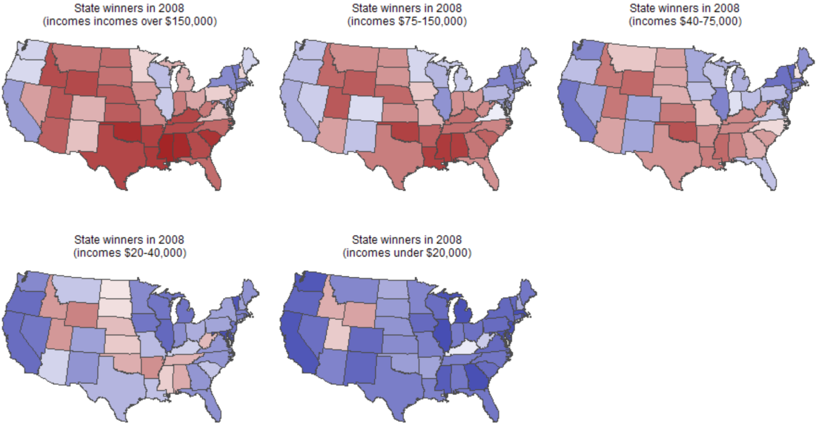
But Rich People are More Republican



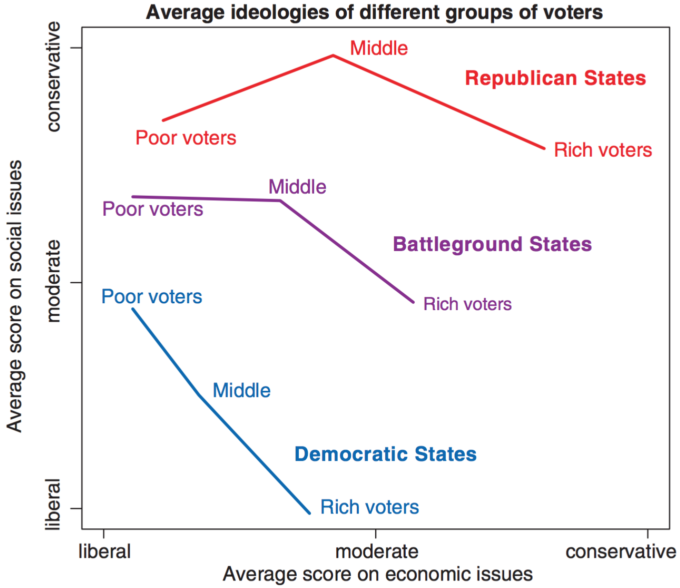
Paradox Resolved



If Only Rich People Voted, it Would Be a Landslide



A Possible Explanation



References

Acemoglu, Daron, Simon Johnson, and James A. Robinson. "The colonial origins of comparative development: An empirical investigation." *American Economic Review*. 91(5). 2001: 1369-1401.

Fish, M. Steven. "Islam and authoritarianism." *World politics* 55(01). 2002: 4-37.

Gelman, Andrew. *Red state, blue state, rich state, poor state: why Americans vote the way they do*. Princeton University Press, 2009.

Where We've Been and Where We're Going...

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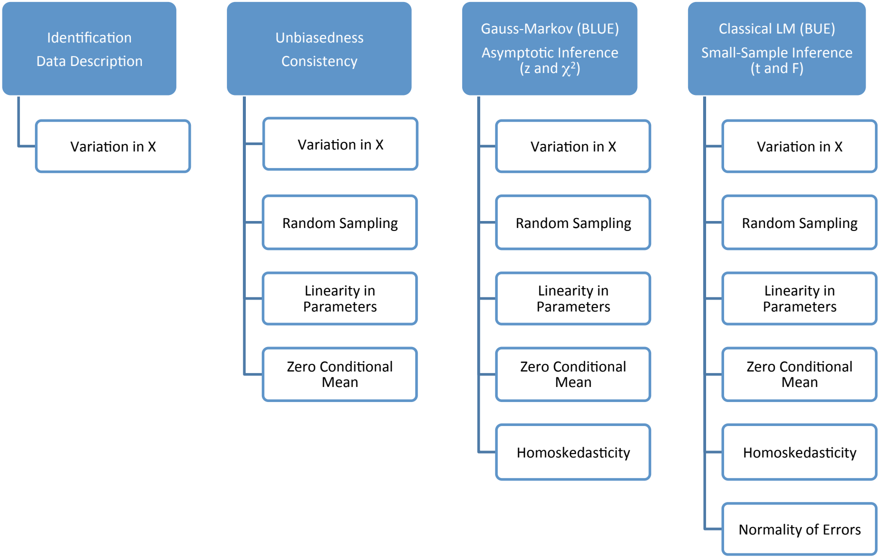
- Last Week
  - mechanics of OLS with one variable
  - properties of OLS
- This Week
  - Monday:
    - adding a second variable
    - new mechanics
  - Wednesday:
    - omitted variable bias
    - multicollinearity
    - interactions
- Next Week
  - multiple regression
- Long Run
  - probability → inference → regression → causal inference

Questions?

1. Two Examples
2. Adding a Binary Variable
3. Adding a Continuous Covariate
4. Once More With Feeling
5. OLS Mechanics and Partialing Out
6. Fun With Red and Blue
7. Omitted Variables
8. Multicollinearity
9. Dummy Variables
10. Interaction Terms
11. Polynomials
12. Conclusion
13. Fun With Interactions

Remember This?



Unbiasedness revisited

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Answer: $\tilde{\beta}_1$ is upward biased since being a strong republican is positively correlated with both watching fox news and voting republican. We have $\beta_1 < E[\tilde{\beta}_1]$.

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Answer: The negative coefficient $\tilde{\beta}_1$ is downward biased compared to the true β_1 so $\beta_1 > E[\tilde{\beta}_1]$. Being hospitalized is negatively correlated with health, and health is positively correlated with survival.

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A. If $\tilde{\delta} = 0$ or $\hat{\beta}_2 = 0$.

Omitted Variable Bias: Simple Case

We take expectations to see what the bias will be:

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So the bias depends on the relationship between x_2 and x_1, our $\tilde{\delta}$, and the relationship between x_2 and y, our β_2.

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- imbalance is how the expectation of the unobserved confounder x_2 varies across levels of x_1.

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Direction of the bias of $\tilde{\beta}_1$ compared to β_1 is given by:

	$\text{cov}(X_1, X_2) > 0$	$\text{cov}(X_1, X_2) < 0$	$\text{cov}(X_1, X_2) = 0$
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- In the more general case with more than two covariates the bias is more difficult to discern. It depends on all the pairwise correlations.

Including an Irrelevant Variable: Simple Case

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True Population Model:

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + u \quad \text{where} \quad \beta_2 = 0$$

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Q: Which statement is correct?

1. $\beta_1 > E[\tilde{\beta}_1]$
2. $\beta_1 < E[\tilde{\beta}_1]$
3. $\beta_1 = E[\tilde{\beta}_1]$
4. Can't tell

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and thus including the irrelevant variable does not generally affect the unbiasedness. The sampling distribution of $\tilde{\beta}_2$ will be centered about zero.

Sampling variance for simple linear regression

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- Regression with an additional independent variable:

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- What happens with perfect collinearity? $R_1^2 = 1$ and the variances are infinite.

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- Given the symmetry, it will also increase $\text{var}(\widehat{\beta}_2)$ as well.

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- Basically, there is less residual variation left in X_i after "partialling out" the effect of Z_i

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In R, `vif()` in the `car` package.

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- If X_1 and X_2 are almost the same, why would you want a unique β_1 and a unique β_2? Think about how you would interpret that?
- Relax, you got way more important things to worry about!
- If possible, get more data
- Drop one of the variables, or combine them
- Or maybe linear regression is not the right tool

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  - E.g. does the effect of education differ by gender?

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- For this we use a single dummy variable which is coded like:

$$D_i = \begin{cases} 1 & \text{if country } i \text{ has a Majoritarian Electoral System} \\ 0 & \text{if country } i \text{ has a Proportional Electoral System} \end{cases}$$

How Can I Use a Dummy Variable?

- Consider the easiest case with two categories. The type of electoral system of country i is given by:

  $X_i \in \{Proportional, Majoritarian\}$

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- Let's regress GDP on this dummy variable and a constant:

  $Y = \beta_0 + \beta_1 D + u$

Example: GDP per capita on Electoral System

R Code

```
> summary(lm(REALGDPCAP ~ MAJORITARIAN, data = D))

Call:
lm(formula = REALGDPCAP ~ MAJORITARIAN, data = D)

Residuals:
   Min     1Q Median     3Q    Max
 -5982  -4592  -2112   4293  13685

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)    7097.7      763.2    9.30 1.64e-14 ***
MAJORITARIAN  -1053.8     1224.9   -0.86    0.392
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 5504 on 83 degrees of freedom
Multiple R-squared: 0.008838,        Adjusted R-squared: -0.003104
F-statistic: 0.7401 on 1 and 83 DF,  p-value: 0.3921
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> summary(gdp.pro)
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Dummy Variables for Multiple Categories

- More generally, let's say X measures which of m categories each unit i belongs to. E.g. the type of electoral system or region of country i is given by:
  - $X_i \in \{Proportional, Majoritarian\}$ so $m = 2$
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- The omitted category is our baseline case (also called a reference category) against which we compare the conditional means of Y for the other $m - 1$ categories.

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Our regression equation is:

$$Y = \beta_0 + \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + u$$

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- Interactions often confuses researchers and mistakes in use and interpretation occur frequently (even in top journals)

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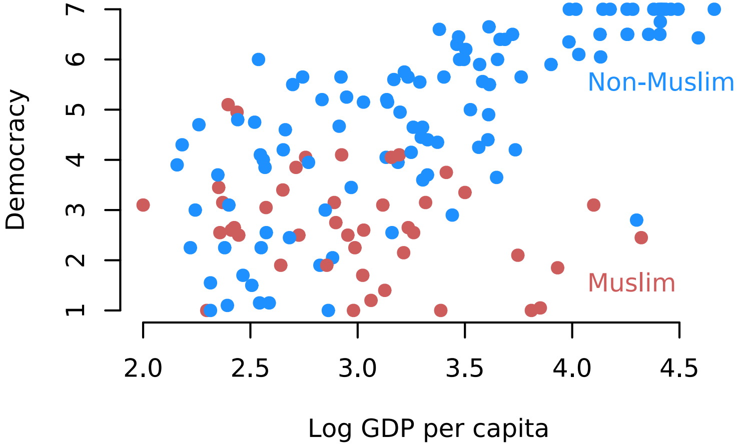
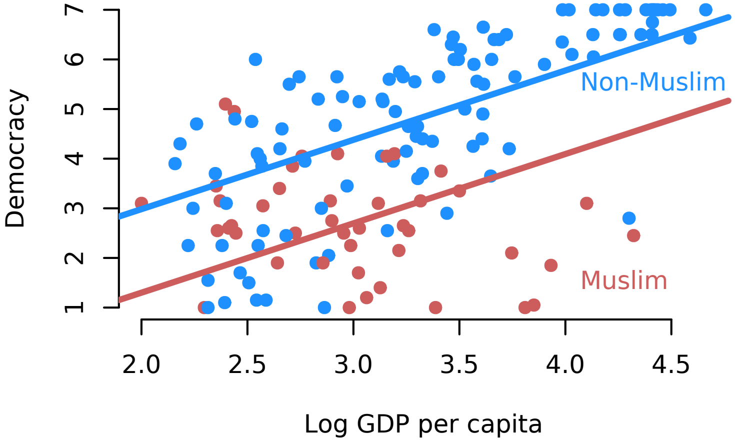
- We measure democracy with a Freedom House score, 1 (less free) to 7 (more free)

Let's see the data



Fish argues that Muslim countries are less likely to be democratic no matter their economic development

Controlling for Religion Additively



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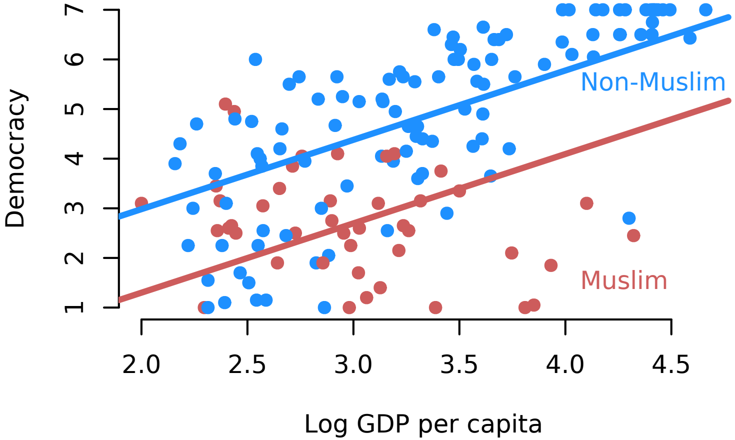

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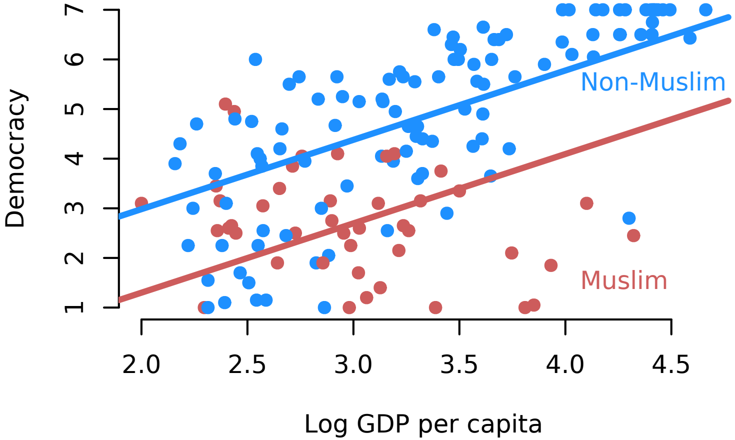

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Can we allow for different slopes for each group?

Interactions with a binary variable

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- This covariate is called an interaction term and it is the product of the two marginal variables of interest: $income_i \times muslim_i$
- Here is the model with the interaction term:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

Two lines in one regression

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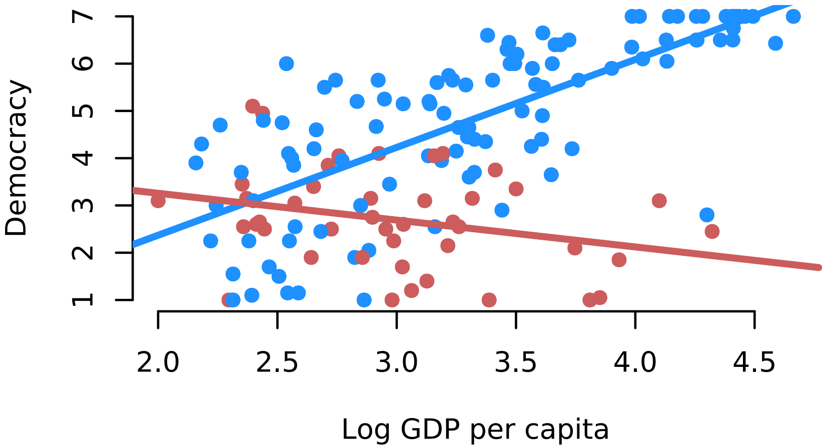

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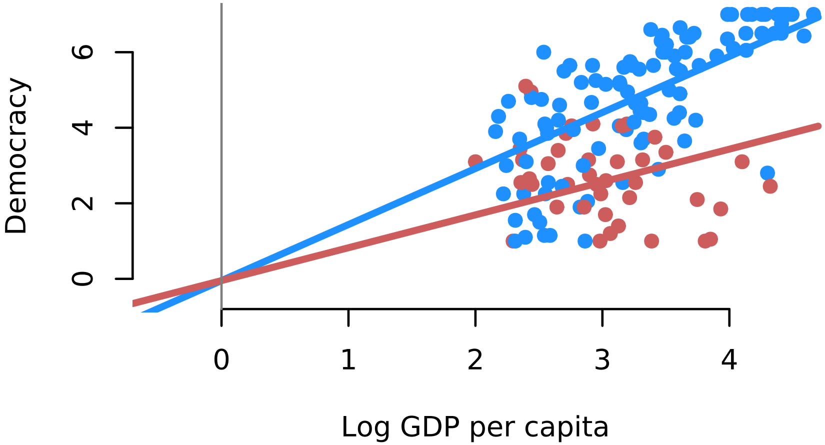

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- And include it in the regression:

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Interpretation

- With a continuous Z_i, we can have more than two values that it can take on:

	Intercept for X_i	Slope for X_i
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General interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

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We will talk about checking these assumptions in a few weeks.

How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice

Jens Hainmueller Jonathan Mummolo Yiqing Xu*

April 20, 2018

(*Political Analysis*, forthcoming)

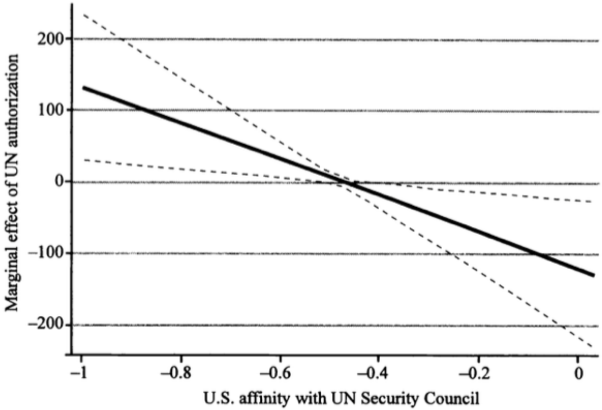
Abstract

Multiplicative interaction models are widely used in social science to examine whether the relationship between an outcome and an independent variable changes with a moderating variable. Current empirical practice tends to overlook two important problems. First, these models assume a linear interaction effect that changes at a constant rate with the moderator. Second, estimates of the conditional effects of the independent variable can be misleading if there is a lack of common support of the moderator. Replicating 46 interaction effects from 22 recent publications in five top political science journals, we find that these core assumptions often fail in practice, suggesting that a large portion of findings across all political science subfields based on interaction models are modeling artifacts or are at best highly model dependent. We propose a checklist of simple diagnostics to assess the validity of these assumptions and offer flexible estimation strategies that allow for nonlinear interaction effects and safeguard against excessive extrapolation. These statistical routines are available in both R and STATA.

Example: Common Support

Chapman 2009 analysis

example and reanalysis from Hainmueller, Mummolo, Xu 2016



Note: Dashed lines give 95 percent confidence interval.

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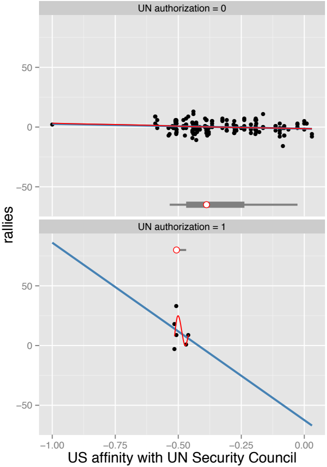

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Further Reading: Brambor, Clark, and Golder. 2006. Understanding Interaction Models: Improving Empirical Analyses. *Political Analysis* 14 (1): 63-82.

Hainmueller, Mummolo, Xu. 2016. How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice. *Working Paper*

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Y &= \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u \\
Y &= \beta_0 + \tilde{\beta}_1 X_1 + \tilde{\beta}_2 X_1^2 + u
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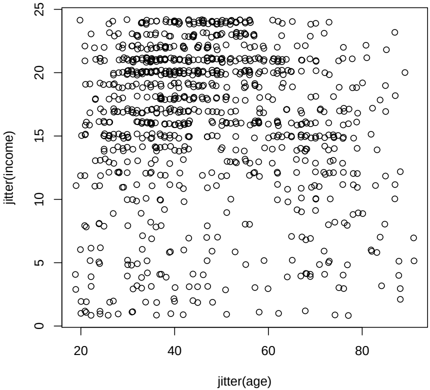
$$Y = \beta_0 + \tilde{\beta}_1 X_1 + \tilde{\beta}_2 X_1^2 + u$$

- This is called a second order polynomial in X_1
- A third order polynomial is given by: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + u$

Polynomial Example: Income and Age

- Let's look at data from the U.S. and examine the relationship between Y: income and X: age



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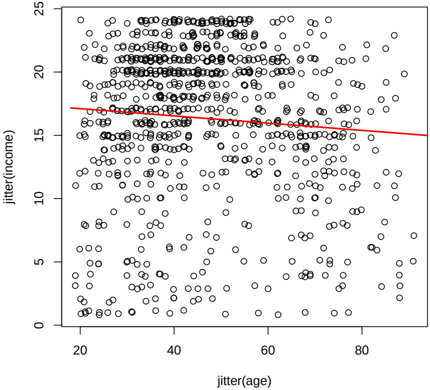

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- We see that a simple linear specification does not fit the data very well:
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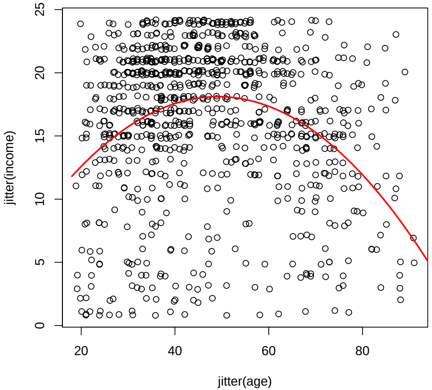
- A second order polynomial in age fits the data a lot better:
  $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$



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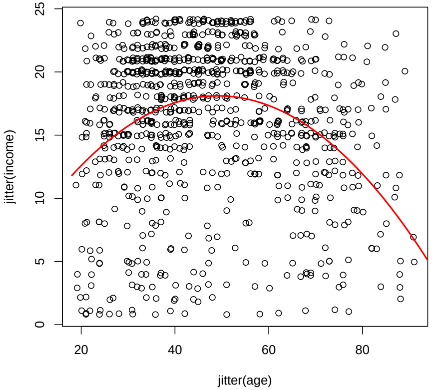

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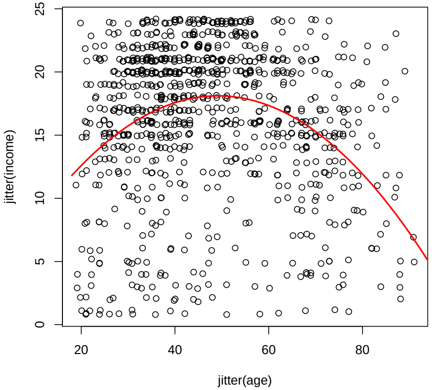
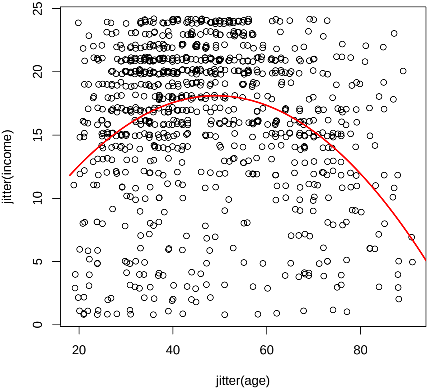


Polynomial Example: Income and Age

- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$
- Is β_1 the marginal effect of age on income?
- No! The marginal effect of age depends on the level of age: $\frac{\partial Y}{\partial X_1} = \widehat{\beta}_1 + 2\,\widehat{\beta}_2\,X_1$
  Here the effect of age changes monotonically from positive to negative with income.



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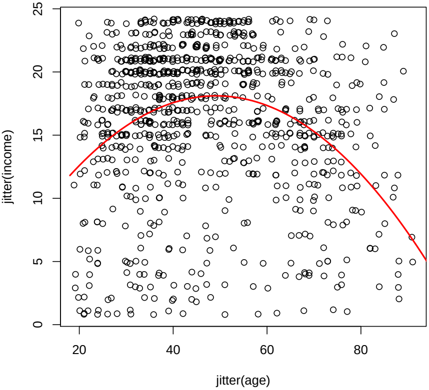
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.



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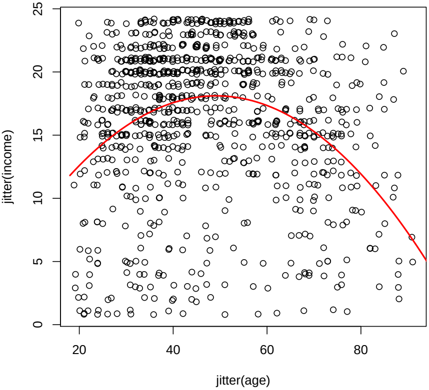
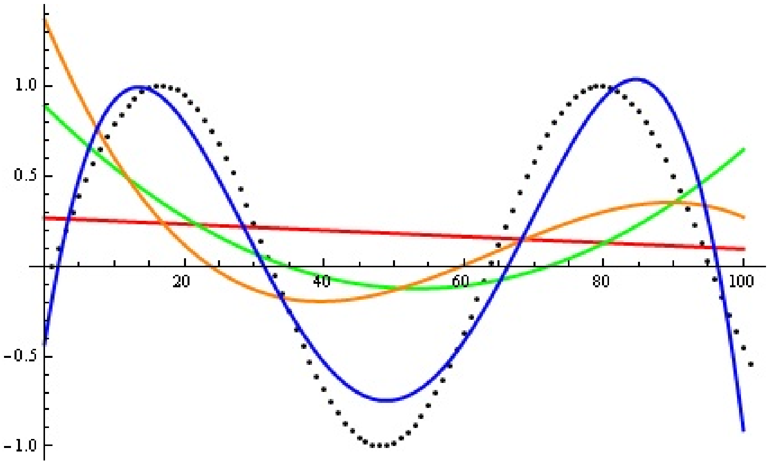
  Here the effect of age changes monotonically from positive to negative with income.
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $|\frac{\beta_1}{2\beta_2}|$. Here turning point is at $X_1 = 50$.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

Conclusion

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Conclusion

In this brave new world with 2 independent variables:

1. β's have slightly different interpretations
2. OLS still minimizing the sum of the squared residuals
3. Small adjustments to OLS assumptions and inference
4. Adding or omitting variables in a regression can affect the bias and the variance of OLS
5. We can optionally consider interactions, but must take care to interpret them correctly

Next Week

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- OLS in its full glory

Next Week

- OLS in its full glory
- Reading:
  - Practice up on matrices - we won't spend time reviewing matrix multiplication and inverses.
  - Aronow and Miller 4.1.3 Regression with Matrix Algebra
  - Optional: Fox Chapter 9.1-9.4 (skip 9.1.1-9.1.2) Linear Models in Matrix Form
  - Optional: Fox Chapter 10 Geometry of Regression
  - Optional: Imai Chapter 4.3-4.3.3
  - Optional: Angrist and Pischke Chapter 3.1 Regression Fundamentals

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Breznau (2015) "The Missing Main Effect of Welfare State Regimes: A Replication of 'Social Policy Responsiveness in Developed Democracies."' *Sociological Science*.

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- Democracy empowers the masses, and that empowerment helps define social outcomes
- Key model is interaction between liberal/non-liberal and public preferences on social spending
- but. . . they leave out a main effect.

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- They mean center so the 0 represents the average over the entire sample

What Happens?

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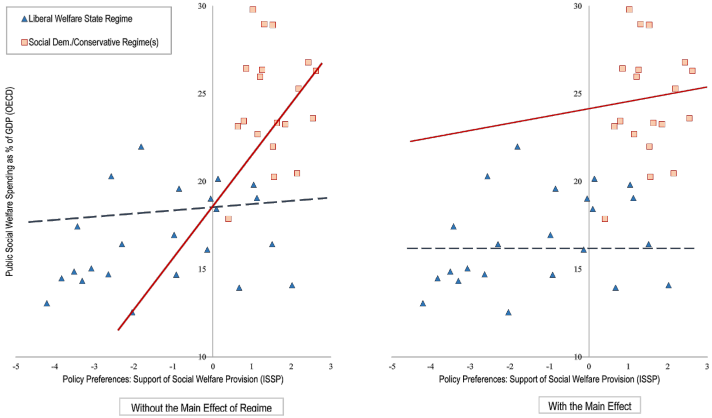

Figure 1: Predicted Regression Lines for the Effect of Policy Preferences on Social Welfare Spending, without and with the Main Effect of Regime

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<PLEASE>

References

Acemoglu, Daron, Simon Johnson, and James A. Robinson. "The colonial origins of comparative development: An empirical investigation." *American Economic Review*. 91(5). 2001: 1369-1401.

Fish, M. Steven. "Islam and authoritarianism." *World politics* 55(01). 2002: 4-37.

Gelman, Andrew. *Red state, blue state, rich state, poor state: why Americans vote the way they do*. Princeton University Press, 2009.