Week 3: Learning from Random Samples

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Princeton

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[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller, Erin Hartman and Matt Salganik. Some illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

- Last Week

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  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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    - Populations and Samples
    - Estimators
    - Analytical
2. Weak Law of Large Numbers and the Central Limit Theorem
    - Chebychev's Inequality
    - Weak Law of Large Numbers
    - The Central Limit Theorem
3. Properties of Estimators
    - Four Desirable Properties
    - Example
4. Interval Estimation
    - Intervals
    - Large Sample Intervals for a Mean
    - Small Sample Intervals for a Mean
    - Comparing Two Groups
    - Interval Estimation for a Proportion
5. Plug-In Principle

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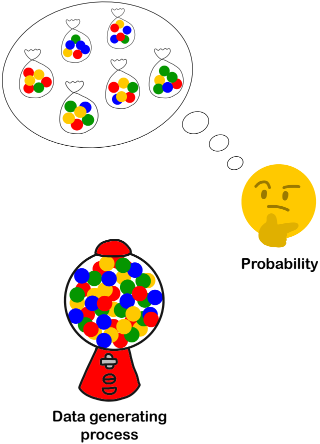

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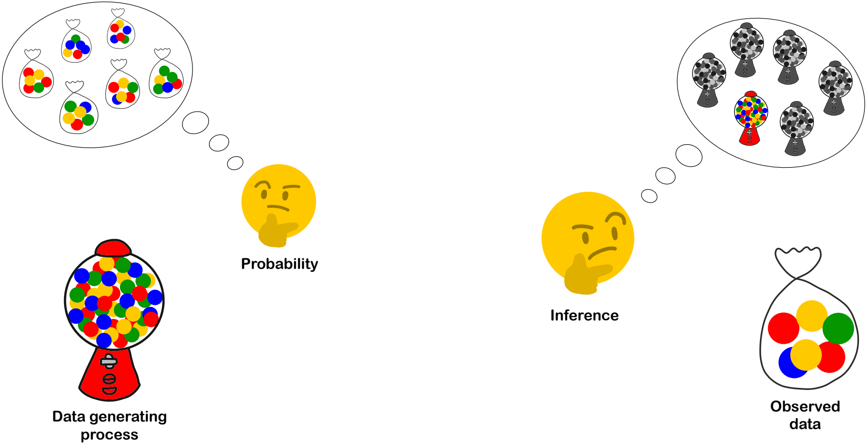

Primary Goal for This Week

*Racial Prejudice and Attitudes Toward Affirmative Action**

James H. Kuklinski, *University of Illinois at Urbana-Champaign*
Paul M. Sniderman, *Stanford University*
Kathleen Knight, *University of Houston*
Thomas Piazza, *University of California-Berkeley*
Philip E. Tetlock, *Ohio State University*
Gordon R. Lawrence, *Williams College*
Barbara Mellers, *Ohio State University*

`https://www.jstor.org/stable/2111770`

Primary Goal for This Week

Theory: We examine the relationship between blatant racial prejudice and anger toward affirmative action.
Hypotheses: (1) Blatantly prejudiced attitudes continue to pervade the white population in the United States. (2) Resistance to affirmative action is more than an extension of this prejudice. (3) White resistance to affirmative action is not unyielding and unalterably fixed.
Methods: Analysis of experiments embedded in a national survey of racial attitudes. Some of these experiments are designed to measure racial prejudice unobtrusively.
Results: Racial prejudice remains a major problem in the United States, but this prejudice alone cannot explain all of the anger toward affirmative action among whites. Although many whites strongly resist affirmative action, they express support for making extra efforts to help African-Americans.

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Primary Goal for This Week

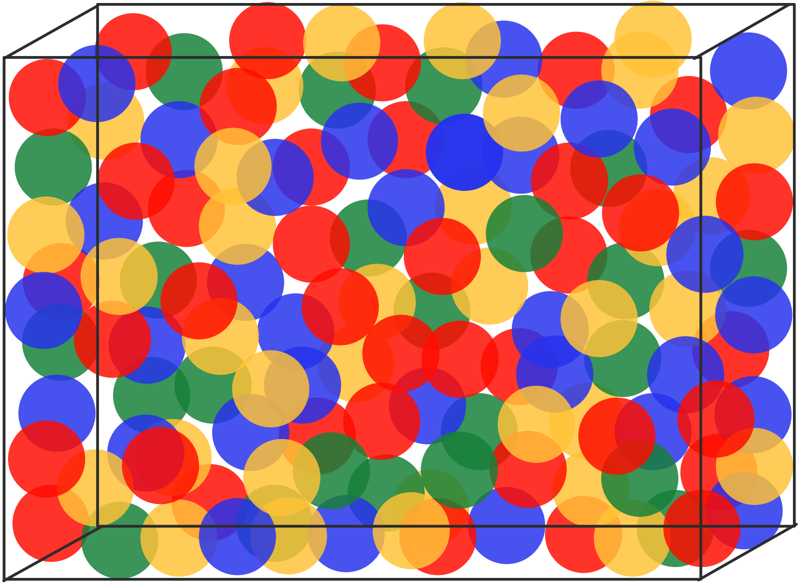
We want to be able to interpret the numbers in this table (and a couple of numbers that can be derived from these numbers).

Table 1. Mean Level of Anger Toward A Black Family Moving in Next Door, by Region (Whites Only)

Region	Experimental Condition: Baseline	Experimental Condition: Black Family	Estimated Percent Angry
Non-South	2.28[a]	2.24	0
	(.07)	(.05)	
	425[b]	461	
South	1.95	2.37	42
	(.06)	(.08)	
	139	136	

[a]Standard error of the estimate.
[b]Number of cases.

An Overview



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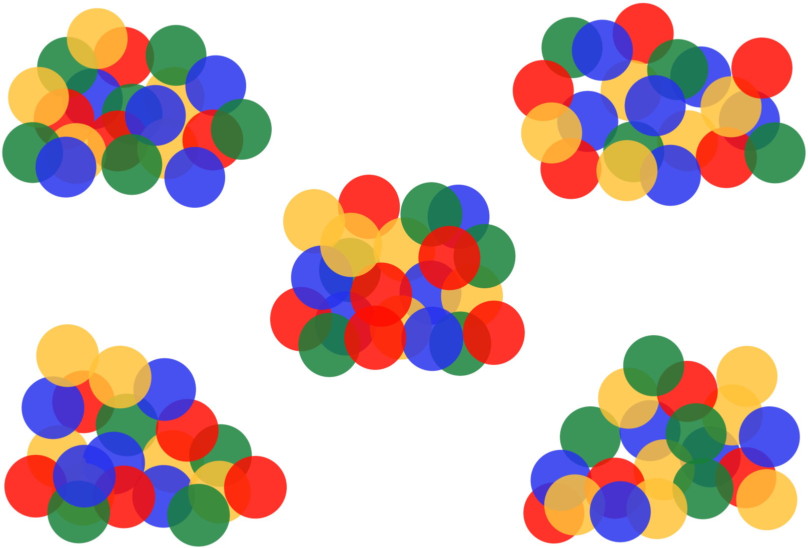

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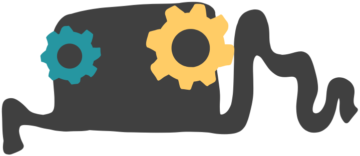

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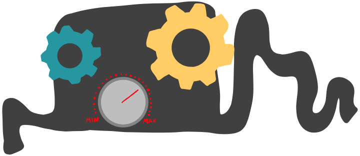

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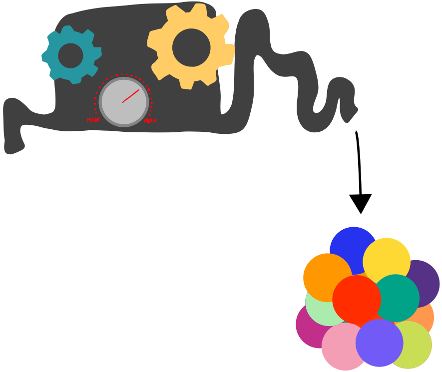

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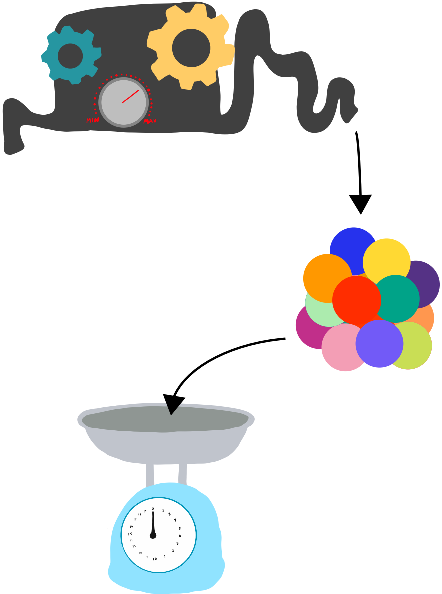

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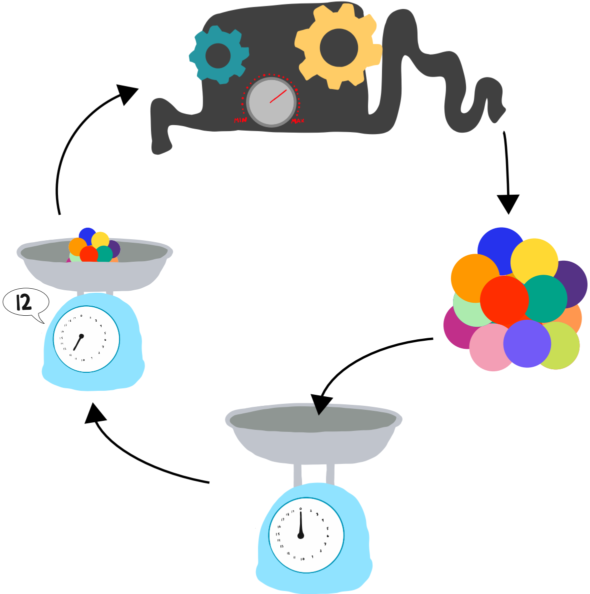

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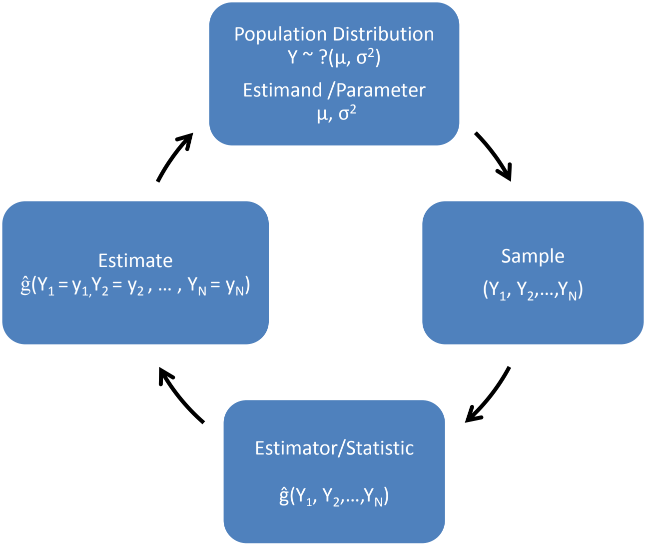
Population Distribution
Y ~ ?(μ, σ²)
Estimand /Parameter
μ, σ²
Sample
(Y₁, Y₂,…,Y_N)
Estimator/Statistic
ĝ(Y₁, Y₂,…,Y_N)
Estimate
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- With either a finite or infinite population our main goal in inference is to learn about the population distribution f_X via summaries, like $E[X]$ or $V[X]$, which we call a population parameter (or just parameter).
- Ideally we assume as little as possible about the form of f_X.

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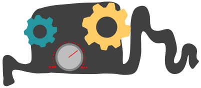

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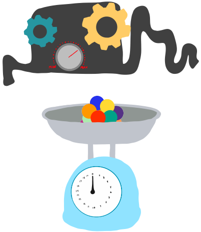

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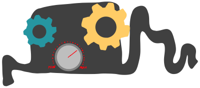

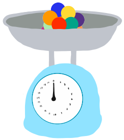

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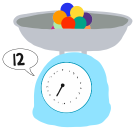
- Estimates are particular values of estimators that are realized in a given sample (e.g. 12)



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- We will leverage the powerful assumption that we are observing IID—independent and identically distributed—samples of the random variable of interest.

Plain language:
Data are sampled IID when each observation is drawn from the same distribution, and the way an observation is drawn does not depend on the values of any other draw.

IID Formal Definition

Definition (Independent and Identically Distributed)

Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ be random variables with CDFs $F_1, F_2, \ldots, F_n$, respectively. Let F_A denote the joint CDF of the random variables with indices in the set A. Then $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ are independently and identically distributed if they satisfy the following:

- Mutually independent: $\forall A \subseteq \{1, 2, \ldots, n\}, \forall (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, F_A\left((x_i)_{i \in A}\right) = \prod_{i \in A} F_i(x_i)$
- Identically distributed: $\forall i, j \in \{1, 2, \ldots, n\}$ *and* $\forall x \in \mathbb{R}, F_i(x) = F_j(x)$

(Aronow and Miller Definition 3.1.1)

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- IID tells us that each one is produced under the same random process. This is how we get leverage to do estimation!
- We we will usually use unsubscripted capital letters, X, to refer to properties that all these draws share.
  e.g. $E[X] = E[X_1] = E[X_2] = \cdots = E[X_n]$

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We will return to these issues more in later videos and in future weeks.

Sampling from Finite Populations

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- If the population is small relative to the sample size, it will be necessary to think carefully through the implications (see e.g. the challenge problem in problem set 3).

Sampling in R

```
## draw a sample of size 10 from our population
## drawn without replacement
my_sample <- dplyr::sample_n(my_data, size = 10,
                             replace = FALSE)
## this is a wrapper around sample.int()
my_sample <- my_data[sample.int(nrow(my_data),
                             size = 10, replace = FALSE), ]
```

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- In real applications, we cannot draw repeated samples, so we approximate the sampling distribution.

The Sampling Distribution of the Sample Mean

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Say we have the following population:

```
pop <- c(4, 2, 3, 6, 9, 2, 3, 6, 8, 5, 2, 9, 6, 3,
         4, 7, 6, 1, 2, 6, 9, 3, 1, 1, 1, 5, 7, 9)
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```
choose(length(pop), 10)
```

```
## [1] 13123110
```

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```
sim_res <- replicate(10000, {
  mean(pop[sample.int(length(pop), 10)])
}) %>% tibble(sample_mean = .) %>%
  rownames_to_column(var = "replicate")

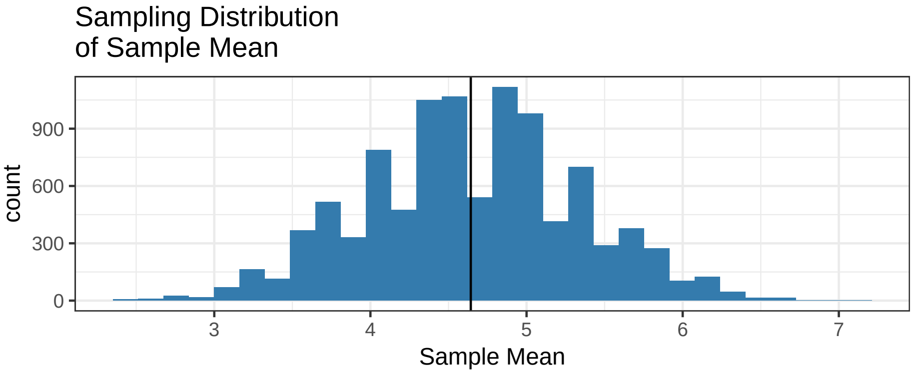
sim_res[1:5, ]
```

```
## # A tibble: 5 x 2
##   replicate sample_mean
##       <chr>       <dbl>
## 1         1         5.4
## 2         2         4.9
## 3         3         3.7
## 4         4         3.6
## 5         5         5.3
```

The Sampling Distribution

And we can plot this sampling distribution

```
true_pop_mean = mean(pop)
ggplot(sim_res, aes(x = sample_mean)) +
  geom_histogram(fill = blue) +
  geom_vline(xintercept = true_pop_mean) +
  ggtitle("Sampling Distribution\nof Sample Mean") +
  xlab("Sample Mean") + theme_bw()
```



An Analytical Approach to the Sampling Distributions

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- We will start with a common estimator, the sample mean, $\overline{X}_n = \frac{1}{n}\sum_{i=1} X_i$.
- Under the identically and independently distributed assumption we can characterize properties of the distribution like the expectation and variance.

Describing the Sampling Distribution for the Sample Mean

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If we assume that $X_1, \ldots X_n \sim_{i.i.d} \ ?(\mu, \sigma^2)$, then we would like to identify the following things about $\overline{X}_n$.

- $E[\overline{X}_n]$
- $V[\overline{X}_n]$
- $f_{\overline{X}_n} \sim ?$

Expectation of $\overline{X}_n$

Let $X_1, X_2, \dots X_n$ be identically and independently distributed from a population distribution with mean ($E[X_i] = \mu$) and variance ($V[X_i] = \sigma^2$).

$$E[\overline{X}_n] \quad =$$

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&= \frac{1}{n}\sum_{i=1}^{n} \mu \\
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$$

Variance of $\overline{X}_n$

Let $X_1, X_2, \ldots X_n$ be identically and independently distributed from a population distribution with mean ($E[X_i] = \mu$) and variance ($V[X_i] = \sigma^2$).

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Note the n in the denominator: as we have more observations, the variance of the sampling distribution will shrink.

What about the "?"

If $X_1, \ldots, X_n \sim_{i.i.d.} N(\mu, \sigma^2)$, then

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What if $X_1, \ldots, X_n$ are not normally distributed?

Bernoulli (Coin Flip) Distribution

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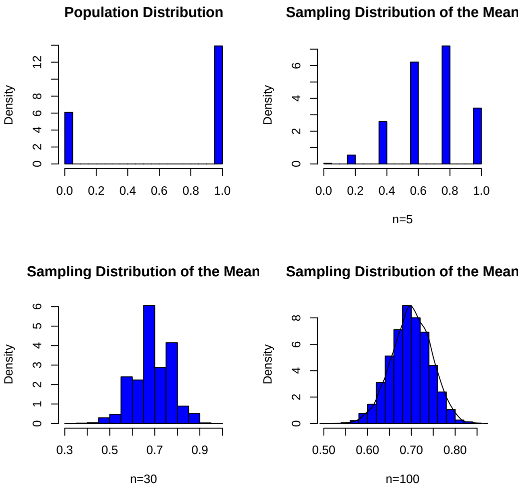
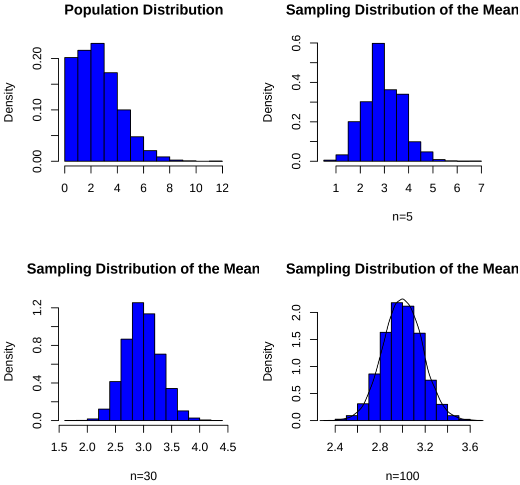
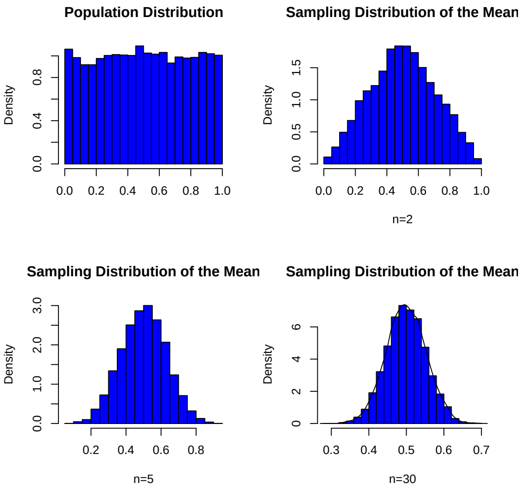


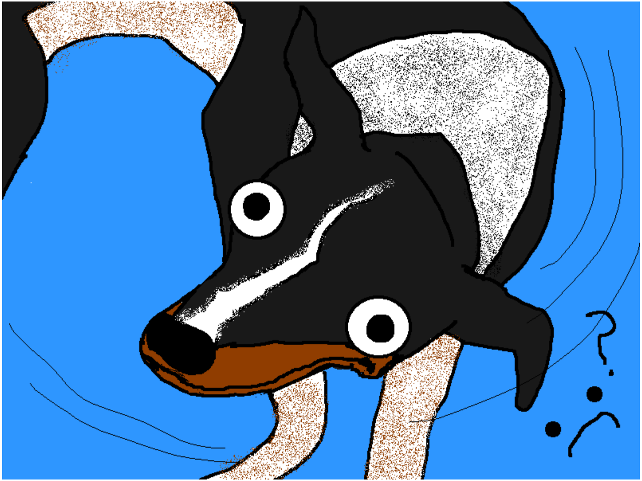
Poisson (Count) Distribution



Uniform Distribution



Why would this be true?



Images from *Hyperbole and a Half* by Allie Brosh.

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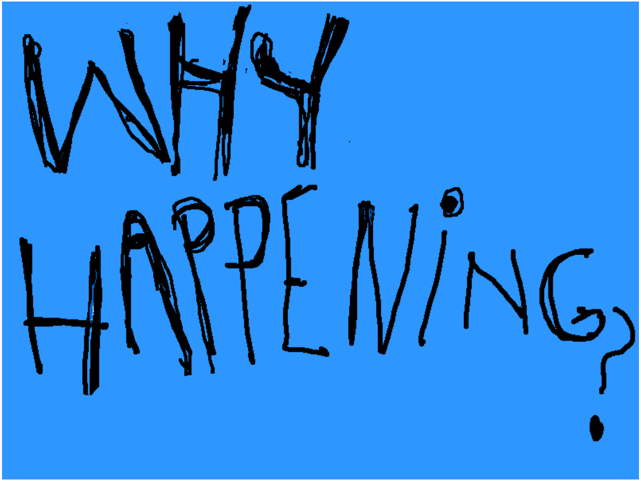

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Next time: the answer to 'why happening?' and the most important theorem in statistics.

Where We've Been and Where We're Going...

- Last Week
  - random variables
  - joint distributions
- This Week
  - estimators and sampling distributions
  - estimator properties (bias, variance, consistency)
  - confidence intervals
- Next Week
  - hypothesis testing
  - what is regression?
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Estimation
   - Populations and Samples
   - Estimators
   - Analytical
2. Weak Law of Large Numbers and the Central Limit Theorem
   - Chebychev's Inequality
   - Weak Law of Large Numbers
   - The Central Limit Theorem
3. Properties of Estimators
   - Four Desirable Properties
   - Example
4. Interval Estimation
   - Intervals
   - Large Sample Intervals for a Mean
   - Small Sample Intervals for a Mean
   - Comparing Two Groups
   - Interval Estimation for a Proportion
5. Plug-In Principle

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The key thing to remember is that the sample mean is itself a random variable.

Warning: This video is a little bit mathier. At the end I'll wrap up with things you need to know so don't stress out your first watch through.

Bounding a Random Variable

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This let's us bound the behavior of a random variable knowing only the expectation and variance (regardless of distributional shape!).

Chebychev's Inequality for the Sample Mean

To apply this to the sample mean, we plug in the expectation and variance to the Chebychev's inequality and re-arranging terms, we get the following handy result.

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Theorem (Chebychev's Inequality for the Sample Mean)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with finite variance $V[X] > 0$. Then, $\forall \epsilon > 0$,

$$P\left[|\overline{X}_n - E[X]| \geq \epsilon\right] \leq \frac{V[X]}{\epsilon^2 n}$$

(Aronow and Miller Theorem 3.2.5)

This allows us to put an upper bound on the probability that the sample mean for a given sample size and known variance will be some arbitrary distance from the true mean.

Planning a Survey

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  - π is proportion of voters expressing support for Biden (the estimand).
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- What do we know?
  - $E[\overline{X}_n] = E[X]$ and $V[\overline{X}_n] = \frac{V[X]}{n}$
  - We know Bernoulli has variance of $\pi(1-\pi)$ which is maximized at $\pi = .5$.

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We want to bound the probability by 0.05 which requires $\frac{1}{0.0016n} \leq 0.05$ which means...

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We want to bound the probability by 0.05 which requires $\frac{1}{0.0016n} \leq 0.05$ which means…we need $n \geq 12,500$ respondents!

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```
nsims <- 10000
holder <- vector(mode="numeric", length=nsims)
for (i in 1:nsims) {
  my.sample <- rbinom(n=12500, size=1, prob=.55)
  holder[i] <- mean(my.sample)
}
mean(abs(holder - .55) > 0.02)
```

Planning a Survey

That's an expensive survey! Do we really need that many people?

No! Chebyshev provides a bound that is guaranteed to hold (and in the worst case variance), but actual probabilities are much smaller.

We can use simulation to assess. Let's simulate a survey with $\pi = .55$ and 12,500 respondents and see how many are far away?

```
nsims <- 10000
holder <- vector(mode="numeric", length=nsims)
for (i in 1:nsims) {
  my.sample <- rbinom(n=12500, size=1, prob=.55)
  holder[i] <- mean(my.sample)
}
mean(abs(holder - .55) > 0.02)
```

None were outside the range!

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- What does this sequence converge to?

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Definition (Convergence in Probability)

Let $(T_{(1)}, T_{(2)}, T_{(3)}, \dots)$ be a sequence of random variables and let $c \in \mathbb{R}$. Then $T_{(n)}$ converges in probability to c if for all accuracy levels satisfying $\epsilon > 0$,

$$\lim_{n\to\infty} P\left[|T_{(n)} - c| \geq \epsilon\right] = 0$$

We will write this as

$$T_{(n)} \xrightarrow{p} c \quad \text{or} \quad \operatorname*{plim}_{n\to\infty} T_{(n)} = c.$$

(Aronow and Miller Theorem 3.2.6)

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NB: Any continuous function of the sequence itself convergence to the value of the function at the probability limit by the Continuous Mapping Theorem (Aronow and Miller Theorem 3.2.7)

(Weak) Law of Large Numbers (WLLN)

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Definition (Weak Law of Large Numbers)

Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with finite variance $V[X] > 0$, and let $\overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then

$$\overline{X}_n \xrightarrow{p} E[X]$$

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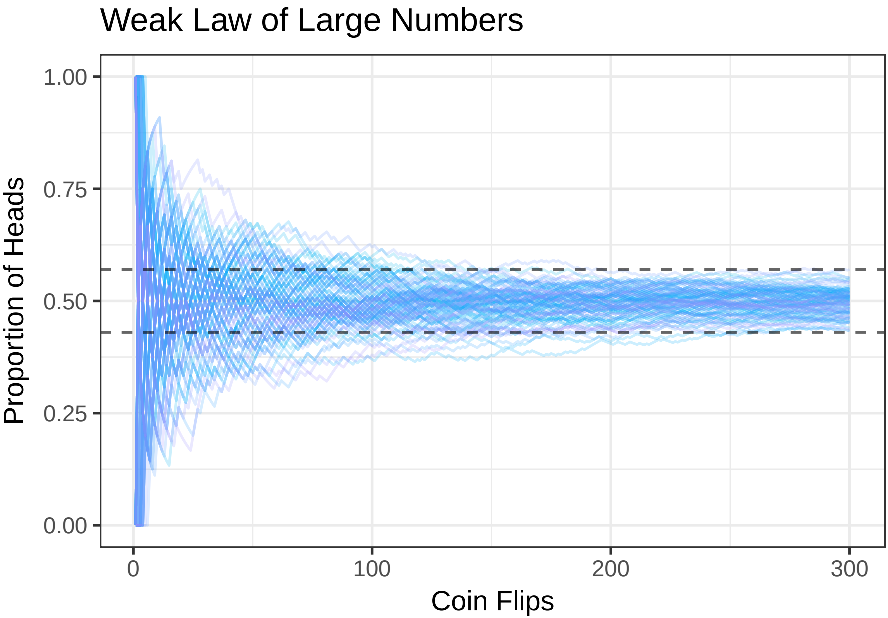
- The distribution of $\overline{X}_n$ collapses on $E[X]$.
- No assumptions necessary about the distribution of X beyond i.i.d. sampling and a finite variance!


Weak Law of Large Numbers
Proportion of Heads
1.00
0.75
0.50
0.25
0.00
0
100
200
300
Coin Flips

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- An implication of the Weak Law of Large Numbers is that the CDF of X can be estimated to arbitrary precision with random iid samples from X. We will return to this result in two videos.

Okay that's pretty cool, but we are almost ready to state the coolest result in statistics.

Convergence in Distribution

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Let $(T_{(1)}, T_{(2)}, T_{(3)}, \dots)$ be a sequence of random variables with CDFs $(F_{(1)}, F_{(2)}, F_{(3)}, \dots)$ and let T be a random variable with CDF F_T. Then $T_{(n)}$ converges in distribution to T if for all $t \in \mathbb{R}$at which F_T is continuous

$$\lim_{n\to\infty} F_{(n)}(t) = F_T(t).$$

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- We will call this the asymptotic distribution or the limit distribution.
- NB: convergence in probability is a special case of convergence in distribution with a degenerate distribution.

Standardizing the Sample Mean

Last prerequisite!

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Definition (Standardizing a Random Variable)

For i.i.d. random variables $X_1, X_2, \ldots, X_n$ with finite $E[X] = \mu$ and finite $V[X] = \sigma^2 > 0$, the standardized sample mean is

$$Z = \frac{(\overline{X} - E[\overline{X}])}{\sigma[\overline{X}]} = \frac{\sqrt{n}\,(\overline{X} - \mu)}{\sigma}$$

(Aronow and Miller Definition 3.2.23)

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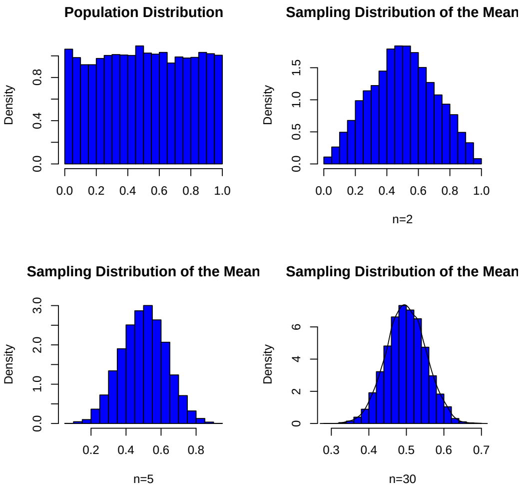
- This is often called the Z-score.

The Puzzle



Population Distribution
Density
0.0
0.4
0.8
0.0 0.2 0.4 0.6 0.8 1.0
Sampling Distribution of the Mean
Density
0.0
0.5
1.0
1.5
0.0 0.2 0.4 0.6 0.8 1.0
n=2
Sampling Distribution of the Mean
Density
0.0
1.0
2.0
3.0
0.2 0.4 0.6 0.8
n=5
Sampling Distribution of the Mean
Density
0
2
4
6
0.3 0.4 0.5 0.6 0.7
n=30

The Central Limit Theorem

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Let $X_1, ..., X_n$ be i.i.d. random variables each with (finite) $E[X] = \mu$ and finite variance $\sigma^2 > 0$. Then, for *any* population distribution of X,

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- CLT also implies that the standardized sample mean converges to a standard normal random variable:

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- This is free of distribution assumptions on X!
- This makes it easy to characterize the sampling distribution of the sample mean for large n.
- NB: the equivalence of the two forms is due to Slutsky's Theorem (see e.g. Aronow and Miller Theorem 3.2.25).

Question:

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As the number of observations in a dataset increases, which of the following is true?

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C) Both statements are true.

Replanning that Survey

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It is easier to work with this standardized variable so:

$$P(|Z| > 0.02(2\sqrt{n})) \leq 0.05$$

Replanning that Survey

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$$P(|Z| > 0.04\sqrt{n}) \quad \leq \quad 0.05$$

Replanning that Survey

$$
\begin{aligned}
P(|Z| > 0.04\sqrt{n}) &\leq 0.05 \\
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This is much lower than the 12,500 from Chebyshev, but that makes sense here because we used more information.

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We get 0.0485!

Real Talk: this has been a mathy video.

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What you need to know:

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- two types of stochastic convergence

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- two types of stochastic convergence
  - Convergence in probability: values in the sequence eventually take a constant value
    (i.e. the limiting distribution is a point mass)

Real Talk: this has been a mathy video.

What you need to know:

- two types of stochastic convergence
  - Convergence in probability: values in the sequence eventually take a constant value
    (i.e. the limiting distribution is a point mass)
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- what asymptotic properties are

It is okay if you didn't follow all the math here. We will keep coming back to these ideas.

The Central Limit Theorem is deep and amazing. Want to learn more about Central Limit Theorem?

Watch this video (Joe Blitzstein):
https://www.youtube.com/watch?v=OprNqnHsVIA&list=PLLVplP8OIVc8EktkrD3Q8td0GmId7DjW0&index=31&t=0s

There are many CLT variants that deal with non-iid random variables as well!

We Covered. . .

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- Chebychev's Inequality

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Next time: properties of estimators.

Where We've Been and Where We're Going...

- Last Week
  - random variables
  - joint distributions
- This Week
  - estimators and sampling distributions
  - estimator properties (bias, variance, consistency)
  - confidence intervals
- Next Week
  - hypothesis testing
  - what is regression?
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1 Estimation
- Populations and Samples
- Estimators
- Analytical

2 Weak Law of Large Numbers and the Central Limit Theorem
- Chebychev's Inequality
- Weak Law of Large Numbers
- The Central Limit Theorem

3 Properties of Estimators
- Four Desirable Properties
- Example

4 Interval Estimation
- Intervals
- Large Sample Intervals for a Mean
- Small Sample Intervals for a Mean
- Comparing Two Groups
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5 Plug-In Principle

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Example: The scariest pieces of mail ever!

American Political Science Review Vol. 102, No. 1 February 2008

DOI: 10.1017/S000305540808009X

Social Pressure and Voter Turnout: Evidence from a Large-Scale Field Experiment

ALAN S. GERBER *Yale University*
DONALD P. GREEN *Yale University*
CHRISTOPHER W. LARIMER *University of Northern Iowa*

`https://doi.org/10.1017/S000305540808009X`

Example: The scariest pieces of mail ever!

Voter turnout theories based on rational self-interested behavior generally fail to predict significant turnout unless they account for the utility that citizens receive from performing their civic duty. We distinguish between two aspects of this type of utility, intrinsic satisfaction from behaving in accordance with a norm and extrinsic incentives to comply, and test the effects of priming intrinsic motives and applying varying degrees of extrinsic pressure. A large-scale field experiment involving several hundred thousand registered voters used a series of mailings to gauge these effects. Substantially higher turnout was observed among those who received mailings promising to publicize their turnout to their household or their neighbors. These findings demonstrate the profound importance of social pressure as an inducement to political participation.

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Example: The scariest pieces of mail ever!

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY — VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

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Basic Analysis

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They make their data available (https://isps.yale.edu/research/data/d001). We can analyze it.

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load("gerber_green_larimer.RData")
## turn turnout variable into a numeric
social$voted <- 1 * (social$voted == "Yes")
neigh.mean <- mean(social$voted[social$treatment == "Neighbors"])
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Is this a "real" effect? Is it big?

Desirable Properties of Estimators

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- We'd like an estimator that has a known sampling distribution (approximately) when the sample size is large.

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Asymptotic Properties (kick in when n is large):

- Consistency: As our sample size grows to infinity, does the sampling distribution of our estimator converge to the true parameter value?
- Asymptotic Normality: As our sample size grows large, does the sampling distribution of our estimator approach a normal distribution?

1: Bias

(not getting the right answer on average)

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An estimator is unbiased if and only if:

$$\text{Bias}(\hat{\mu}) = 0$$

Example: Estimators for Population Mean

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Candidate estimators:

1. $\hat{\mu_1} = Y_1$ (the first observation)
2. $\hat{\mu_2} = \frac{1}{2}(Y_1 + Y_n)$ (average of the first and last observation)
3. $\hat{\mu_3} = 42$
4. $\hat{\mu_4} = \overline{Y}_n$ (the sample average)

How do we choose between these estimators?

Bias of Example Estimators

Which of these estimators are unbiased?

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2. $E[\frac{1}{2}(Y_1 + Y_n) - \mu] =$
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Bias of Example Estimators

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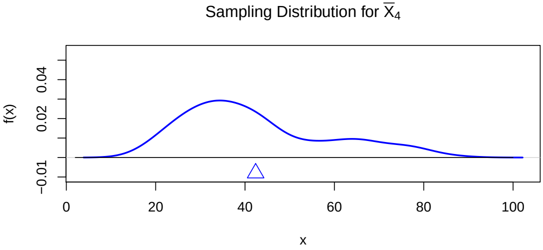
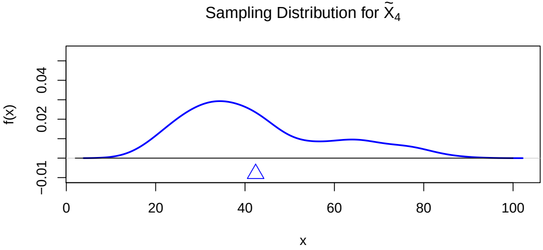
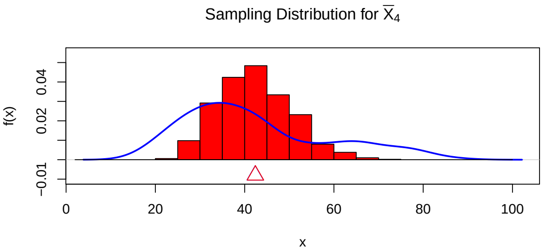
- Estimator 3 is biased.

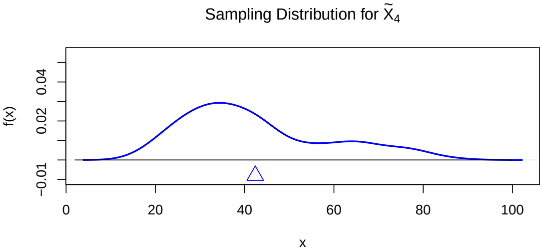
Age population distribution in blue





Age population distribution in blue, sampling distributions in red





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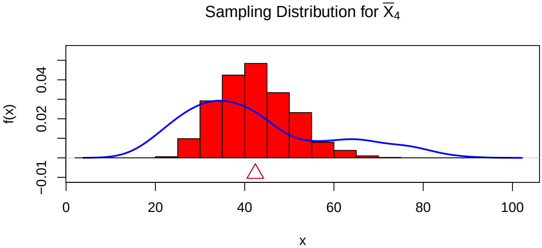

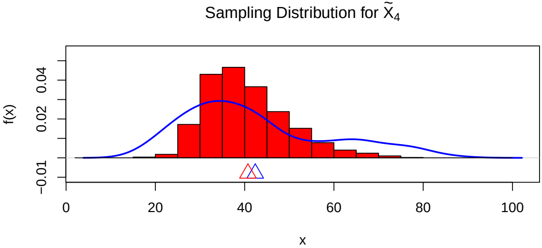

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If $\hat{\theta}_1$ and $\hat{\theta}_2$ are unbiased estimators of θ, then $\hat{\theta}_1$ is more efficient relative to $\hat{\theta}_2$ iff

$$V[\hat{\theta}_1] < V[\hat{\theta}_2]$$

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Aronow and Miller discuss efficiency in terms of MSE (more on this in a second).

Variance of Example Estimators

What is the variance of our estimators?

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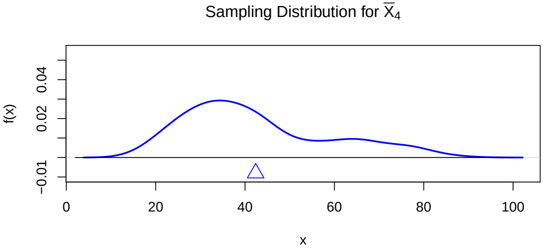
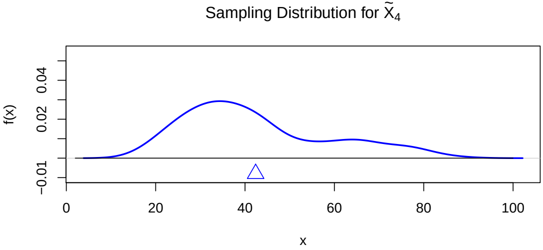
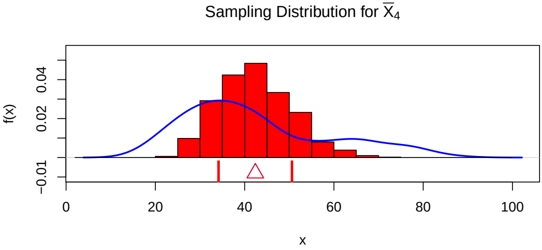
Among the unbiased estimators, the sample average has the smallest variance. This means that Estimator 4 (the sample average) is likely to be closer to the true value μ, than Estimators 1 and 2.

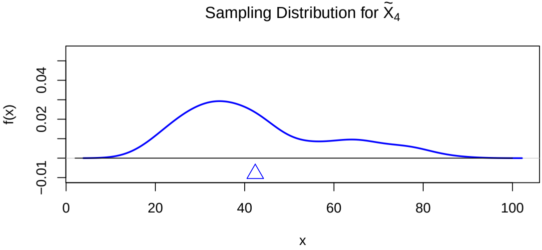
Age population distribution in blue





Age population distribution in blue, sampling distributions in red





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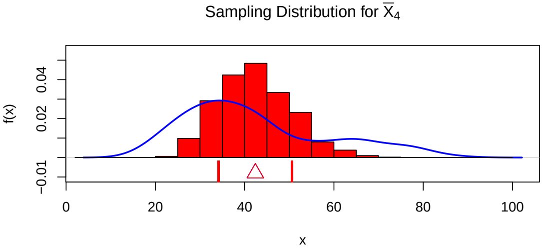
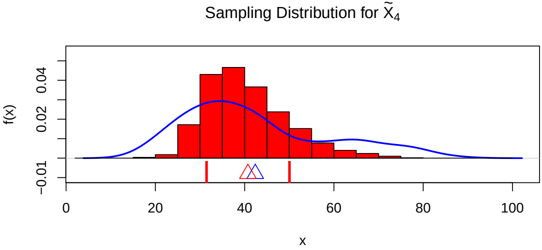
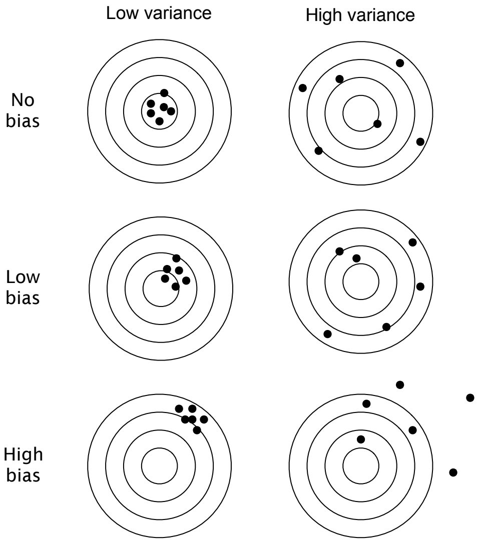




Trading Off Bias and Variance



Salganik (2018), Figure 3.1

Mean Squared Error

How can we choose between an unbiased estimator and a biased, but lower variance estimator?

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Definition (Mean Squared Error)

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Sometimes (as in Aronow and Miller Deinition 3.2.16) efficiency is defined as having lower MSE.

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(what happens as sample size increases)

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- Asymptotic properties of an estimator are defined by the behavior of $\hat{\theta}_1, ...\hat{\theta}_n$ when n goes to infinity.

3: Consistency

(does it get closer to the right answer as sample size increases)

Definition

An estimator $\hat{\theta}_n$ is consistent if the sequence $\hat{\theta}_1, ..., \hat{\theta}_n$ converges in probability to the true parameter value θ as sample size n grows to infinity:

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Deriving Consistency of Estimators

Our candidate estimators:

1. $\widehat{\mu}_1 = Y_1$
2. $\widehat{\mu}_2 = 4$
3. $\widehat{\mu}_3 = \overline{Y}_n \equiv \frac{1}{n}(Y_1 + \cdots + Y_n)$
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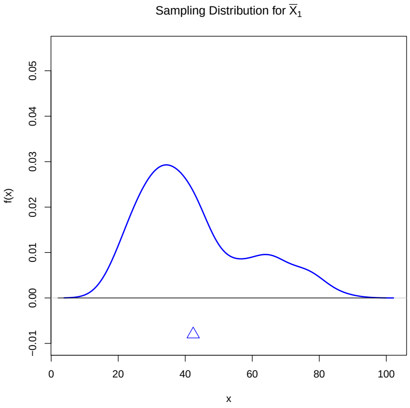

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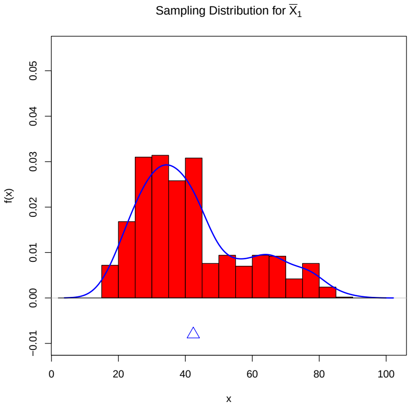

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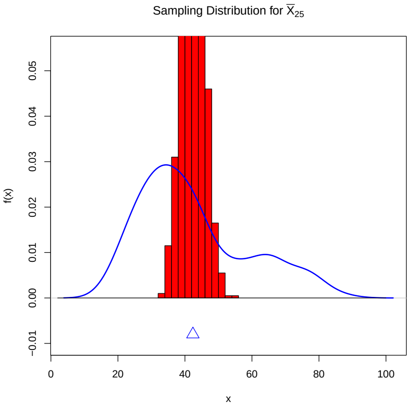

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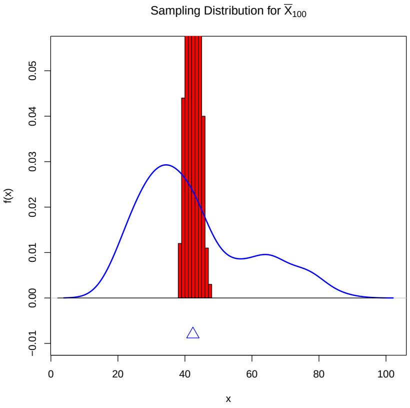

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- The sampling distribution never collapses around anything

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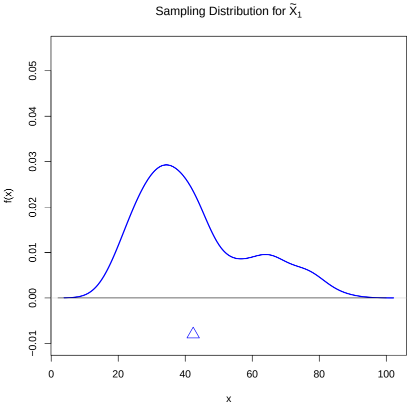

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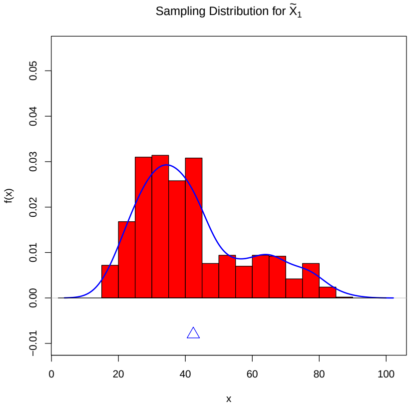

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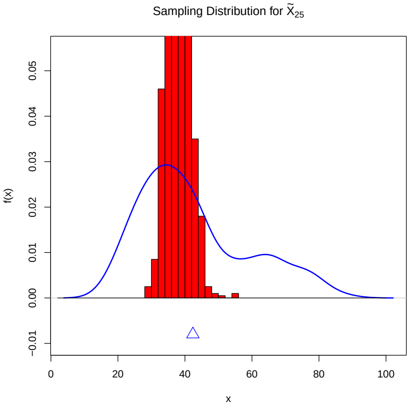

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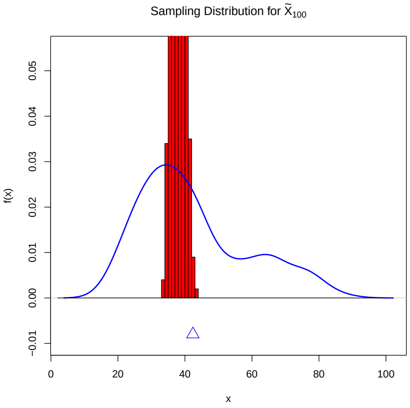

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This will play a crucial role in our ability to form confidence intervals.

Summary of Properties

Concept	Criteria	Intuition
Unbiasedness	$E[\hat{\mu}] = \mu$	Right on average
Efficiency	$V[\hat{\mu}_1] < V[\hat{\mu}_2]$	Low variance
Consistency	$\hat{\mu}_n \xrightarrow{p} \mu$	Converge to estimand as $n \rightarrow \infty$
Asymptotic Normality	$\hat{\mu}_n \overset{\text{approx.}}{\sim} N(\mu, \frac{\sigma^2}{n})$	Approximately normal in large n

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Repeating theme in this class—how to characterize an estimator.

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Back to the Example

American Political Science Review Vol. 102, No. 1 February 2008
DOI: 10.1017/S000305540808009X

Social Pressure and Voter Turnout: Evidence from a Large-Scale Field Experiment

ALAN S. GERBER *Yale University*
DONALD P. GREEN *Yale University*
CHRISTOPHER W. LARIMER *University of Northern Iowa*

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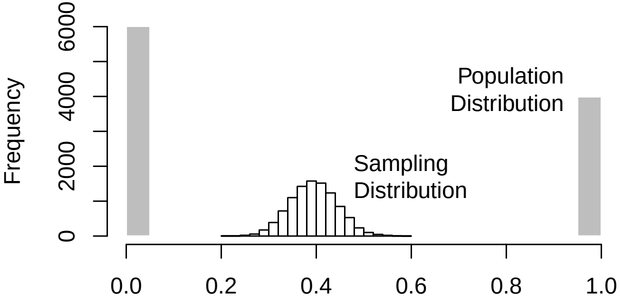

Population vs. Sampling Distribution

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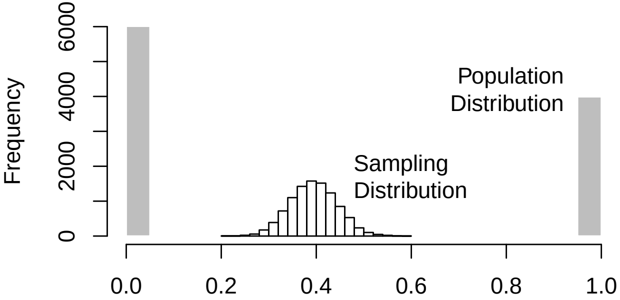

But remember that we only get to see one draw from the sampling distribution. Thus ideally we want an estimator with good properties.

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- By asymptotic Normality $(\hat{\theta} - 0)/\text{SE}(\hat{\theta}) \sim N(0, 1)$
- By the properties of Normals, we know that this implies that $\hat{\theta} \sim \mathcal{N}(0, \text{SE}(\hat{\theta}))$

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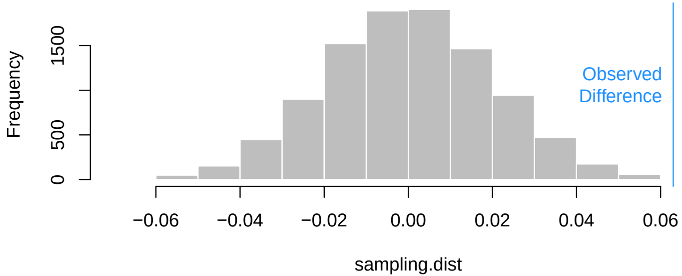

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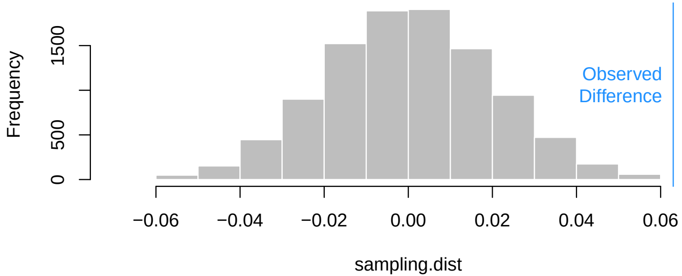
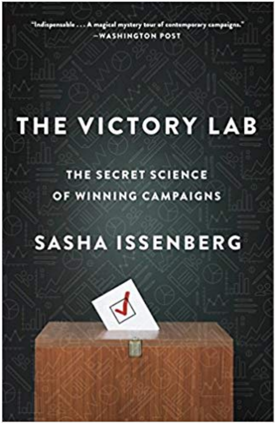


Does the observed difference in means seem plausible if there really were no difference between the two groups in the population?

The scariest pieces of mail ever! continued



Summarizes the relationships between political science research and campaigns. Also, attempts to weaponize the results of Gerber et al (2008).

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Next Time: interval estimation

Where We've Been and Where We're Going...

- Last Week

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  - random variables

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Where We've Been and Where We're Going...

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- This Week
  - estimators and sampling distributions
  - estimator properties (bias, variance, consistency)

Where We've Been and Where We're Going...

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   - Four Desirable Properties
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- An interval estimate is a realized value from an interval estimator. The estimated interval typically forms what we call a confidence interval, which we will define shortly.

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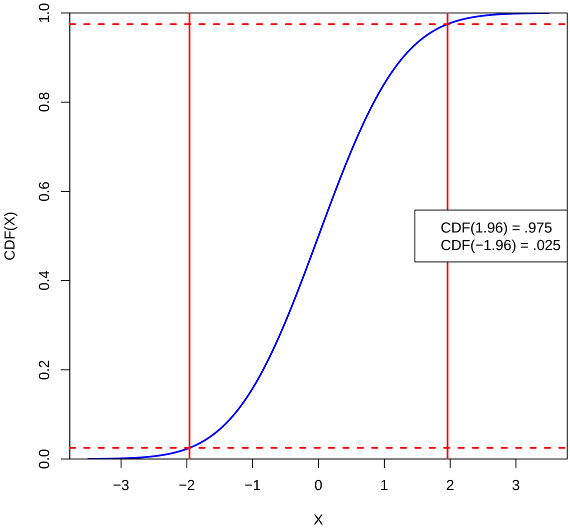

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We call this estimator a 95% confidence interval for μ.

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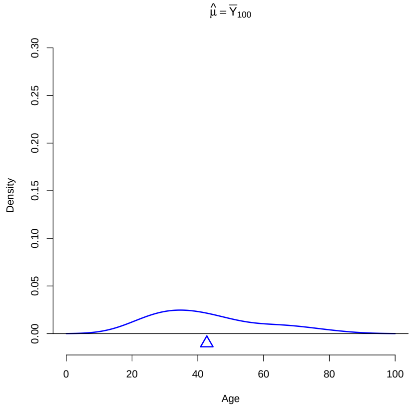

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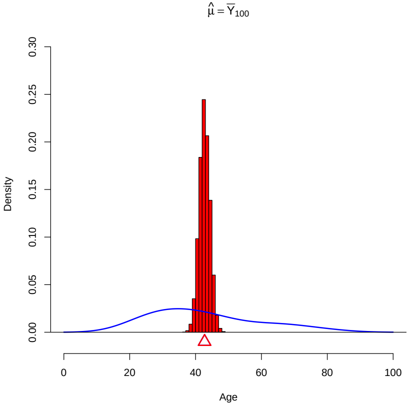

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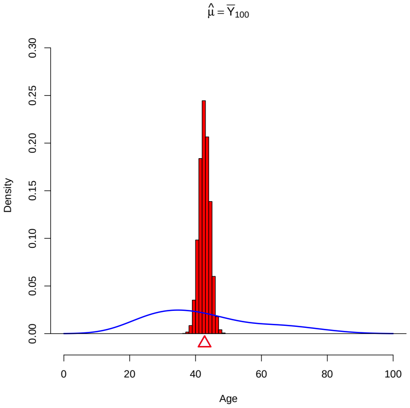

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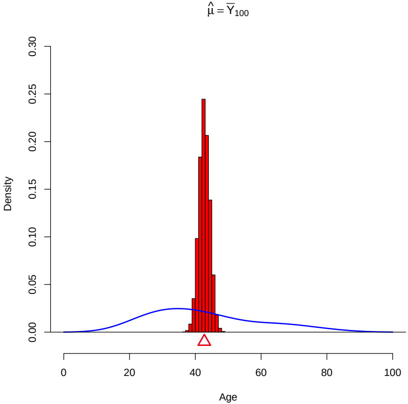

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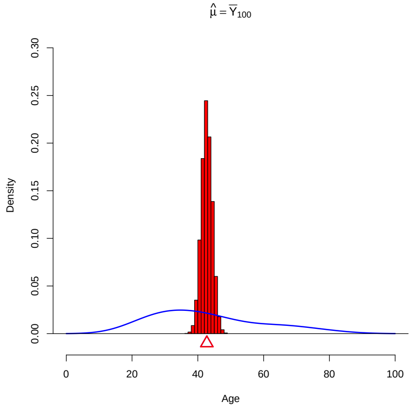

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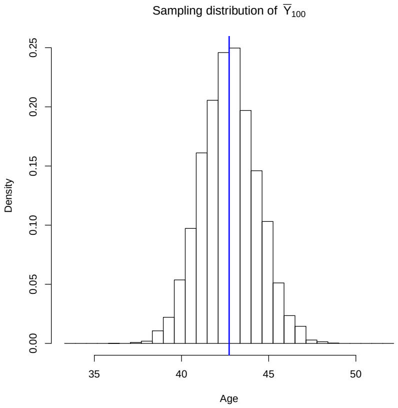

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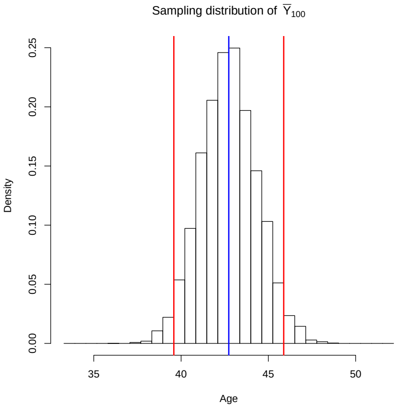

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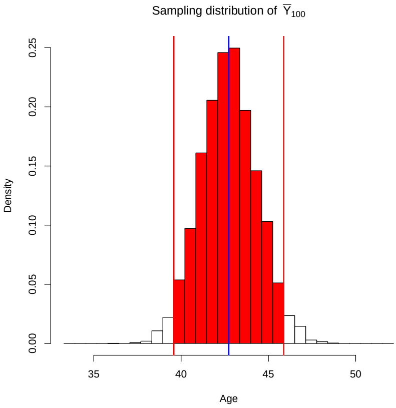

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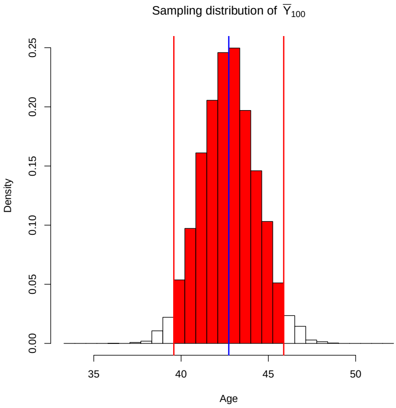

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Instead, we need an estimator of σ^2, $\hat{\sigma}^2$.

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S_{1n}^2 (unbiased and consistent) is commonly called the sample variance.

Estimating σ and the SE

Returning to Kulinski et. al. . .

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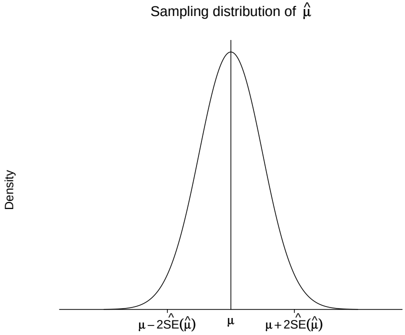
We will plug in S for σ and our estimated standard error will be

$$\widehat{SE}[\hat{\mu}] = \frac{S}{\sqrt{n}}$$

95% Confidence Intervals

If $X_1, ..., X_n$ are i.i.d. and n is large, then

$$\widehat{\mu} \quad \sim \quad N(\mu, (\widehat{SE}[\hat{\mu}])^2)$$



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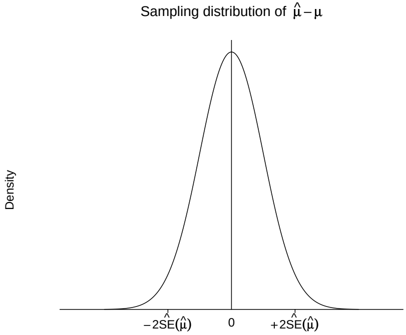

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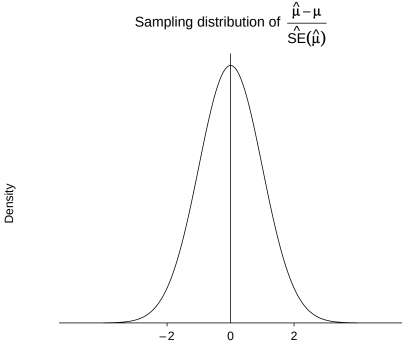

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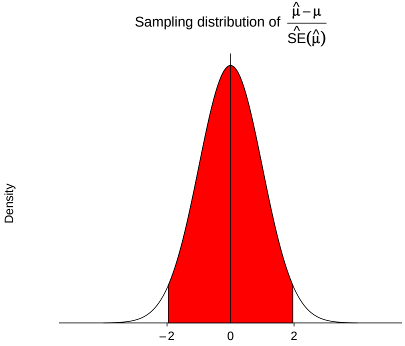

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P\left(-1.96 \leq \frac{\widehat{\mu} - \mu}{\widehat{SE}[\hat{\mu}]} \leq 1.96\right) &= 95\% \\
P\left(-1.96\widehat{SE}[\hat{\mu}] \leq \widehat{\mu} - \mu \leq 1.96\widehat{SE}[\hat{\mu}]\right) &= 95\% \\
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\end{aligned}
$$

The random quantities in this statement are $\widehat{\mu}$ and $\widehat{SE}[\hat{\mu}]$.

95% Confidence Intervals

We can work backwards from this:

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P\left(-1.96 \leq \frac{\widehat{\mu} - \mu}{\widehat{SE}[\hat{\mu}]} \leq 1.96\right) &= 95\% \\
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\end{aligned}
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The random quantities in this statement are $\widehat{\mu}$ and $\widehat{SE}[\hat{\mu}]$.
Once the data are observed, nothing is random!

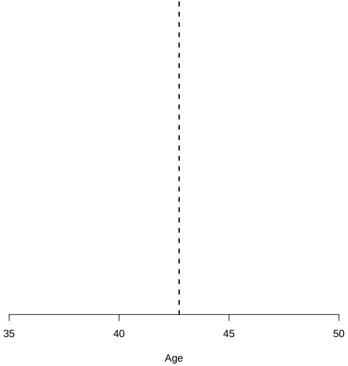
What does this mean?

We can simulate this process using the Kuklinski data:

1) Draw a sample of size 100:

2) Calculate $\hat{\mu}$ and $\widehat{SE}[\hat{\mu}]$:

3) Construct the 95% CI:



What does this mean?

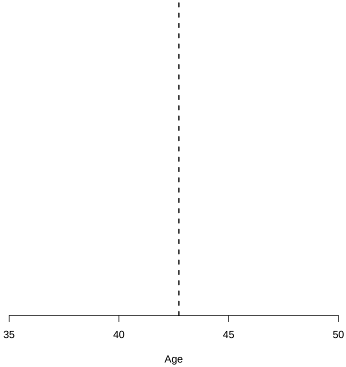
We can simulate this process using the Kuklinski data:

1) Draw a sample of size 100:

2) Calculate $\hat{\mu}$ and $\widehat{SE}[\hat{\mu}]$:

$$\hat{\mu} = 42.32 \quad \widehat{SE}[\hat{\mu}] = 1.498$$

3) Construct the 95% CI:



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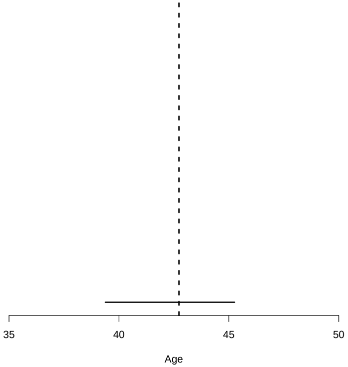

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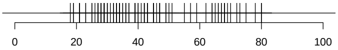

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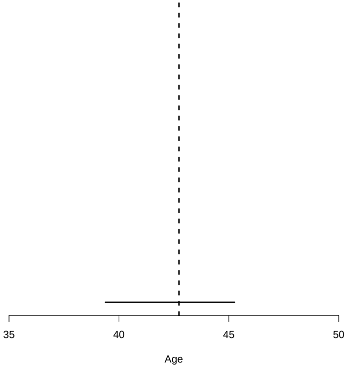

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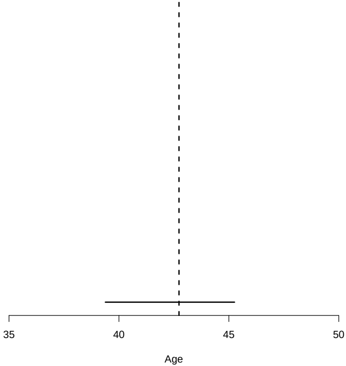

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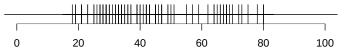

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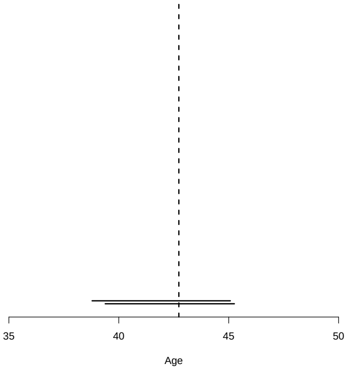

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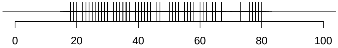

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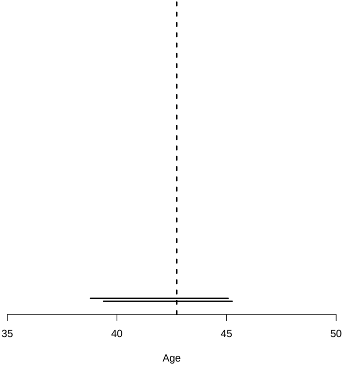

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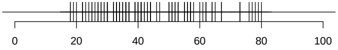

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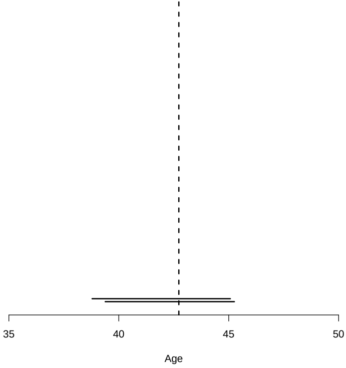

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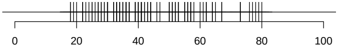

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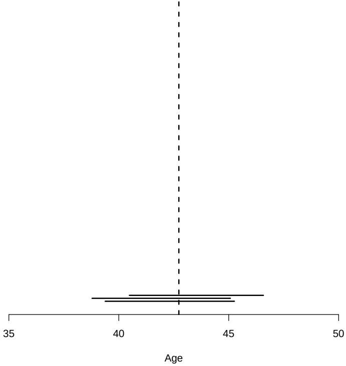

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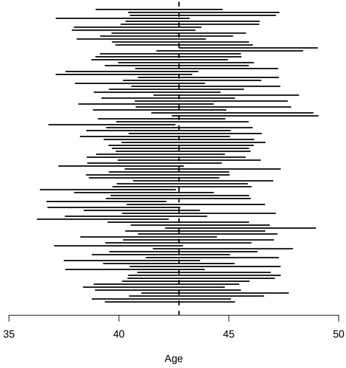
By repeating this process, we generate the sampling distribution of the 95% CIs.



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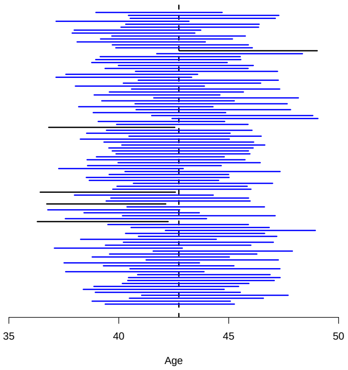
Most of the CIs cover the true μ;



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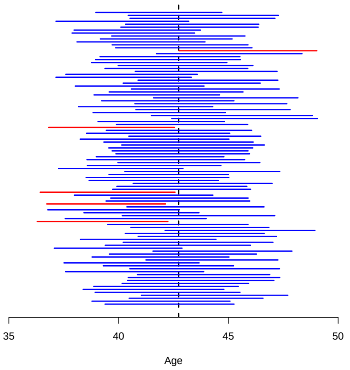

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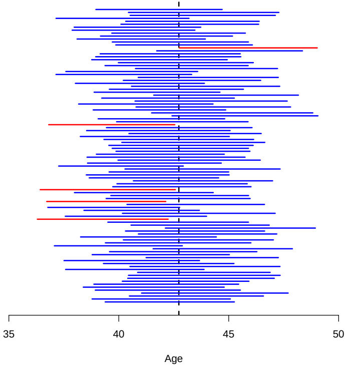
In the long run, we expect 95% of the CIs generated to contain the true value.



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  - Therefore, we refer to .95 as the coverage probability

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- You want the the shortest confidence interval with the desired coverage probability.

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$$P\left(-z \leq \frac{\widehat{\mu} - \mu}{\widehat{SE}[\hat{\mu}]} \leq z\right) = (1 - \alpha)\%$$

How can we find z?

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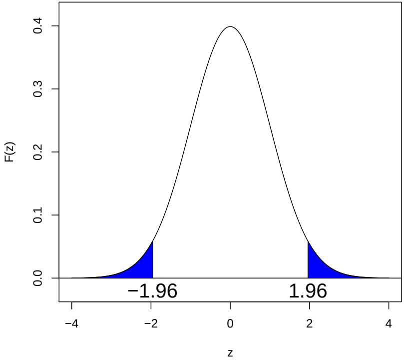
This gives us a value of 1.96 for z.



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When $(1-\alpha) = 0.50$, we want to pick z so that 25% of the probability is in each tail.

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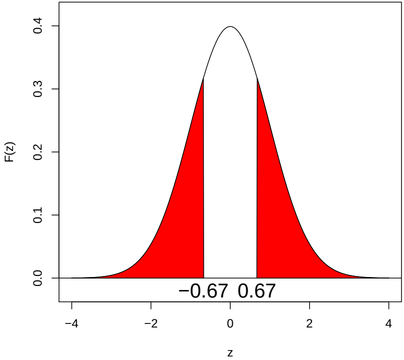

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Statistical problems emerge from real science

Comparing different methods of growing barley (Full history: `https://www.jstor.org/stable/2245613`)

`https://en.wikipedia.org/wiki/Guinness#/media/File:Guinness.jpg`

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When the sample size is small, we need to know something about the distribution in order to construct confidence intervals with the correct coverage (because we can't appeal to the CLT or assume that S is a good approximation of σ).

VOLUME VI MARCH, 1908 No. 1

BIOMETRIKA.

THE PROBABLE ERROR OF A MEAN.

BY STUDENT.

https://www.jstor.org/stable/2331554

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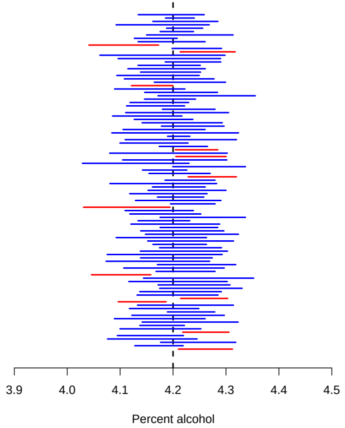
In this sample, only 88 of the 100 CIs cover the true value.



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We rarely know σ and have to use an estimate instead:

$$\frac{\overline{X} - \mu}{\frac{s}{\sqrt{n}}} \sim t_{n-1}$$

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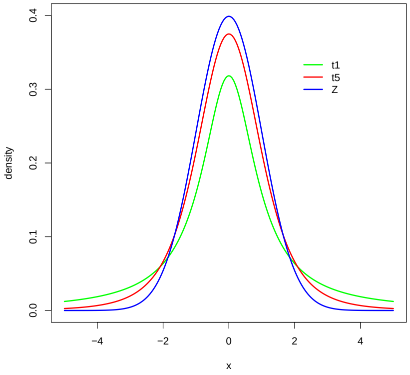

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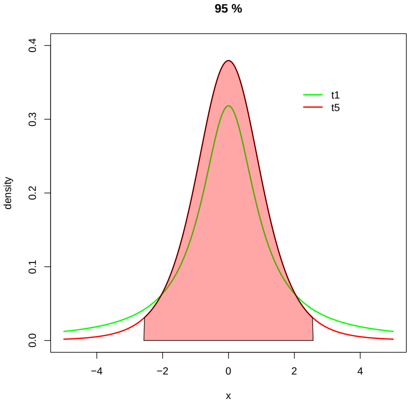

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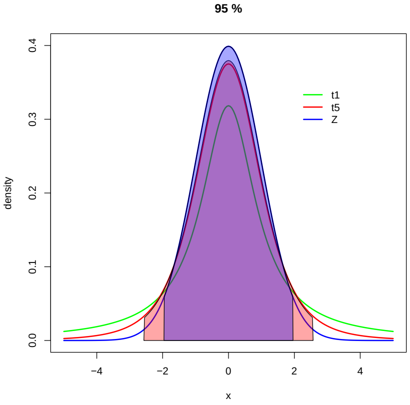

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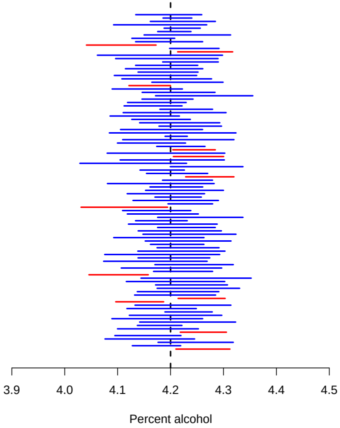

Small Sample Example

When we generated 95% CIs with the large sample formula

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When we use the correct small-sample formula

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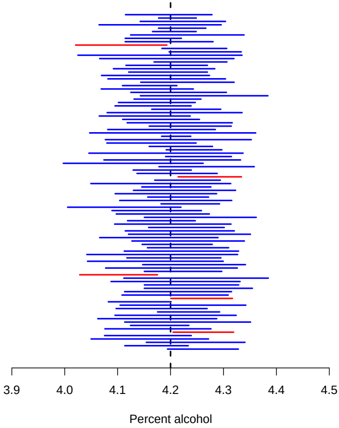

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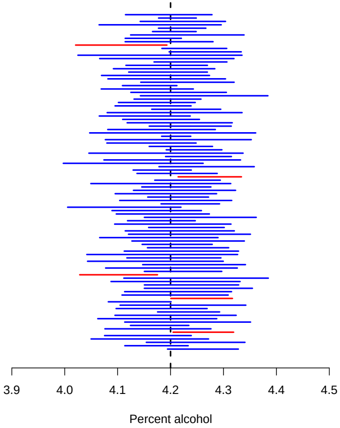

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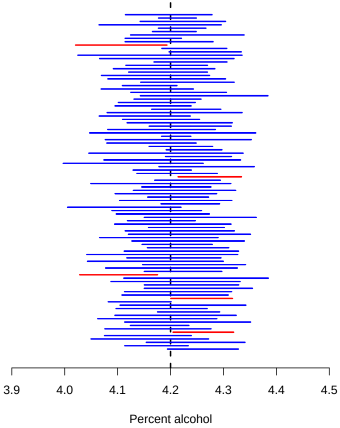
95 of the 100 CIs in this sample cover the truth.



Another Rationale for the t-Distribution

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Thus, we need to derive the sampling distribution of the new random variable. It turns out that T_n follows Student's t-distribution with $n-1$ degrees of freedom.

Theorem (Distribution of t-Value from a Normal Population)

Suppose we have an i.i.d. random sample of size n from $N(\mu, \sigma^2)$. Then, the sample mean $\overline{X}_n$ standardized with the estimated standard error $S_n/\sqrt{n}$ satisfies,

$$T_n \equiv \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}} \sim \tau_{n-1}$$

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We will usually be interested in comparing μ_1 to μ_2, although we will sometimes need to compare σ_1^2 to σ_2^2 in order to make the first comparison.

Sampling Distribution for $\overline{X}_1 - \overline{X}_2$

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- $\overline{X}_1 - \overline{X}_2$ is distributed $\sim N(\mu_1 - \mu_2, \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2})$.

CIs for $\mu_1 - \mu_2$

Using the same type of argument that we used for the univariate case, we write a $(1 - \alpha)$% CI as the following:

$$\overline{X}_1 - \overline{X}_2 \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

Interval estimation of the population proportion

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- Let's say that we have a sample of iid Bernoulli random variables, $Y_1, \ldots, Y_n$, where each takes $Y_i = 1$ with probability π. Note that this is also the population proportion of ones. We have shown in previous weeks that the expectation of one of these variable is just the probability of seeing a 1: $E[Y_i] = \pi$.

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- **Problem** Show that the sample proportion, $\hat{\pi} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, of the above iid Bernoulli sample, is unbiased for the true population proportion, π, and that the sampling variance is equal to $\frac{\pi(1-\pi)}{n}$.

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- Note that if we have an estimate of the population proportion, $\hat{\pi}$, then we also have an estimate of the sampling variance: $\frac{\hat{\pi}(1-\hat{\pi})}{n}$.
- Given the facts from the previous problem, we just apply the same logic from the population mean to show the following confidence interval:

$$P\left(\hat{\pi} - z_{\alpha/2} \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} \leq \pi \leq \hat{\pi} + z_{\alpha/2} \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}\right) = (1-\alpha)$$

Gerber, Green, and Larimer experiment

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- Let's use what we have learned up until now and the information in the table to calculate a 95% confidence interval for the difference in proportions voting between the Neighbors group and the Civic Duty group.

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- Remember that we can calculate the sample variance for a sample proportion like so: $(\hat{\pi}_C(1-\hat{\pi}_C))/n_C$

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```
n.n <- 38201
samp.var.n <- (0.378 * (1 - 0.378))/n.n
n.c <- 38218
samp.var.c <- (0.315 * (1 - 0.315))/n.c
se.diff <- sqrt(samp.var.n + samp.var.c)
## lower bound
(0.378 - 0.315) - 1.96 * se.diff
## [1] 0.05626701
## upper bound
(0.378 - 0.315) + 1.96 * se.diff
## [1] 0.06973299
```

Thus, the confidence interval for the effect is $[0.056267, 0.069733]$.

We can use our analytic samples to find a confidence interval

Our estimate

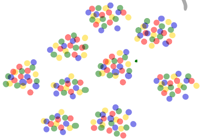
$$CI(\alpha) = [r - z_{\alpha/2} * SE, r + z_{\alpha/2} * SE]$$

Alpha

$\alpha/2$ because we're looking for a two-sided interval

Standard error of our estimate

Critical value



Review

To use the confidence interval formula, we need to find:

1. The distribution
2. Confidence level
   - Alpha
3. Sidedness
4. Critical value(s)
5. Standard error of our estimate

```
##Calculating our critical value
cv <- qnorm(.975)
cv

## [1] 1.959964
```

```
##Finding the standard error of our estimate
se <- sqrt(red.sample*(1-red.sample)/n.samp)
se

## [1] 0.01966499
```

for a proportion, the formula is:

$$SE(\hat{p}) = \sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

Review

Calculating the confidence interval

$$CI(\alpha) = [r - z_{\alpha/2} * SE, r + z_{\alpha/2} * SE]$$

```
##Finding and printing the confidence interval
c(red.sample - cv*se,
  red.sample + cv*se)
```

```
## [1] 0.2234573 0.3005427
```

Review

Our results

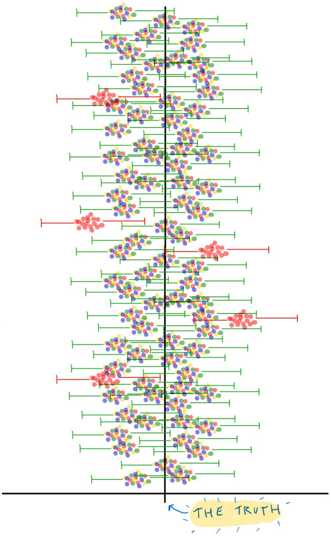
26.2% red with a 95 percent confidence interval of **[22.3, 30.1]**

Review

We hope our sample is in the 95%

samples whose confidence intervals contain the truth (95%)

samples whose confidence intervals do not contain the truth (5%)



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Next Time: The plug-in principle!

Where We've Been and Where We're Going...

- Last Week
  - random variables
  - joint distributions
- This Week
  - estimators and sampling distributions
  - estimator properties (bias, variance, consistency)
  - confidence intervals
- Next Week
  - hypothesis testing
  - what is regression?
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Estimation
   - Populations and Samples
   - Estimators
   - Analytical
2. Weak Law of Large Numbers and the Central Limit Theorem
   - Chebychev's Inequality
   - Weak Law of Large Numbers
   - The Central Limit Theorem
3. Properties of Estimators
   - Four Desirable Properties
   - Example
4. Interval Estimation
   - Intervals
   - Large Sample Intervals for a Mean
   - Small Sample Intervals for a Mean
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5. Plug-In Principle

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5 Plug-In Principle

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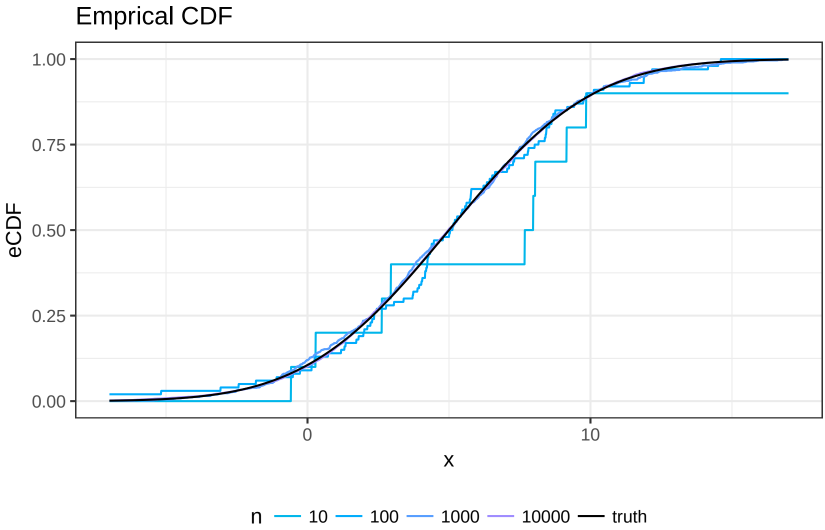
- The simplest way is to use the sample analog.
- Ex: If we're interested in the population mean, we use the sample mean
- This is justified because of the plug-in principle.
- The Weak Law of Large Numbers tells us that the empirical CDF is a good sample analog of the true CDF (which fully describes a distribution).

The Plug-in Principle in Action

Say we have a $\mathcal{N}(5, 4)$ distribution



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if T is well-behaved, then $\hat{\theta}$ is also asymptotically normal.

Bootstrapped Sampling Distributions

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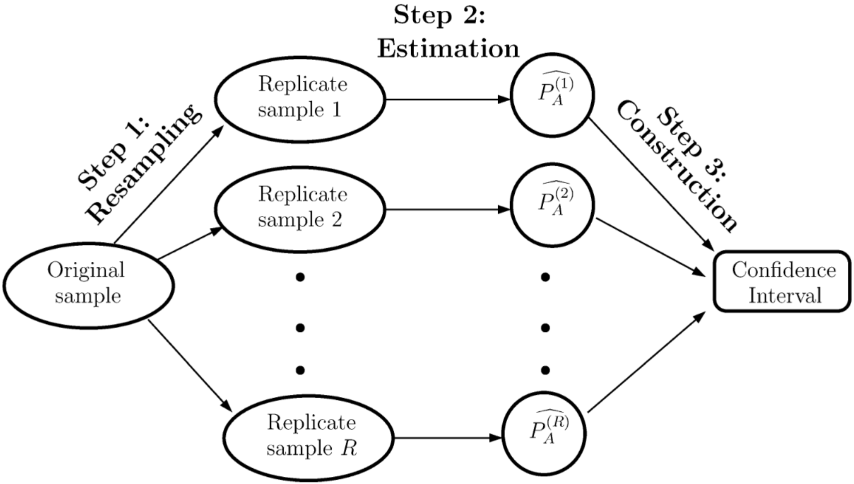
The plug-in principle gives us a way forward.



Source: Salganik (2006)

This works for almost* any estimator

*basically it works when plug-in estimation works

Statistical Science
1986, Vol. 1, No. 1, 54–77

Bootstrap Methods for Standard Errors, Confidence Intervals, and Other Measures of Statistical Accuracy

B. Efron and R. Tibshirani

Efron and Tibshirani (1986), `http://www.jstor.org/stable/2245500`

The Bootstrap

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Bootstrap: Use the eCDF as a plug-in for the CDF, and resample from that. I.e. we are pretending our sample eCDF looks sufficiently close to our true CDF, and so we're sampling from the eCDF as an approximation to repeated sampling from the true CDF. This is called a resampling method.

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1. Take a with replacement sample of size n from our sample.
2. Calculate our would-be estimate using this bootstrap sample.
3. Repeat steps 1 and 2 many (B) times.
4. Using the resulting collection of bootstrap estimates, calculate the standard deviation of the bootstrap distribution of our estimator. This serves our estimate of the standard deviation of the sampling distribution

Example of a Bootstrap

```r
samp <- c(9.7, 4.99, 5.9, 3.58, 8.15, 5.54, 4.77, 5.01, 4.89,
          3.42, 8.63, 7.17, 8.93, 7.5, 4.93, 8.6, 6.26, 7.31,
          8.96, 3.95)

obs_mean = mean(samp)

obs_mean
```

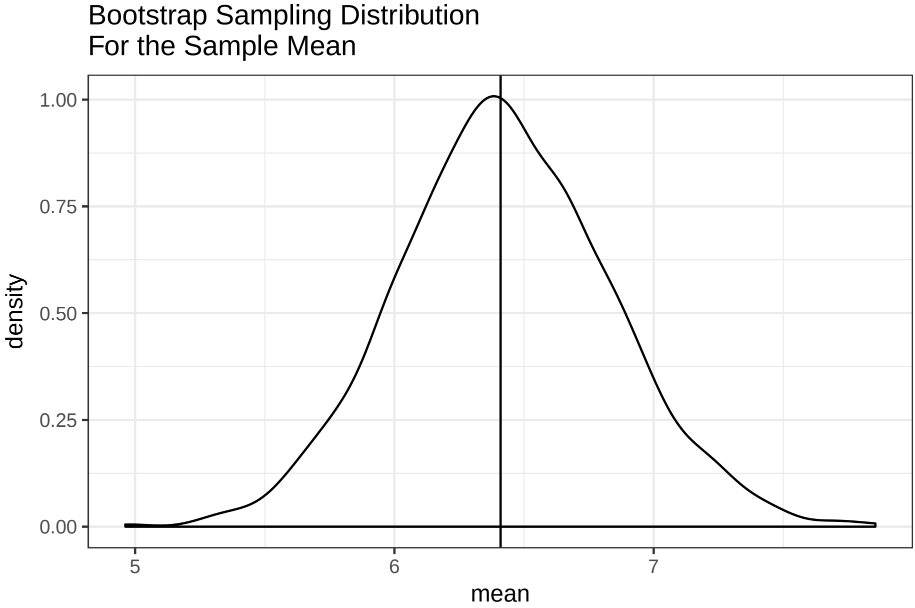
```
## [1] 6.4095
```

Example of a Bootstrap

```
# resample WITH REPLACEMENT reps times
# recalculate the mean within each bootstrap replicate
boot_samp_dist <- replicate(2000, {
      mean(samp[sample.int(length(samp), replace = TRUE)])
    })
```

```
ggplot(tibble(boot_samp_dist = boot_samp_dist),
       aes(x = boot_samp_dist)) +
  geom_density() +
  geom_vline(xintercept = obs_mean) +
  theme_bw() + ggtitle("Bootstrap Sampling Distribution
                       For the Sample Mean") +
  xlab("mean")
```

Example of a Bootstrap



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1) Using normal approximation intervals, use the estimates from step 4.

$$\left[\overline{X} - \Phi^{-1}(1-\alpha/2) * \hat{\sigma}_{\text{boot}}, \overline{X} + \Phi^{-1}(1-\alpha/2) * \hat{\sigma}_{\text{boot}}\right]$$

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2) Percentile method for the CI: Sort B bootstrap estimates from smallest to largest. α interval is constructed as

$$CI_{1-\alpha} = \left[\alpha/2 * B \text{ sample}, (1-\alpha/2) * B \text{ sample}\right]$$

- Percentile method does not rely on normal approximation, and behaves better with small n.

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- We will return to both in future weeks.

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- Central Limit Theorem!
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Going Deeper:

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Next week: hypothesis testing and regression!