Week 7: Multiple Regression

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller and Erin Hartman.

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  - regression with two variables
  - omitted variables, multicollinearity, interactions

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- Next Week
  - break!
  - then ... diagnostics
- Long Run
  - probability → inference → regression → causal inference

Questions?

1. Matrix Form of Regression
2. OLS inference in matrix form
3. Standard Hypothesis Tests
4. Testing Joint Significance
5. Testing Linear Hypotheses: The General Case
6. Fun With(out) Weights
7. Appendix: Derivations and Consistency
8. The Problems with p-values
9. Agnostic Regression
10. Inference via the Bootstrap
11. Fun With Weights
12. Appendix: Tricky p-value Example

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- Outcome is a linear combination of the the $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ vectors

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- We can also write this at the individual level, where $\mathbf{x}_i'$ is the ith row of $\mathbf{X}$:

$$y_i = \mathbf{x}_i'\boldsymbol{\beta} + u_i$$

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- We've learned about matrix multiplication, but what about matrix "division"?

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Disclaimer: the final equation is exactly true for all non-intercept coefficients if you remove the intercept from $\mathbf{X}$ such that $\hat{\boldsymbol{\beta}}_{-0} = \text{Var}(\mathbf{X}_{-0})^{-1}\text{Cov}(\mathbf{X}_{-0}, \mathbf{y})$. The numerator and denominator are the variances and covariances if $\mathbf{X}$ and $\mathbf{y}$ are demeaned and normalized by the sample size minus 1.

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1. Linearity: $\mathbf{y} = \mathbf{X}\beta + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (k+1)$ matrix with rank $k+1$
4. Zero conditional mean: $\mathbb{E}[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
6. Normality: $\mathbf{u}|\mathbf{X} \sim N(\mathbf{0}, \sigma_u^2 \mathbf{I}_n)$

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- Just like variation in X led us to be able to divide by the variance in simple OLS

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- The stated homoskedasticity assumption is: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
- To really understand this we need to know what $\text{var}(\mathbf{u}|\mathbf{X})$ is in full generality.

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- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

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- This matrix is always symmetric since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$ by definition.

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- In less matrix notation:
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So, yes!

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Now we know the sampling distribution is centered on β we want to derive the variance of the sampling distribution conditional on X.

Rule: Variance of Linear Function of Random Vector

Recall that for a linear transformation of a random variable X we have $V[aX + b] = a^2 V[X]$ with constants a and b.

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Definition (Variance of Linear Transformation of Random Vector)

Let $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[f(\mathbf{u})] = V[\mathbf{A}\mathbf{u} + \mathbf{B}] = \mathbf{A}V[\mathbf{u}]\mathbf{A}' = \mathbf{A}\Sigma_{\mathbf{u}}\mathbf{A}'$$

Conditional Variance of $\hat{\beta}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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\end{aligned}
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This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\boldsymbol{\beta}}$.

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&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'V[\mathbf{u}|\mathbf{X}]\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\sigma^2\mathbf{I}\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \quad \text{(by homoskedasticity)} \\
&= \sigma^2\mathbf{I}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}(\mathbf{X}'\mathbf{X})^{-1} \\
&= \sigma^2(\mathbf{X}'\mathbf{X})^{-1}
\end{aligned}
$$

This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\beta}$.

To estimate $V[\hat{\beta}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$
\hat{\sigma}^2 = \frac{\sum_i \hat{u}_i^2}{n-(k+1)} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}
$$

Sampling Variance for $\hat{\beta}$

Under assumptions 1-5, the variance-covariance matrix of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} =$$

$$\begin{array}{c|cccccc}
 & \widehat{\beta}_0 & \widehat{\beta}_1 & \widehat{\beta}_2 & \cdots & \widehat{\beta}_k \\
\hline
\widehat{\beta}_0 & V[\widehat{\beta}_0] & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1] & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2] & \cdots & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k] \\
\widehat{\beta}_1 & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1] & V[\widehat{\beta}_1] & \text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2] & \cdots & \text{Cov}[\widehat{\beta}_1, \widehat{\beta}_k] \\
\widehat{\beta}_2 & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2] & \text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2] & V[\widehat{\beta}_2] & \cdots & \text{Cov}[\widehat{\beta}_2, \widehat{\beta}_k] \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\widehat{\beta}_k & \text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k] & \text{Cov}[\widehat{\beta}_k, \widehat{\beta}_1] & \text{Cov}[\widehat{\beta}_k, \widehat{\beta}_2] & \cdots & V[\widehat{\beta}_k]
\end{array}$$

Recall that standard errors are the square root of the diagonals of this matrix.

Overview of Inference in the General Setting

- Under assumption 1-5 in large samples:

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}[\widehat{\beta}_j]} \sim N(0,1)$$

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$$\widehat{SE}[\hat{\beta}_j] = \sqrt{\widehat{\text{var}}[\widehat{\boldsymbol{\beta}}]_{jj}}$$

$$\widehat{\text{var}}[\widehat{\boldsymbol{\beta}}] = \widehat{\sigma}^2_u(\mathbf{X}'\mathbf{X})^{-1}$$

$$\widehat{\sigma}^2_u = \frac{\widehat{\mathbf{u}}'\widehat{\mathbf{u}}}{n-(k+1)}$$

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$$\widehat{\sigma}_u^2 = \frac{\widehat{\mathbf{u}}'\widehat{\mathbf{u}}}{n-(k+1)}$$

- Thus, confidence intervals and hypothesis tests proceed in essentially the same way.

Properties of the OLS Estimator: Summary

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Theorem

Under Assumptions 1–6, the $(k+1)\times 1$ *vector of OLS estimators* $\hat{\boldsymbol{\beta}}$*, conditional on* $\mathbf{X}$*, follows a multivariate normal distribution with mean* $\boldsymbol{\beta}$ *and variance-covariance matrix* $\sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$*:*

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$$\hat{\boldsymbol{\beta}}|\mathbf{X} \sim \mathcal{N}\left(\boldsymbol{\beta}, \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}\right)$$

- *Each element of* $\hat{\boldsymbol{\beta}}$ *(i.e.* $\hat{\beta}_0, \ldots, \hat{\beta}_{k+1}$*) is normally distributed, and* $\hat{\boldsymbol{\beta}}$ *is an unbiased estimator of* $\boldsymbol{\beta}$ *as* $E[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$

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$$\widehat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n - (k+1)}$$

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$$\widehat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n - (k+1)}$$

- *With a large sample, $\hat{\boldsymbol{\beta}}$ approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of $\mathbf{u}$.*

Implications of the Variance-Covariance Matrix

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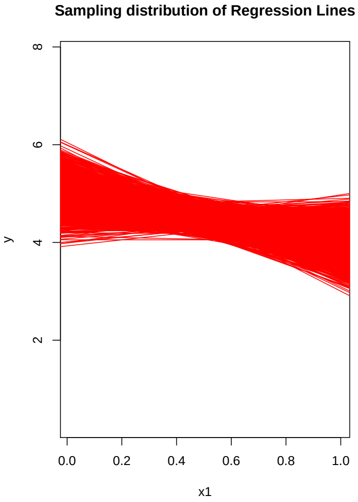
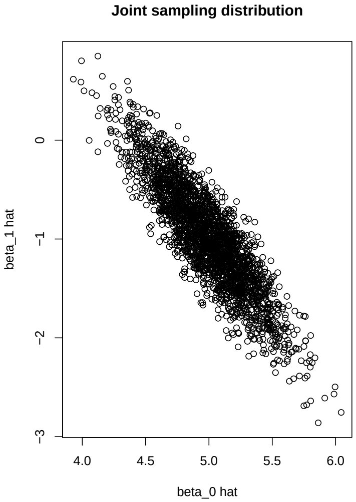
- Note that the sampling distribution is a joint distribution because it involves multiple random variables.
- This is because the sampling distribution of the terms in $\hat{\beta}$ are correlated.
- In a practical sense, this means that our uncertainty about coefficients is correlated across variables.

Multivariate Normal: Simulation

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:





Marginals of Multivariate Normal RVs are Normal

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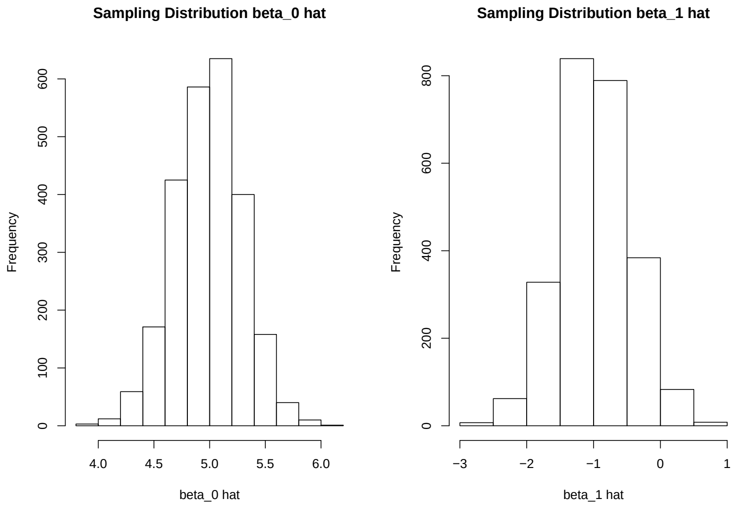

Running Example: Chilean Referendum on Pinochet

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- Plebiscite was held on October 5, 1988. The No side won with 56% of the vote, with 44% voting Yes.
- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

Model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

The t-Value for Multiple Linear Regression

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$$H_0 : \ \beta_j \ = \ c$$

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$$T \ = \ \frac{\hat{\beta}_j - c}{\hat{SE}(\hat{\beta}_j)}$$

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$$\hat{SE}(\hat{\beta}_j) = \sqrt{\widehat{V}(\hat{\beta}_j)} = \sqrt{\widehat{V}(\hat{\boldsymbol{\beta}})_{(j,j)}} = \sqrt{\hat{\sigma}^2(\mathbf{X}'\mathbf{X})^{-1}_{(j,j)}}$$

where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.

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where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.
That is, take the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and square root the diagonal element corresponding to j.

Getting the Standard Errors

R Code

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> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
Coefficients:
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R Code

```
> V <- vcov(fit)
> V
              (Intercept)           fem          educ           age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem         -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ        -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age         -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
 (Intercept)          fem         educ          age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

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2. Compare the value to the critical value $t_{\alpha/2}$ for the α level test, which under the null hypothesis satisfies

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3. Decide whether the realized value of T in our data is unusual given the known distribution of the test statistic.
4. Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

To construct confidence intervals, there is again no difference compared to the case of $k = 1$, except that we need to use t_{n-k-1} instead of t_{n-2}

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So we also know the probability that the value of our test statistics falls into a given interval:

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We rearrange:

$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\,,\, \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

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We rearrange:

$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\,,\; \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
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---
```

R Code

```
> confint(fit)
                   2.5 %      97.5 %
(Intercept)  0.303407780  0.50504909
fem          0.089493169  0.18251357
educ        -0.087954435 -0.03356629
age          0.002147755  0.00540954
```

Testing Hypothesis About a Linear Combination of β_j

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4452.7      783.4   5.684 2.07e-07 ***
RegionAfrica     -2552.8     1204.5  -2.119   0.0372 *
RegionAsia         148.9     1149.8   0.129   0.8973
RegionLatAmerica  -271.3     1007.0  -0.269   0.7883
RegionOecd        9671.3     1007.0   9.604 5.74e-15 ***
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```

- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0: \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

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RegionAfrica     -2552.8     1204.5  -2.119   0.0372 *
RegionAsia         148.9     1149.8   0.129   0.8973
RegionLatAmerica  -271.3     1007.0  -0.269   0.7883
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- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0 : \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

against the alternative of

$$H_1 : \ \beta_{LAm} \neq \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} \neq 0$$

Testing Hypothesis About a Linear Combination of β_j

R Code

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- What would be an appropriate test statistic for this hypothesis?

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- Let's consider a t-value:

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We will reject H_0 if T is sufficiently different from zero.

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- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About a Linear Combination of β_j

- Our test statistic:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \sim t_{n-k-1}$$

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$$V(X \pm Y) = V(X) + V(Y) \pm 2\text{Cov}(X, Y)$$

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Therefore, the standard error for a linear combination of coefficients is:

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which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.

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  which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.
- Since the estimates of the coefficients are correlated, we need the covariance term.

Example: GDP per capita on Regions

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
                (Intercept) RegionAfrica RegionAsia RegionLatAmerica
(Intercept)        613769.9    -613769.9  -613769.9        -613769.9
RegionAfrica      -613769.9    1450728.8   613769.9         613769.9
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RegionLatAmerica  -613769.9     613769.9   613769.9        1014054.6
RegionOecd        -613769.9     613769.9   613769.9         613769.9
                 RegionOecd
(Intercept)       -613769.9
RegionAfrica       613769.9
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> se <- sqrt(V[4,4] + V[3,3] - 2*V[3,4])
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[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
      -0.3990977
```

$$t = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \quad \text{where}$$

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Plugging in we get $t \approx -0.40$. So what do we conclude?

We cannot reject the null that the difference in average GDP resulted from chance.

Aside: Adjusted R^2

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R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,        Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

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$$R^2 = 1 - \frac{\text{SS}_{\text{res}}}{\text{SS}_{\text{tot}}}$$

where SS_{res} are the sum of squared residuals and the SS_{tot} are the sum of the squared deviations from the mean.

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- Still since people report it, let's quickly derive adjusted R^2,

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$$R^2_{adj} = R^2 - \underbrace{(1-R^2)\frac{k-1}{n-k}}_{\text{model complexity penalty}}$$

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- Adjusted R^2 will always be smaller than R^2 and can sometimes be negative!

F Test for Joint Significance of Coefficients

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- F tests allows us to to test joint hypothesis

The χ^2 Distribution

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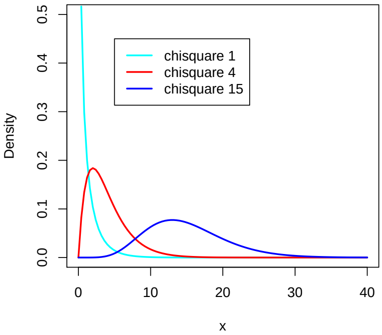

Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

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The F distribution arises as a ratio of two independent chi-squared distributed random variables:

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df_1 and df_2 are called the numerator degrees of freedom and the denominator degrees of freedom.

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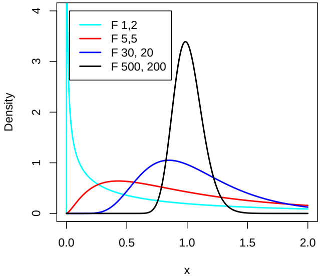
The F distribution arises as a ratio of two independent chi-squared distributed random variables:

$$F = \frac{X_1/df_1}{X_2/df_2} \sim \mathcal{F}_{df_1, df_2}$$

where $X_1 \sim \chi^2_{df_1}$, $X_2 \sim \chi^2_{df_2}$, and $X_1 \perp\!\!\!\perp X_2$.

df_1 and df_2 are called the numerator degrees of freedom and the denominator degrees of freedom.



In R: `df()`, `pf()`, `rf()`

F Test against $H_0: \gamma_1 = \gamma_2 = \gamma_3 = 0.$

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3. From the two results, compute the F Statistic:

$$F_0 = \frac{(SSR_r - SSR_{ur})/q}{SSR_{ur}/(n-k-1)}$$

   where SSR=sum of squared residuals, q=number of restrictions, k=number of predictors in the unrestricted model, and n= # of observations.

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   $$\frac{\text{increase in prediction error}}{\text{original prediction error}}$$

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1. Fit the Unrestricted Model (UR) which *does not* impose H_0:

   $$Vote = \beta_0 + \gamma_1 FEM + \beta_1 EDUC + \gamma_2 (FEM * EDUC) + \beta_2 AGE + \gamma_3 (FEM * AGE) + u$$

2. Fit the Restricted Model (R) which *does* impose H_0:

   $$Vote = \beta_0 + \beta_1 EDUC + \beta_2 AGE + u$$

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   where SSR=sum of squared residuals, q=number of restrictions, k=number of predictors in the unrestricted model, and n= # of observations.

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   $$\frac{\text{increase in prediction error}}{\text{original prediction error}}$$

The F statistics have the following sampling distributions:

- Under Assumptions 1–6, $F_0 \sim \mathcal{F}_{q,n-k-1}$ regardless of the sample size.
- Under Assumptions 1–5, $qF_0 \overset{a.}{\sim} \chi^2_q$ as $n \to \infty$ (see next section).

Unrestricted Model (UR)

R Code

```
> fit.UR <- lm(vote1 ~ fem + educ + age + fem:age + fem:educ, data = Chile)
> summary(fit.UR)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,       Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

R Code

```
> fit.R <- lm(vote1 ~ educ + age, data = Chile)
> summary(fit.R)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4878039  0.0497550   9.804  < 2e-16 ***
educ        -0.0662022  0.0139615  -4.742 2.30e-06 ***
age          0.0035783  0.0008385   4.267 2.09e-05 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.4921 on 1700 degrees of freedom
Multiple R-squared:  0.03275,       Adjusted R-squared:  0.03161
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

R Code

```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

> DFdenom <- df.residual(fit.UR)  # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

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```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
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> F
[1] 13.01581

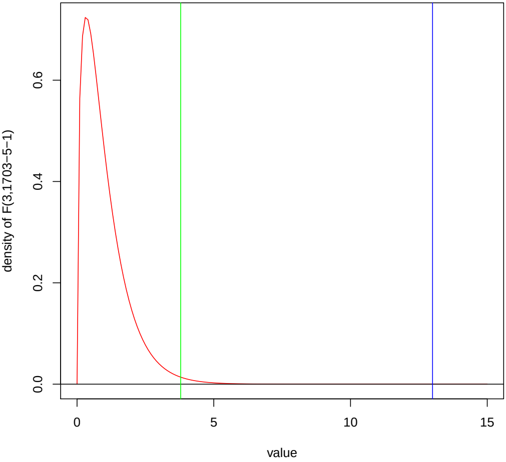
> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

$F_0 = 13$ is greater than the critical value for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



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The F test can be used to test various joint hypotheses which involve multiple linear restrictions.

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$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

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We may want to test:

$$H_0 : \beta_1 = \beta_2 = ... = \beta_k = 0$$

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  $\rightarrow$ Does any of the X variables help to predict Y?

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We may want to test:

$$H_0 : \beta_1 = \beta_2 = ... = \beta_k = 0$$

- What question are we asking?

  $\rightarrow$ Does any of the X variables help to predict Y?

- This is called the omnibus test and is routinely reported by statistical software.

Omnibus Test in R

R Code

```
> summary(fit.UR)
~~~~~

Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
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---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,        Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Omnibus Test in R with Random Noise

R Code

```
> set.seed(08540)
> p <- 10; x <- matrix(rnorm(p*1000), nrow=1000)
> y <- rnorm(1000); summary(lm(y~x))
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0115475  0.0320874  -0.360   0.7190
x1          -0.0019803  0.0333524  -0.059   0.9527
x2           0.0666275  0.0314087   2.121   0.0341 *
x3          -0.0008594  0.0321270  -0.027   0.9787
x4           0.0051185  0.0333678   0.153   0.8781
x5           0.0136656  0.0322592   0.424   0.6719
x6           0.0102115  0.0332045   0.308   0.7585
x7          -0.0103903  0.0307639  -0.338   0.7356
x8          -0.0401722  0.0318317  -1.262   0.2072
x9           0.0553019  0.0315548   1.753   0.0800 .
x10          0.0410906  0.0319742   1.285   0.1991
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 1.011 on 989 degrees of freedom
Multiple R-squared:  0.01129,       Adjusted R-squared:  0.001294
F-statistic: 1.129 on 10 and 989 DF,  p-value: 0.3364
```

F Test Examples II

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

Next, let's consider:

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

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- How many restrictions?

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Next, let's consider:

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- What question are we asking?
  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?
- How many restrictions?
  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)

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- How do we fit the restricted model?

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  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?
- How many restrictions?
  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)
- How do we fit the restricted model?
  $\rightarrow$ The null hypothesis implies that the model can be written as:

  $$Y = \beta_0 + \beta_1 (X_1 + X_2 + X_3) + ... + \beta_k X_k + u$$

  So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:

  $$Y = \beta_0 + \beta_1 X^* + ... + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
~~~~~

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1899.9      914.9   2.077   0.0410 *
Asia          2701.7     1243.0   2.173   0.0327 *
LatAmerica    2281.5     1112.3   2.051   0.0435 *
Transit       2552.8     1204.5   2.119   0.0372 *
Oecd         12224.2     1112.3  10.990   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,        Adjusted R-squared: 0.6951
F-statistic: 48.88 on 4 and 80 DF,  p-value: < 2.2e-16
```

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

R Code

```
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)

> SSR.UR2 <- sum(resid(fit.UR2)^2)
> SSR.R2 <- sum(resid(fit.R2)^2)

> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

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R Code

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> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
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> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.
- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})\big/(SSR_R/(n-k-1))$ and do the F test.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})/(SSR_R/(n-k-1))$ and do the F test.

It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \Longleftrightarrow \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

t Test vs. F Test

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})/(SSR_R/(n-k-1))$ and do the F test.

It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \iff \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.

t Test vs. F Test

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- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR})/(SSR_R/(n-k-1))$ and do the F test.

It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \Longleftrightarrow \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k-1)}$$

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- The F-value can also be calculated from R^2:

$$F \;=\; \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k-1)}$$

- F tests only work for testing nested models, i.e. the restricted model must be a special case of the unrestricted model.

  For example F tests cannot be used to test

$$Y \;=\; \beta_0 + \beta_1 X_1 \qquad\qquad + \beta_3 X_3 + u$$

  against

$$Y \;=\; \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \qquad\quad + u$$

Some More Notes on F Tests

Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

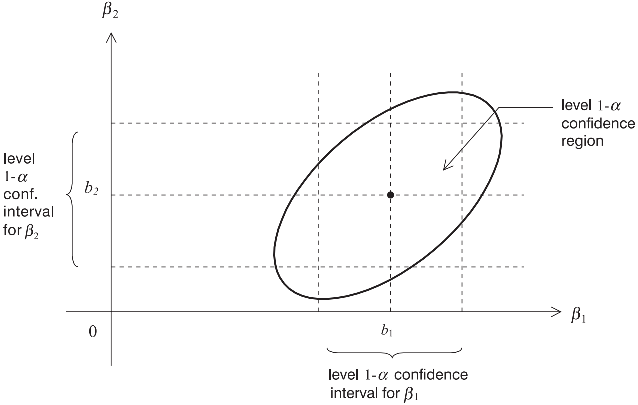

Figure 1.5: t- versus F-Tests

Image Credit: Hayashi (2011) *Econometrics*

Limitation of the F Formula

Consider the following null hypothesis:

$$H_0:\ \beta_1 = \beta_2 = \beta_3 = 3$$

or

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- Is there a general solution?

General Procedure for Testing Linear Hypotheses

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- $q\mathcal{F}_{q,n-k-1} \xrightarrow{d} \chi^2_q$ as $n \to \infty$, so the difference disappears when n large.

```
> pf(3.1, 2, 500,lower.tail=F) [1] 0.04591619

> pchisq(2*3.1, 2,lower.tail=F) [1] 0.0450492

> pf(3.1, 2, 50000,lower.tail=F) [1] 0.04505786
```

Testing General Linear Hypotheses in R

In R, the `linearHypothesis()` function in the car package does the Wald test for general linear hypotheses.

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> R <- matrix(c(0,1,-1,0,0, 0,1,0,-1,0), nrow = 2, byrow = T)
> r <- c(0,0)
> linearHypothesis(fit.UR2, R, r)
Linear hypothesis test

Hypothesis:
Asia - LatAmerica = 0
Asia - Transit = 0

Model 1: restricted model
Model 2: REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd

  Res.Df       RSS Df Sum of Sq      F Pr(>F)
1     82 738141635
2     80 736523836  2   1617798 0.0879  0.916
```

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- We showed how to estimate the coefficients and get the variance covariance matrix.
- Much of the hypothesis test infrastructure ports over nicely, plus there are new joint tests we can use.
- Appendix contains material on:
  - Derivation for the estimator (+ some of the math for this)
  - Proof of consistency

Fun Without Weights

The Robust Beauty of Improper Linear Models in Decision Making

ROBYN M. DAWES *University of Oregon*

ABSTRACT: *Proper linear models are those in which predictor variables are given weights in such a way that the resulting linear composite optimally predicts some criterion of interest; examples of proper linear models are standard regression analysis, discriminant function analysis, and ridge regression analysis. Research summarized in Paul Meehl's book on clinical versus statistical prediction—and a plethora of research stimulated in part by that book—all indicates that when a numerical criterion variable (e.g., graduate grade point average) is to be predicted from numerical predictor variables, proper linear models outperform clinical intuition. Improper linear models are those in which the weights of the predictor variables are obtained by some nonoptimal method; for example, they may be obtained on the basis of intuition, derived from simulating a clinical judge's predictions, or set to be equal. This article presents evidence that even such improper linear models are superior to clinical intuition when predicting a numerical criterion from numerical predictors. In fact, unit (i.e., equal) weighting is quite robust for making such predictions. The article discusses, in some detail, the application of unit weights to decide what bullet the Denver Police Department should use. Finally, the article considers commonly raised technical, psychological, and ethical resistances to using linear models to make important social decisions and presents arguments that could weaken these resistances.*

A *proper linear model* is one in which the weights given to the predictor variables are chosen in such a way as to optimize the relationship between the prediction and the criterion. Simple regression analysis is the most common example of a proper linear model; the predictor variables are weighted in such a way as to maximize the correlation between the subsequent weighted composite and the actual criterion. Discriminant function analysis is another example of a proper linear model; weights are given to the predictor variables in such a way that the resulting linear composites maximize the discrepancy between two or more groups. Ridge regression analysis, another example (Darlington, 1978; Marquardt & Snee, 1975), attempts to assign weights in such a way that the linear composites correlate maximally with the criterion of interest in a new set of data.

Thus, there are many types of proper linear models and they have been used in a variety of contexts. One example (Dawes, 1971) was presented in this Journal; it involved the prediction of faculty ratings of graduate students. All gradu-

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- Dawes argues that even improper linear models (those where weights are set by hand or set to be equal), outperform clinical intuition.
- Equal weight models are argued to be quite robust for these predictions

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- Correlation of faculty ratings with average rating of admissions committee was .19
- Standardized and equally weighted improper linear model, correlated at .48

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- Einhorn (1972) study of doctors coding biopsies of patients with Hodgkin's disease and then rated severity. Their rating of severity was essentially uncorrelated with survival times, but the variables they coded predicted outcomes using a regression model.

Other Examples

TABLE 1

Correlations Between Predictions and Criterion Values

Example	Average validity of judge	Average validity of judge model	Average validity of random model	Validity of equal weighting model	Cross-validity of regression analysis	Validity of optimal linear model
Prediction of neurosis vs. psychosis	.28	.31	.30	.34	.46	.46
Illinois students' predictions of GPA	.33	.50	.51	.60	.57	.69
Oregon students' predictions of GPA	.37	.43	.51	.60	.57	.69
Prediction of later faculty ratings at Oregon	.19	.25	.39	.48	.38	.54
Yntema & Torgerson's (1961) experiment	.84	.89	.84	.97	—	.97

Note. GPA = grade point average.

Column descriptions:

- C1) average of human judges
- C2) model based on human judges
- C3) randomly chosen weights preserving signs
- C4) equal weighting
- C5) cross-validated weights
- C6) unattainable optimal linear model

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- The difficulty arises in part because people in general lack a strong reference to the distribution of the predictors.
- Linear models are robust to deviations from the optimal weights (see also Waller 2008 on "Fungible Weights in Multiple Regression")

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- This all applies because predictors well chosen and the sample size is small (so the weight optimization isn't great)
- It is a fascinating paper!

Gradient

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Definition (Gradient)

We can define the column vector of partial derivatives

$$\frac{\partial v(\mathbf{u})}{\partial \mathbf{u}} = \begin{bmatrix} \partial v/\partial u_1 \\ \partial v/\partial u_2 \\ \vdots \\ \partial v/\partial u_n \end{bmatrix}$$

This vector of partial derivatives is called the gradient.

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Vector Derivative Rule I (linear functions)

Theorem (differentiation of linear functions)

Given a linear function $v(\mathbf{u}) = \mathbf{c}'\mathbf{u}$ of an $(n \times 1)$ vector $\mathbf{u}$, the derivative of $v(\mathbf{u})$ w.r.t. $\mathbf{u}$ is given by

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Hence,

$$\frac{\partial v}{\partial \mathbf{u}} = \mathbf{c}$$

Vector Derivative Rule II (quadratic form)

Theorem (quadratic form)

Given a $(n \times n)$ *symmetric matrix* $\mathbf{A}$ *and a scalar-valued function* $v(\mathbf{u}) = \mathbf{u}'\mathbf{A}\mathbf{u}$ *of* $(n \times 1)$ *vector* $\mathbf{u}$, *we have*

$$\frac{\partial v}{\partial \mathbf{u}} = \mathbf{A}'\mathbf{u} + \mathbf{A}\mathbf{u} = 2\mathbf{A}\mathbf{u}$$

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Hessian

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Suppose v is a scalar-valued function $v = f(\mathbf{u})$ of a $(k+1) \times 1$ column vector $\mathbf{u} = \begin{bmatrix} u_1 & u_2 & \cdots & u_{k+1} \end{bmatrix}'$

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Definition (Hessian)

The $(k+1) \times (k+1)$ matrix of second-order partial derivatives of $v = f(\mathbf{u})$ is called the Hessian matrix and denoted

$$\frac{\partial v^2}{\partial \mathbf{u} \partial \mathbf{u}'} = \begin{bmatrix} \frac{\partial v^2}{\partial u_1 \partial u_1} & \cdots & \frac{\partial v^2}{\partial u_1 \partial u_{k+1}} \\ \vdots & \ddots & \vdots \\ \frac{\partial v^2}{\partial u_{k+1} \partial u_1} & \cdots & \frac{\partial v^2}{\partial u_{k+1} \partial u_{k+1}} \end{bmatrix}$$

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The above rules are used to derive the optimal estimators in the appendix slides.

Derivatives with respect to $\tilde{\boldsymbol{\beta}}$

$$
\begin{aligned}
S(\tilde{\boldsymbol{\beta}}, \mathbf{X}, \mathbf{y}) &= (\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}})'(\mathbf{y} - \mathbf{X}\tilde{\boldsymbol{\beta}}) \\
&= \mathbf{y}'\mathbf{y} - 2\mathbf{y}'\mathbf{X}\tilde{\boldsymbol{\beta}} + \tilde{\boldsymbol{\beta}}'\mathbf{X}'\mathbf{X}\tilde{\boldsymbol{\beta}}
\end{aligned}
$$

$$
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- The first term does not contain $\tilde{\boldsymbol{\beta}}$
- The second term is an example of rule I from the derivative section
- The third term is an example of rule II from the derivative section

Derivatives with respect to $\tilde{\boldsymbol{\beta}}$

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And while we are at it the Hessian is:

$$\frac{\partial^2 S(\tilde{\boldsymbol{\beta}}, \mathbf{X}, \mathbf{y})}{\partial \tilde{\boldsymbol{\beta}} \partial \tilde{\boldsymbol{\beta}}'} = 2\mathbf{X}'\mathbf{X}$$

Solving for $\hat{\boldsymbol{\beta}}$

$$\frac{\partial S(\tilde{\boldsymbol{\beta}}, \mathbf{X}, \mathbf{y})}{\partial \tilde{\boldsymbol{\beta}}} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\tilde{\boldsymbol{\beta}}$$

Setting the vector of partial derivatives equal to zero and substituting $\hat{\boldsymbol{\beta}}$ for $\tilde{\boldsymbol{\beta}}$, we can solve for the OLS estimator.

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$$\mathbf{0} = -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}}$$

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$$\begin{aligned} \mathbf{0} &= -2\mathbf{X}'\mathbf{y} + 2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} \\ -2\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} &= -2\mathbf{X}'\mathbf{y} \end{aligned}$$

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(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{y}
\end{aligned}$$

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Setting the vector of partial derivatives equal to zero and substituting $\hat{\boldsymbol{\beta}}$ for $\tilde{\boldsymbol{\beta}}$, we can solve for the OLS estimator.

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\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{X}\hat{\boldsymbol{\beta}} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} \\
\mathbf{I}\hat{\boldsymbol{\beta}} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}
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Solving for $\hat{\boldsymbol{\beta}}$

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\hat{\boldsymbol{\beta}} &= \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}
\end{aligned}$$

Note that we implicitly assumed that $\mathbf{X}'\mathbf{X}$ is invertible.

Consistency of $\hat{\beta}$

To show consistency, we rewrite the OLS estimator in terms of sample means so that we can apply LLN.

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First, note that a matrix cross product can be written as a sum of vector products:

$$\mathbf{X}'\mathbf{X} = \sum_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i \quad \text{and} \quad \mathbf{X}'\mathbf{y} = \sum_{i=1}^{n} \mathbf{x}_i' y_i$$

where $\mathbf{x}_i$ is the $1 \times (k+1)$ row vector of predictor values for unit i.

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where $\mathbf{x}_i$ is the $1 \times (k+1)$ row vector of predictor values for unit i.

Now we can rewrite the OLS estimator as,

$$\begin{aligned}
\hat{\boldsymbol{\beta}} &= \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i' y_i\right) \\
&= \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i'(\mathbf{x}_i\boldsymbol{\beta} + u_i)\right) \\
&= \boldsymbol{\beta} + \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\sum\nolimits_{i=1}^{n} \mathbf{x}_i' u_i\right) \\
&= \boldsymbol{\beta} + \left(\frac{1}{n}\sum\nolimits_{i=1}^{n} \mathbf{x}_i'\mathbf{x}_i\right)^{-1} \left(\frac{1}{n}\sum\nolimits_{i=1}^{n} \mathbf{x}_i' u_i\right)
\end{aligned}$$

Consistency of $\hat{\beta}$

Now let's apply the LLN to the sample means:

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right) \xrightarrow{p} E[\mathbf{x}_i'\mathbf{x}_i], \text{ a } (k+1)\times(k+1) \text{ nonsingular matrix.}$$

$$\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'u_i\right) \xrightarrow{p} E[\mathbf{x}_i'u_i] = 0, \text{ by the zero cond. mean assumption.}$$

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Therefore, we have

$$\begin{aligned}\text{plim}(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + \left(E[\mathbf{x}_i'\mathbf{x}_i]\right)^{-1}\cdot 0\\ &= \boldsymbol{\beta}.\end{aligned}$$

Consistency of $\hat{\boldsymbol{\beta}}$

Now let's apply the LLN to the sample means:

$$\begin{aligned}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'\mathbf{x}_i\right) &\xrightarrow{p} E[\mathbf{x}_i'\mathbf{x}_i],\ \text{a } (k+1)\times(k+1) \text{ nonsingular matrix.}\\ \left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x}_i'u_i\right) &\xrightarrow{p} E[\mathbf{x}_i'u_i] \ = \ 0,\ \text{by the zero cond. mean assumption.}\end{aligned}$$

Therefore, we have

$$\begin{aligned}\text{plim}(\hat{\boldsymbol{\beta}}) &= \boldsymbol{\beta} + \left(E[\mathbf{x}_i'\mathbf{x}_i]\right)^{-1}\cdot 0\\ &= \boldsymbol{\beta}.\end{aligned}$$

We can also show the asymptotic normality of $\hat{\boldsymbol{\beta}}$ using a similar argument but with the CLT.

References

- Wooldridge, Jeffrey. *Introductory econometrics: A modern approach*. Cengage Learning, 2012.

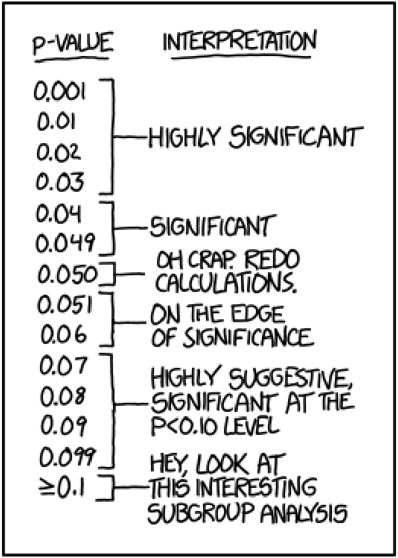
Where We've Been and Where We're Going...

Where We've Been and Where We're Going...

- Last Week
  - regression with two variables
  - omitted variables, multicollinearity, interactions
- This Week
  - Monday:
    - matrix form of linear regression
    - t-tests, F-tests and general linear hypothesis tests
  - Wednesday:
    - problems with p-values
    - agnostic regression
    - the bootstrap
- Next Week
  - break!
  - then . . . diagnostics
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Questions?

p-values (courtesy of XKCD)



The value of the p-value

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Ronald Fisher (1935)

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In social science (and I think in psychology as well), the null hypothesis is almost certainly false, false, false, and you don't need a p-value to tell you this. The p-value tells you the extent to which a certain aspect of your data are consistent with the null hypothesis. A lack of rejection doesn't tell you that the null hypothesis is likely true; rather, it tells you that you don't have enough data to reject the null hypothesis.

Andrew Gelman (2010)

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  - the probability that the alternative hypothesis is false

Problems with p-values

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- p-values are not:
  - an indication of a large substantive effect
  - the probability that the null hypothesis is true
  - the probability that the alternative hypothesis is false
- a large p-value could mean either that we are in the null world OR that we had insufficient power

So what is the basic idea?

The idea was to run an experiment, then see if the results were consistent with what random chance might produce. Researchers would first set up a 'null hypothesis' that they wanted to disprove, such as there being no correlation or no difference between groups. Next, they would play the devil's advocate and, assuming that this null hypothesis was in fact true, calculate the chances of getting results at least as extreme as what was actually observed. This probability was the P value. The smaller it was, suggested Fisher, the greater the likelihood that the straw-man null hypothesis was false.
(Nunzo 2014, emphasis mine)

I've got 99 problems. . .

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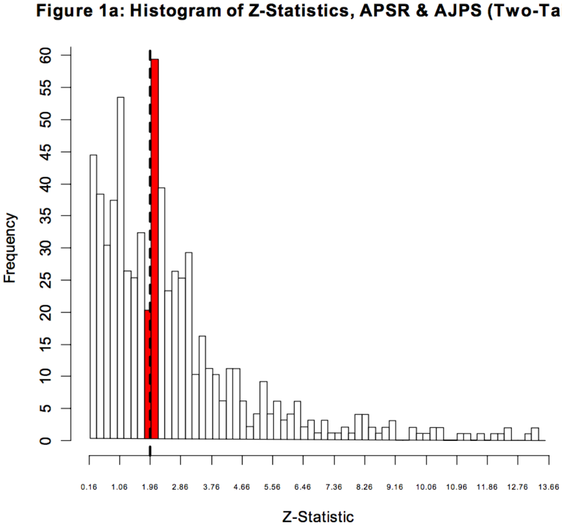
- they lead to publication filtering on arbitrary cutoffs.

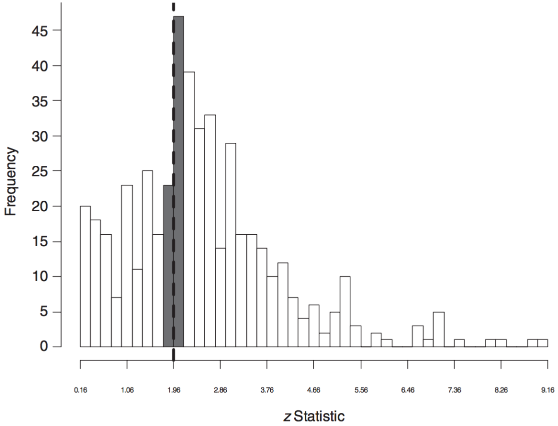
Arbitrary Cutoffs



Gerber and Malhotra (2006) Top Political Science Journals

Arbitrary Cutoffs

Figure 1
Histogram of *z* Statistics From the *American Sociological Review*, the *American Journal of Sociology*, and *The Sociological Quarterly* (Two-Tailed)



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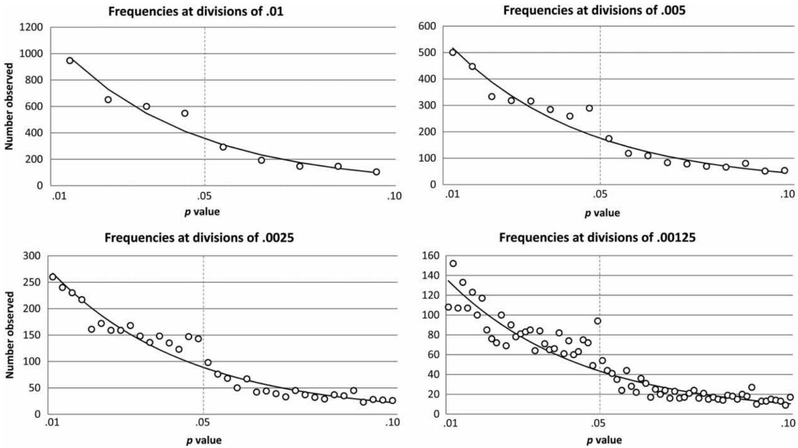

Figure 1.. *The graphs show the distribution of 3,627* p *values from three major psychology journals.*

Masicampo and Lalande (2012) Top Psychology Journals

Still Not Convinced?
The Real Harm of Misinterpreted p-values

ELSEVIER

Accident Analysis and Prevention 36 (2004) 495–500

ACCIDENT ANALYSIS & PREVENTION

www.elsevier.com/locate/aap

Viewpoint

The harm done by tests of significance

Ezra Hauer*

35 Merton Street, Apt. 1706, Toronto, Ont., Canada M4S 3G4

Abstract

Three historical episodes in which the application of null hypothesis significance testing (NHST) led to the mis-interpretation of data are described. It is argued that the pervasive use of this statistical ritual impedes the accumulation of knowledge and is unfit for use.

Keywords: Significance; Statistical hypothesis; Scientific method

Example from Hauer: Right-Turn-On-Red

Table 1
The Virginia RTOR study

	Before RTOR signing	After RTOR signing
Fatal crashes	0	0
Personal injury crashes	43	60
Persons injured	69	72
Property damage crashes	265	277
Property damage (US$)	161243	170807
Total crashes	308	337

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The Point in Hauer

- Two other interesting examples in Hauer (2004)
- Core issue is that lack of significance is not an indication of a zero effect, it could also be a lack of power (i.e. a small sample size relative to the difficulty of detecting the effect)
- On the opposite end, large tech companies rarely use significance testing because they have huge samples which essentially always find some non-zero effect. But that doesn't make the finding significant in a colloquial sense of important.

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  2) Substantive: p-values are divorced from your quantity of interest–which almost always should relate to how much an intervention changes a quantity of social scientific interest (newspaper rule)

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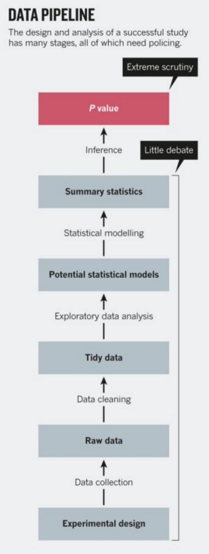
Instead, show quantities you care about with confidence intervals.

Don't misinterpret, or rely too heavily, on your p-values. They are evidence against your null, not evidence in favor of your alternative.

But Let's Not Obsess Too Much About p-values



From Leek and Peng (2015) "*P* values are just the tip of the iceberg" *Nature*.

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  - A2 ensures that we observe independent samples for estimation.

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- NB: this makes no statement about whether or not the CEF you are looking at is the 'right' one.

Justifying linear regression

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- In other words, even a non-linear CEF has a "true" linear approximation, even though that approximation may not be great.

Regression anatomy

- Consider simple linear regression:

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- With more covariates, β is more complicated, but we can still write it like this.
- Let $\tilde{X}_{ki}$ be the residual from a regression of X_{ki} on all the other independent variables. Then, β_k, the coefficient for X_{ki} is:

$$\beta_k = \frac{\text{Cov}(Y_i, \tilde{X}_{ki})}{\mathrm{Var}(\tilde{X}_{ki})}$$

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- A model is saturated if there are as many parameters as there are possible combination of the X_i variables.

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- Two binary variables, X_{1i} for marriage status and X_{2i} for having children.

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- Two binary variables, X_{1i} for marriage status and X_{2i} for having children.
- Four possible values of X_i, four possible values of $\mu(X_i)$:

$$\begin{aligned} E[Y_i|X_{1i} = 0, X_{2i} = 0] &= \alpha \\ E[Y_i|X_{1i} = 1, X_{2i} = 0] &= \alpha + \beta \\ E[Y_i|X_{1i} = 0, X_{2i} = 1] &= \alpha + \gamma \\ E[Y_i|X_{1i} = 1, X_{2i} = 1] &= \alpha + \beta + \gamma + \delta \end{aligned}$$

- We can write the CEF as follows:

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- Requires a set of dummies for each categorical variable plus all interactions.
- Or, a series of dummies for each unique combination of X_i.
- This makes linearity hold mechanically and so linearity is not an assumption.

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```r
girls <- foreign::read.dta("girls.dta")
head(girls[, c("name", "totchi", "aauw")])
```

```
##                       name totchi aauw
## 1    ABERCROMBIE, NEIL      0  100
## 2    ACKERMAN, GARY L.      3   88
## 3 ADERHOLT, ROBERT B.       0    0
## 4      ALLEN, THOMAS H.     2  100
## 5   ANDREWS, ROBERT E.      2  100
## 6          ARCHER, W.R.     7    0
```

Linear model

```
summary(lm(aauw ~ totchi, data = girls))
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)    61.31       1.81   33.81   <2e-16 ***
## totchi         -5.33       0.62   -8.59   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 42 on 1733 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.0408, Adjusted R-squared:  0.0403
## F-statistic: 73.8 on 1 and 1733 DF,  p-value: <2e-16
```

Saturated model

```r
summary(lm(aauw ~ as.factor(totchi), data = girls))
```

```
##
## Coefficients:
##                     Estimate Std. Error t value Pr(>|t|)
## (Intercept)            56.41       2.76   20.42  < 2e-16 ***
## as.factor(totchi)1      5.45       4.11    1.33   0.1851
## as.factor(totchi)2     -3.80       3.27   -1.16   0.2454
## as.factor(totchi)3    -13.65       3.45   -3.95  8.1e-05 ***
## as.factor(totchi)4    -19.31       4.01   -4.82  1.6e-06 ***
## as.factor(totchi)5    -15.46       4.85   -3.19   0.0015 **
## as.factor(totchi)6    -33.59      10.42   -3.22   0.0013 **
## as.factor(totchi)7    -17.13      11.41   -1.50   0.1336
## as.factor(totchi)8    -55.33      12.28   -4.51  7.0e-06 ***
## as.factor(totchi)9    -50.41      24.08   -2.09   0.0364 *
## as.factor(totchi)10   -53.41      20.90   -2.56   0.0107 *
## as.factor(totchi)12   -56.41      41.53   -1.36   0.1745
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 41 on 1723 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.0506, Adjusted R-squared:  0.0446
## F-statistic: 8.36 on 11 and 1723 DF,  p-value: 1.84e-14
```

Saturated model minus the constant

```r
summary(lm(aauw ~ as.factor(totchi) - 1, data = girls))
```

```
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## as.factor(totchi)0      56.41       2.76   20.42   <2e-16 ***
## as.factor(totchi)1      61.86       3.05   20.31   <2e-16 ***
## as.factor(totchi)2      52.62       1.75   30.13   <2e-16 ***
## as.factor(totchi)3      42.76       2.07   20.62   <2e-16 ***
## as.factor(totchi)4      37.11       2.90   12.79   <2e-16 ***
## as.factor(totchi)5      40.95       3.99   10.27   <2e-16 ***
## as.factor(totchi)6      22.82      10.05    2.27   0.0233 *
## as.factor(totchi)7      39.29      11.07    3.55   0.0004 ***
## as.factor(totchi)8       1.08      11.96    0.09   0.9278
## as.factor(totchi)9       6.00      23.92    0.25   0.8020
## as.factor(totchi)10      3.00      20.72    0.14   0.8849
## as.factor(totchi)12      0.00      41.43    0.00   1.0000
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 41 on 1723 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.587,  Adjusted R-squared:  0.584
## F-statistic:  204 on 12 and 1723 DF,  p-value: <2e-16
```

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```
c1 <- coef(lm(aauw ~ as.factor(totchi) - 1, data = girls))
c2 <- with(girls, tapply(aauw, totchi, mean, na.rm = TRUE))
rbind(c1, c2)
```

```
##      0  1  2  3  4  5  6  7   8 9 10 12
## c1 56 62 53 43 37 41 23 39 1.1 6  3  0
## c2 56 62 53 43 37 41 23 39 1.1 6  3  0
```

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- Warning if the CEF is very nonlinear then this approximation could be terrible!!

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- Note the residual e_i is uncorrelated with X_i:

$$\begin{aligned}
\mathbb{E}[X_i e_i] &= \mathbb{E}[X_i(Y_i - X_i'\beta)] \\
&= \mathbb{E}[X_i Y_i] - \mathbb{E}[X_i X_i'\beta] \\
&= \mathbb{E}[X_i Y_i] - \mathbb{E}\left[X_i X_i' \mathbb{E}[X_i X_i']^{-1} \mathbb{E}[X_i Y_i]\right] \\
&= \mathbb{E}[X_i Y_i] - \mathbb{E}[X_i X_i'] \mathbb{E}[X_i X_i']^{-1} \mathbb{E}[X_i Y_i] \\
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\end{aligned}$$

- No assumptions on the linearity of $\mathbb{E}[Y_i | X_i]$.

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$$\beta = \mathbb{E}[X_i X_i']^{-1}\mathbb{E}[X_i Y_i]$$

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- If you work through the matrix algebra, this turns out to be:

$$\hat{\beta} = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y}$$

Asymptotic OLS inference

- With this representation in hand, we can write the OLS estimator as follows:

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- The square root of the diagonals of this covariance matrix are the "robust" or Huber-White standard errors (we will return to this in a few classes).

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- We will come back to re-thinking the implications for finite samples during the diagnostics classes.

The Bootstrap

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3. Repeat steps 1 and 2 many (B) times.
4. Using the resulting collection of bootstrap estimates to calculate estimates of the standard error or confidence intervals (more later).

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Bootstrap is really useful when:

- you want to avoid making a normal approximation
- you want to avoid making certain assumptions (i.e. homoskedasticity)

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This is the closest thing to magic I will show you all semester.

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- Percentile method for the CI: Sort B bootstrap estimates from smallest to largest. Grab the values at $\alpha/2 * B$ and $1 - \alpha/2 * B$ position.
  - Percentile method does not rely on normal approximation but requires very large B and thus more computational time.

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4) Calculate confidence interval by identifying $\alpha/2$ and $1 - \alpha/2$ value of statistic. (percentile method)

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  - Why does this work? Sampling distribution entirely determined by the CDF and n, WLLN says the ECDF will look more and more like the CDF as n gets large.

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Fox Chapter 21 has a nice section on the bootstrap, Aronow and Miller (2016) covers the theory well.

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- Agnostic Regression: how will regression behave in large samples if we don't really buy the assumptions
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- Appendix contains a fun example of the difficulty of thinking through p-values.

Next “Week” of Classes (Three Classes)

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- Reading:
    - Angrist and Pishke Chapter 8 ('Nonstandard Standard Error Issues')
    - Optional: Fox Chapters 11-13
    - Optional: King and Roberts "How Robust Standard Errors Expose Methodological Problems They Do Not Fix, and What to Do About It." *Political Analysis*, 2, 23: 159179.
    - Optional: Aronow and Miller Chapters 4.2-4.4 (Inference, Clustering, Nonlinearity)

Fun With Weights

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- Imagine we care about the possibly heterogeneous causal effect of a treatment D and we control for some covariates X?
- We can express the regression as a weighting over individual observation treatment effects where the weight depends only on X.
- Useful technology for understanding what our models are identifying off of by showing us our effective sample.

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How this works

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$$\hat{\beta} \xrightarrow{p} \frac{E[w_i \tau_i]}{E[w_i]} \text{ where } w_i = (D_i - E[D_i|X])^2 \,,$$

so that $\hat{\beta}$ converges to a reweighted causal effect. As $E[w_i|X_i] = \text{Var}[D_i|X_i]$, we obtain an average causal effect reweighted by conditional variance of the treatment.

Estimation

A simple, consistent plug-in estimator of w_i is available: $\hat{w}_i = \tilde{D}_i^2$ where $\tilde{D}_i$ is the residualized treatment. (the proof is connected to the partialing out strategy we showed last week)

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Easily implemented in R:

```
wts <- (d - predict(lm(d~x)))^2
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Implications

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- The downside is that we have to be aware of what happened!

Application

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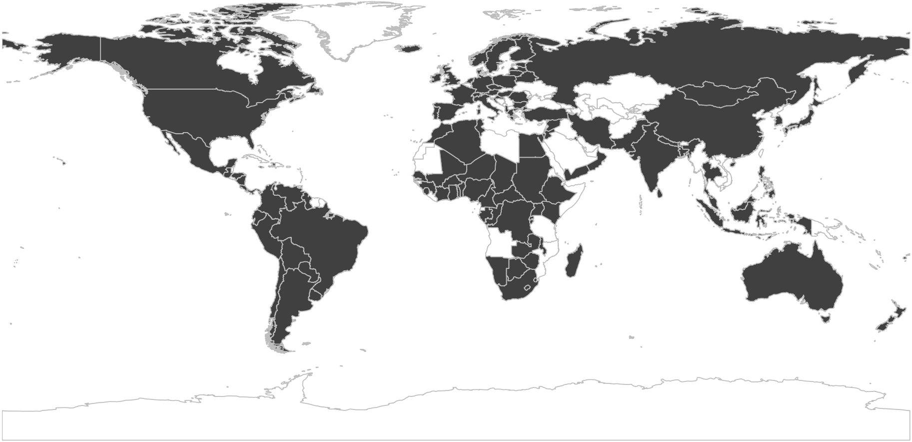
Jensen estimates that a 1 unit increase in polity score corresponds to a 0.020 increase in net FDI inflows as a percentage of GDP ($p < 0.001$).

Nominal and Effective Samples



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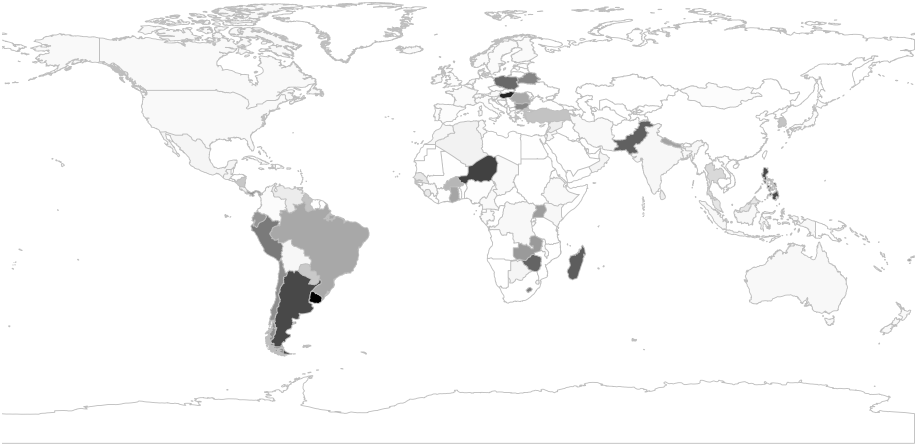

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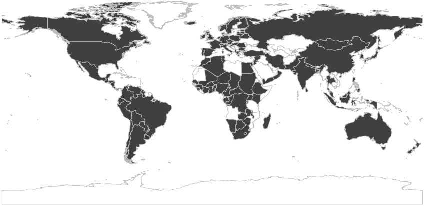

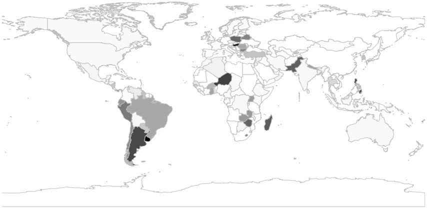

Over 50% of the weight goes to just 12 (out of 114) countries.

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- "Externally valid": perhaps unreliable estimates of ATEs, but for the population of interest
  - large-N analyses, representative surveys

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- The effective sample (upon which causal effects are estimated) may have radically different properties than the nominal sample.
- When there is an underlying natural experiment in the data, a properly specified regression model may reproduce the internally valid estimate associated with the natural experiment.

Still Not Convinced? A Tricky Example: p-values

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[3] From a Comment on Berger and Sellke "Testing a Point Null Hypothesis: The Irreconcilability of P Values and Evidence

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Note: The p values are all about .021.

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Table 1. Data, p Values, Posterior Probabilities, and Power at $\theta_1 = .55$ for the Three Surveys

Survey	(a)	(b)	(c)
n	20	200	2,000
$\hat{\theta}$	.750	.575	.523
t	2.03	2.05	2.03
p value	.021	.020	.021
C_n	.408	.816	.976
$\Pr(H_0 \mid t)$	.204	.047	.024
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