Week 6: Linear Regression with Two Regressors

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Princeton

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller and Erin Hartman.

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- Next Week
  - multiple regression
- Long Run
  - probability → inference → regression → causal inference

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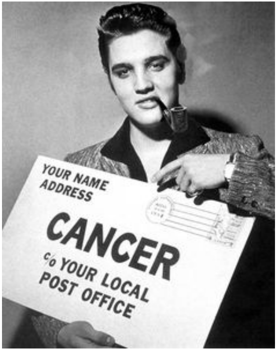
Example 1: Cigarette Smokers and Pipe Smokers



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- What about the conditional relationship within departments?

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- Marginal relationships (admissions and gender) $\neq$ conditional relationship given third variable (department)

Sex Bias in Graduate Admissions: Data from Berkeley

Measuring bias is harder than is usually assumed, and the evidence is sometimes contrary to expectation.

P. J. Bickel, E. A. Hammel, J. W. O'Connell

Determining whether discrimination because of sex or ethnic identity is being practiced against persons seeking passage from one social status or locus to another is an important problem in our society today. It is legally important and morally important. It is also often quite difficult. This article is an exploration of some of the issues of measurement and assessment involved in one example of the general problem, by means of which we hope to shed some light on the difficulties. We deceision to admit or to deny admission. The question we wish to pursue is whether the decision to admit or to deny was influenced by the sex of the applicant. We cannot know with any certainty the influences on the evaluators in the Graduate Admissions Office, or on the faculty reviewing committees, or on any other administrative personnel participating in the chain of actions that led to a decision on an individual application. We can, however, say that if the admissions decision and the sex by using a familiar statistic, chi-square. As already noted, we are aware of the pitfalls ahead in this naive approach, but we intend to stumble into every one of them for didactic reasons.

We must first make clear two assumptions that underlie consideration of the data in this contingency table approach. Assumption 1 is that in any given discipline male and female applicants do not differ in respect of their intelligence, skill, qualifications, promise, or other attribute deemed legitimately pertinent to their acceptance as students. It is precisely this assumption that makes the study of "sex bias" meaningful, for if we did not hold it any differences in acceptance of applicants by sex could be attributed to differences in their qualifications, promise as scholars, and so on. Theoretically one could test the assumption, for example, by examining presumably unbiased estimators of academic qualification such as Graduate Record Examination scores, undergraduate grade point averages, and so on. There are, however, enormous practical difficulties in this. We therefore predicate our discussion on the validity of assumption 1.

Bickel, Peter J., Eugene A. Hammel, and J. William O'Connell. "Sex bias in graduate admissions: Data from Berkeley." *Science* 187, no. 4175 (1975): 398-404.

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- This general pattern repeats in many debates, often because of the limits of data collection.

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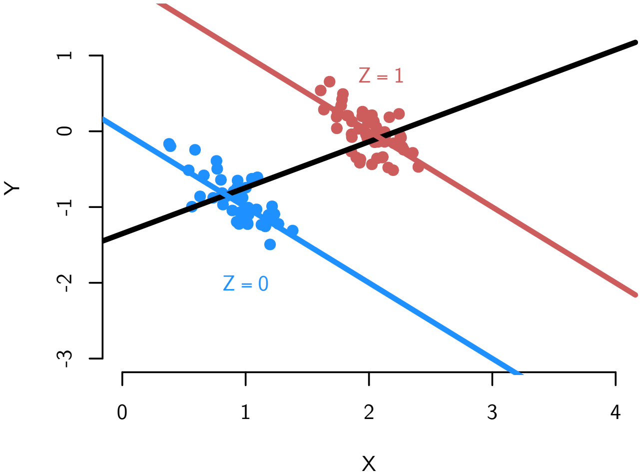

Core idea: a relationship in one direction between Y_i and X_i but the opposite relationship within strata defined by Z_i.

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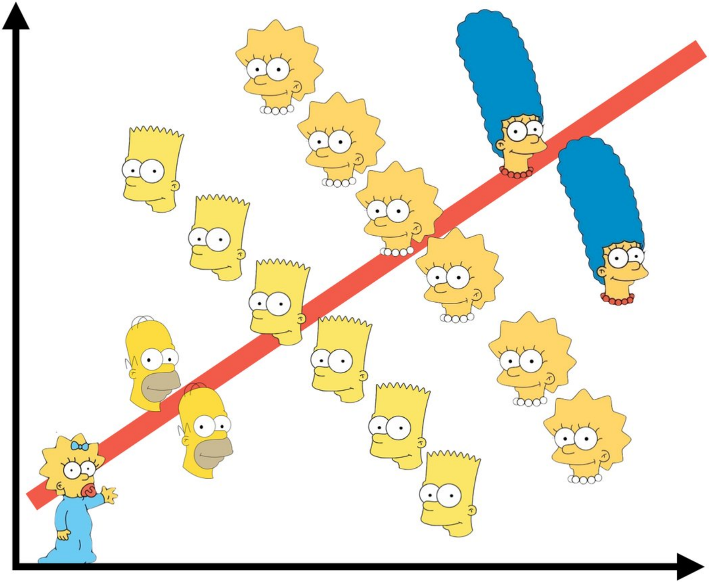

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- Conditioning is just a way of looking at subgroups—we will see later that this plays a key role in making causal inferences but it requires careful assumptions.

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Instance of a more general problem called the ecological inference fallacy.

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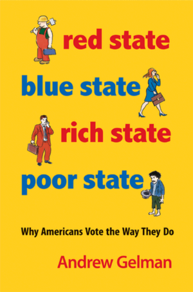
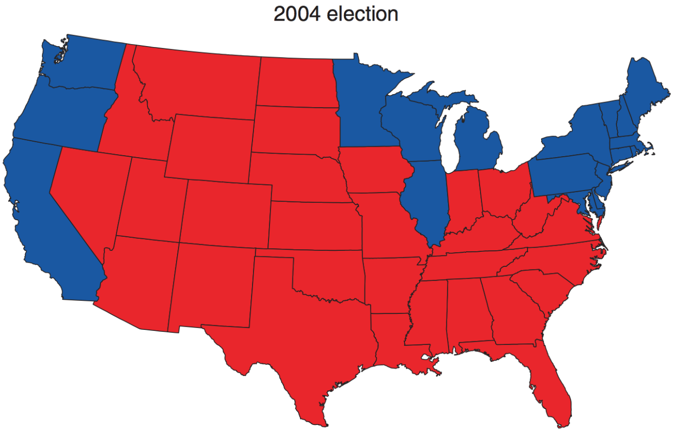
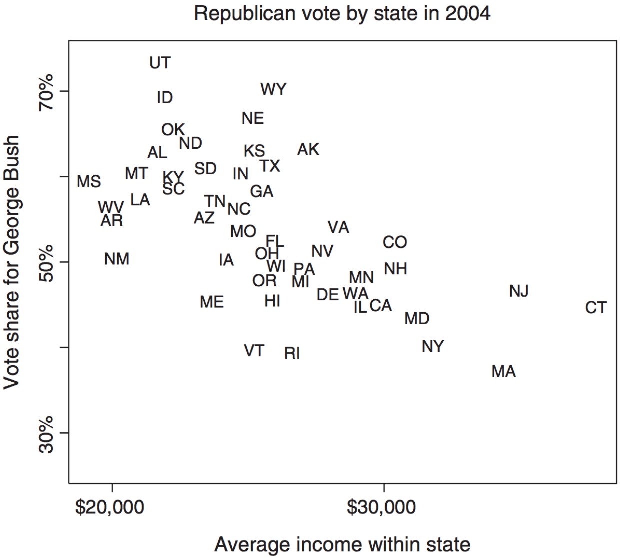


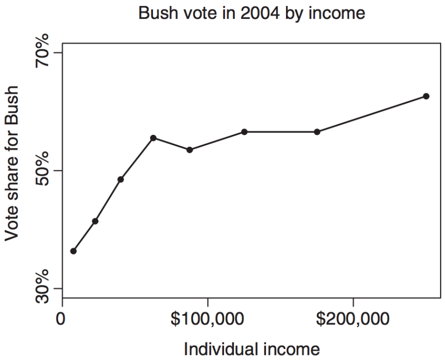
Red and Blue States

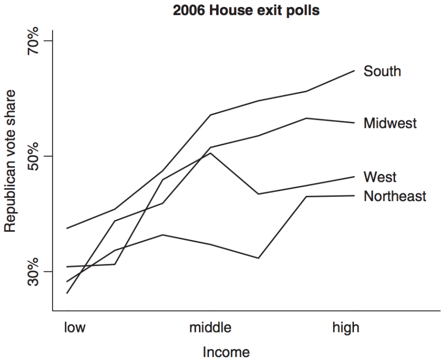


Rich States are More Democratic

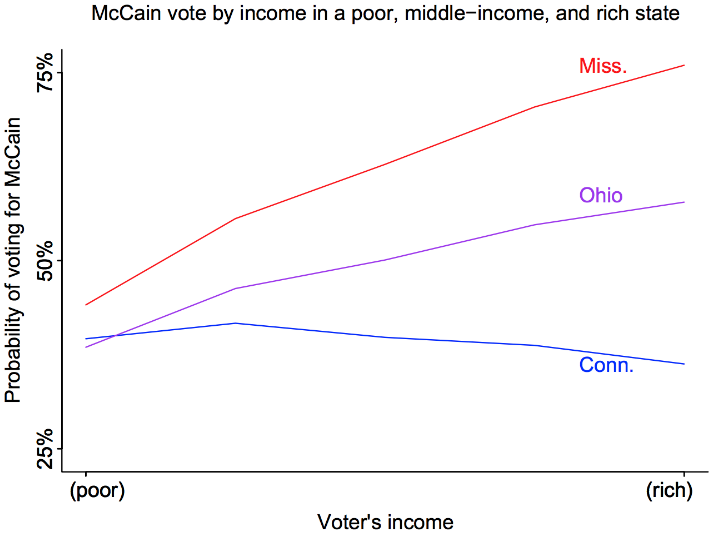


But Rich People are More Republican

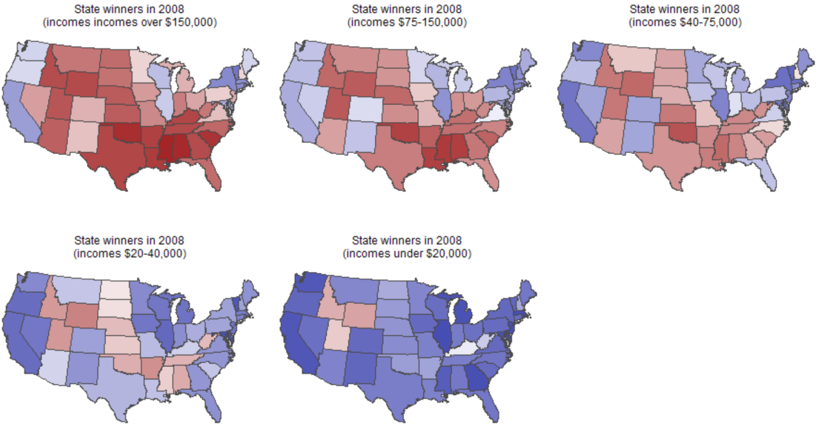




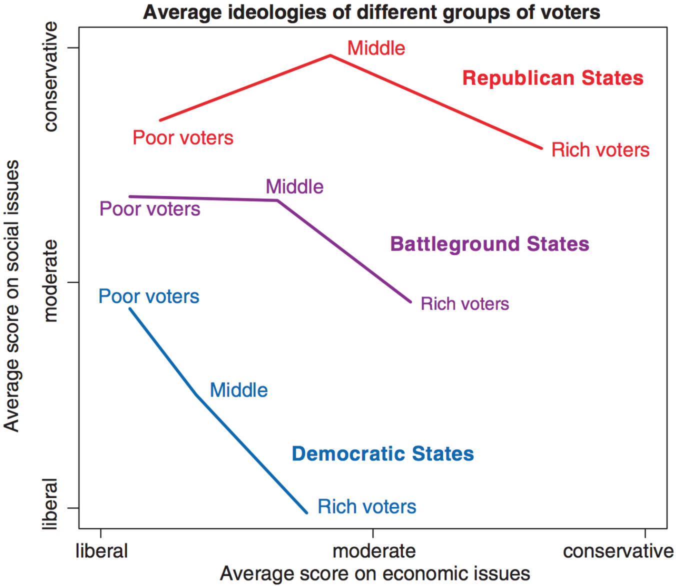
Paradox Resolved



If Only Rich People Voted, it Would Be a Landslide



A Possible Explanation



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Next Time: How to Add a Variable

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- New goal: estimate the relationship of two variables, Y_i and X_i, conditional on a third variable, Z_i:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + u_i$$

Basic Idea of Two Variable Regressions

- Old goal: estimate the mean of Y as a function of one independent variable, X:

$$E[Y_i|X_i]$$

- We modeled the CEF/regression function with a line:

$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

- New goal: estimate the relationship of two variables, Y_i and X_i, conditional on a third variable, Z_i:

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 Z_i + u_i$$

- β's are the population parameters we want to estimate

Regression with Two Explanatory Variables

Example: data from Fish (2002) "Islam and Authoritarianism." *World Politics.* 55: 4-37. Data from 157 countries.

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- With two predictors we ask questions like: Does income (X_1) predict or explain the level of democracy (Y), once we "control" for ethnic heterogeneity or British colonial heritage (X_2)?
- The rest of this lecture is designed to explain what is meant by "controlling for another variable" with linear regression.

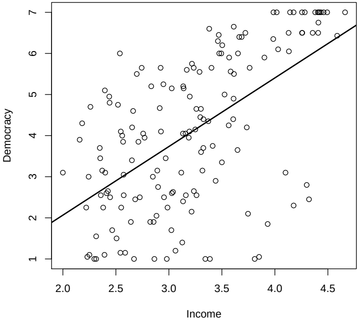
Simple Regression of Democracy on Income

Simple Regression of Democracy on Income

- Let's look at the bivariate regression of Democracy on Income:

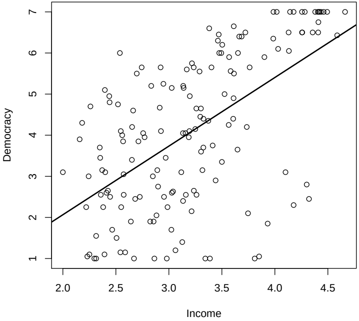
$$\widehat{y_i} = \widehat{\beta}_0 + \widehat{\beta}_1 x_1$$

$$\widehat{Demo} = -1.26 + 1.6\, Log(GDP)$$



Simple Regression of Democracy on Income

- But we can use more information in our prediction equation.



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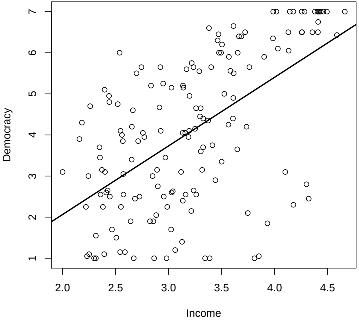

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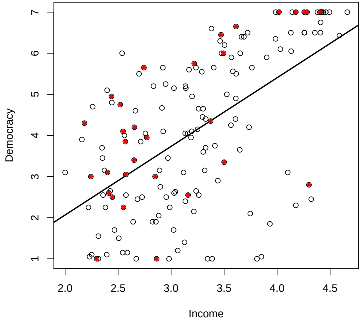

Simple Regression of Democracy on Income

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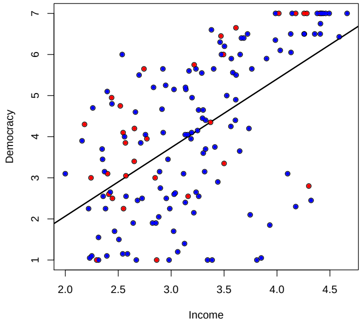
- For example, some countries were originally British colonies and others were not:
  - Former British colonies tend to have higher levels of democracy
  - Non-colony countries tend to have lower levels of democracy



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How do we do this? We can generalize the prediction equation:

$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_{1i} + \widehat{\beta}_2 x_{2i}$$

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In words:

$$\widehat{Democracy} = \widehat{\beta}_0 + \widehat{\beta}_1 \, Log(GDP) + \widehat{\beta}_2 \, Colony$$

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What does this mean?

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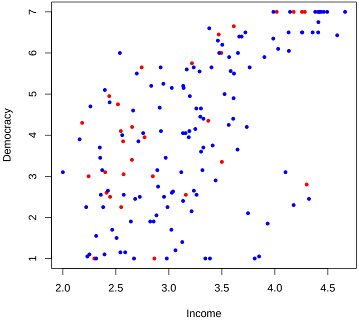
What does this mean? We are fitting two lines with the same slope but different intercepts.

Regression of Democracy on Income

From R, we obtain estimates $\widehat{\beta}_0$, $\widehat{\beta}_1$, $\widehat{\beta}_2$:

```
Coefficients:
             Estimate
(Intercept)   -1.5060
GDP90LGN       1.7059
BRITCOL        0.5881
```



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$$\widehat{y} = \widehat{\beta}_0 + \widehat{\beta}_1 x_1$$
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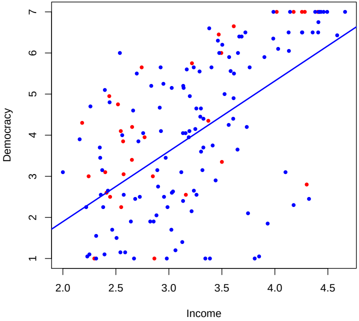

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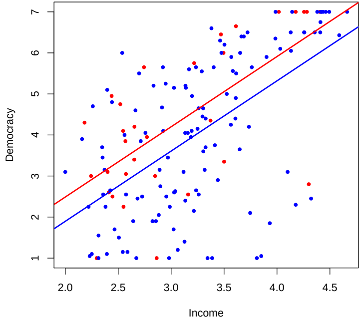
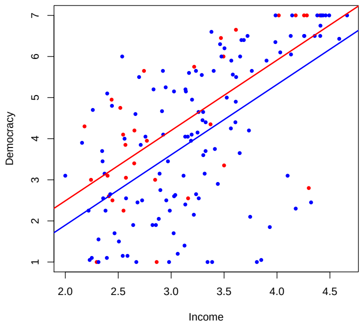


Regression of Democracy on Income

Our prediction equation is:
$\widehat{y} = -1.5 + 1.7\, x_1 + .58\, x_2$

Where do these quantities appear on the graph?



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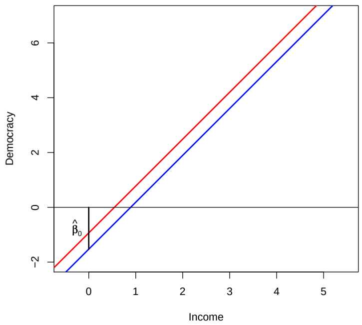

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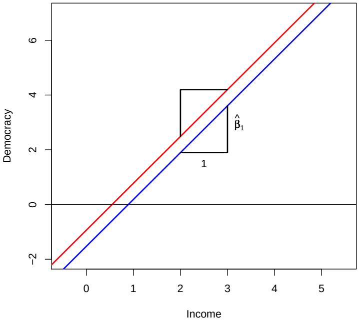

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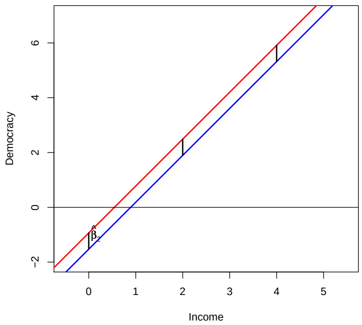
- $\widehat{\beta}_2 = .58$ is the vertical distance between the two lines for Ex-British colonies and non-colonies respectively



Example Interpretation of the Coefficients

- Let's review what we've seen so far:

	Intercept for X_1	Slope for X_1
Non-Colony ($X_2 = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
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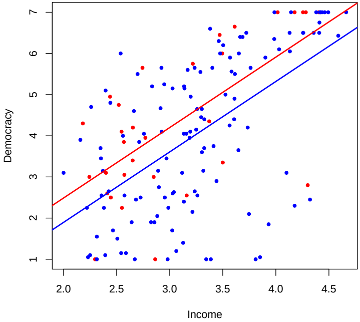
  - $\widehat{\beta}_2$: former british colonies are predicted to have a 0.5881 higher average democracy score than non-british colonies with the same level of income.

Fitting a regression plane

- We have considered an example of multiple regression with one continuous explanatory variable and one binary explanatory variable.



Fitting a regression plane

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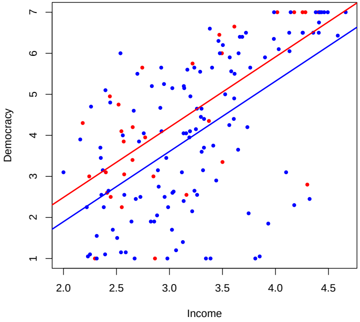
- This is easy to represent graphically in two dimensions because we can use colors to distinguish the two groups in the data.



Regression of Democracy on Income

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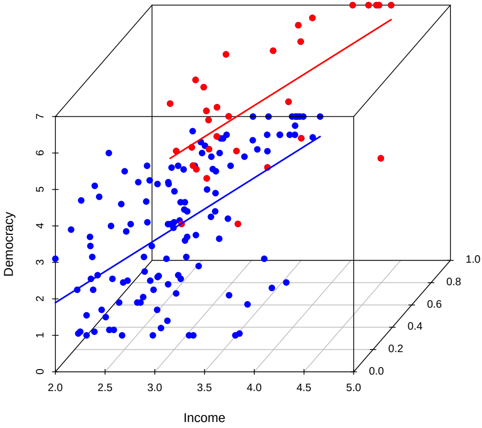

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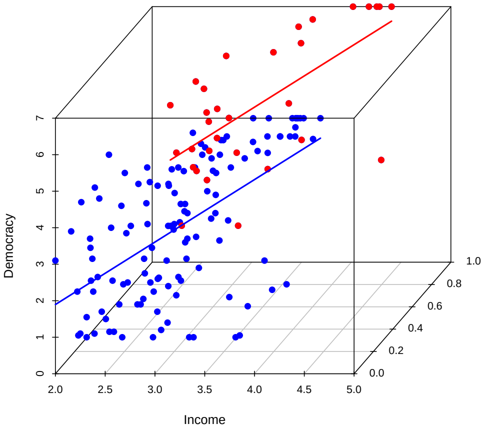

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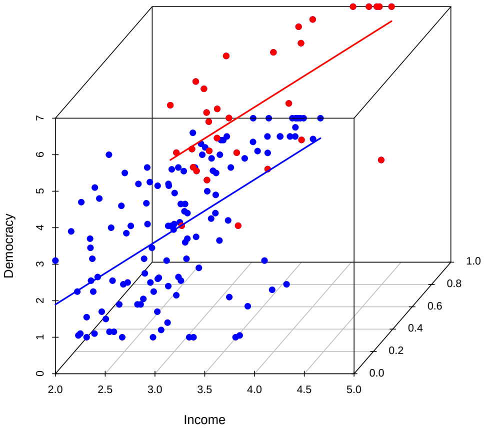
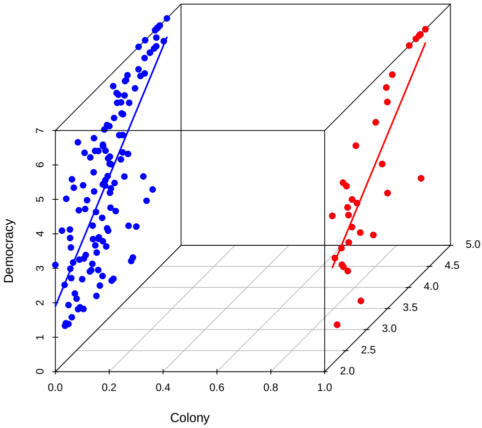
- In this view, we are looking at the data from the Income side; the two regression lines are drawn in the appropriate locations.



Regression of Democracy on Income

- We can also look at the 3D scatterplot from the British colony side.



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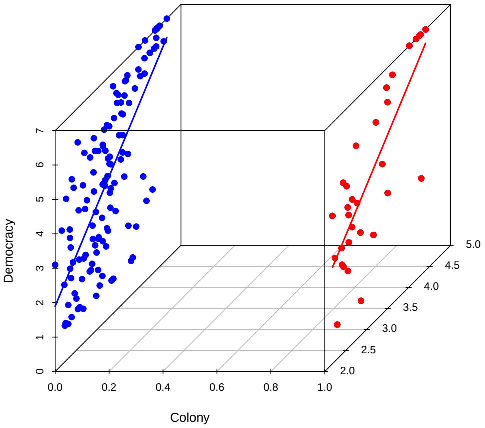

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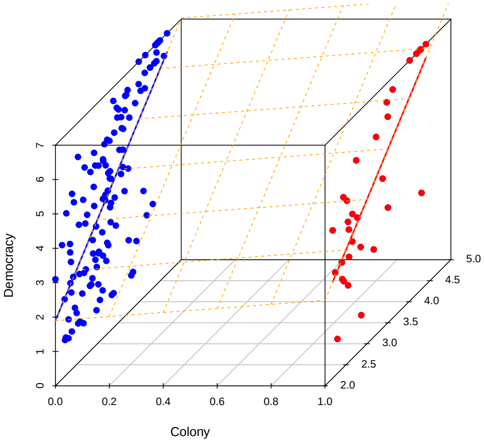
- We can also look at the 3D scatterplot from the British colony side.
- While the British colonial status variable is either 0 or 1, there is nothing in the prediction equation that requires this to be the case.
- In fact, the prediction equation defines a regression plane that connects the lines when $x_2 = 0$ and $x_2 = 1$.



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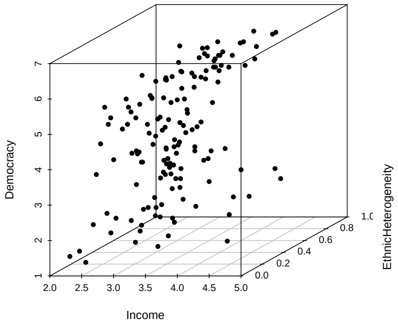
- Since we fit a regression plane to the data whenever we have two explanatory variables, it is easy to move to a case with two continuous explanatory variables.
- For example, we might want to use:
  - X_1 Income and X_2 Ethnic Heterogeneity
  - Y Democracy

$$\widehat{\text{Democracy}} = \hat{\beta}_0 + \hat{\beta}_1 \text{Income} + \hat{\beta}_2 \text{Ethnic Heterogeneity}$$

Regression of Democracy on Income

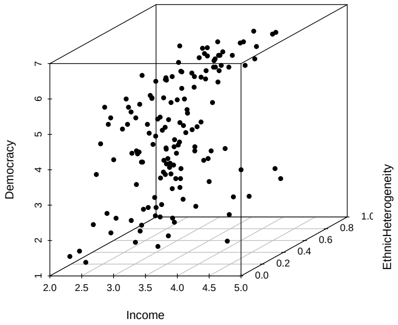
- We can plot the points in a 3D scatterplot.



Regression of Democracy on Income

- We can plot the points in a 3D scatterplot.
- R returns:
  - $\widehat{\beta}_0 = -.71$
  - $\widehat{\beta}_1 = 1.6$ for Income
  - $\widehat{\beta}_2 = -.6$ for Ethnic Heterogeneity

  How does this look graphically?



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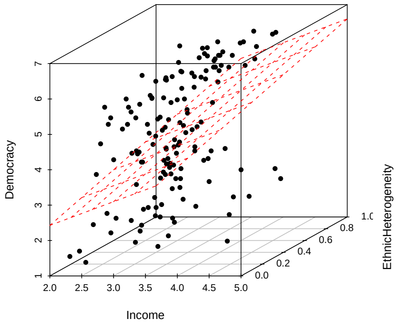

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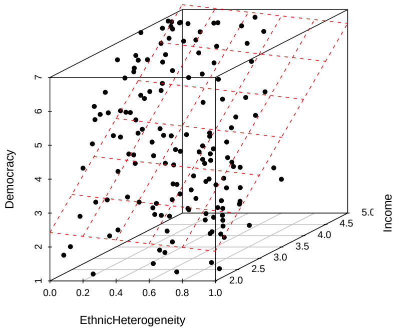

Interpreting a Continuous Covariate

- The coefficient estimates have a similar interpretation in this case as they did in the Income-British Colony example.

Interpreting a Continuous Covariate

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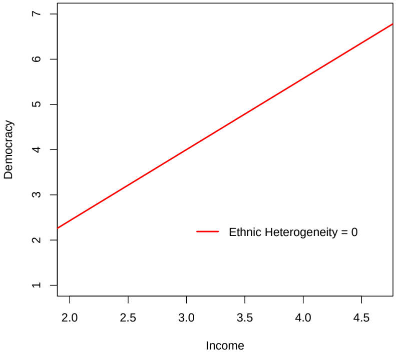

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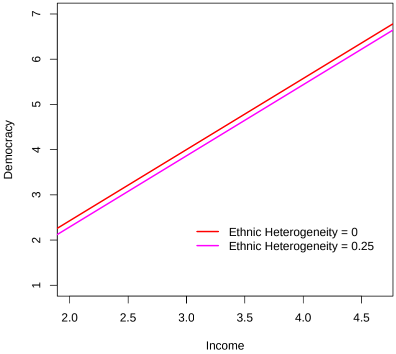

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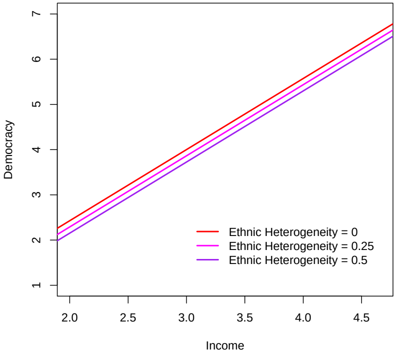

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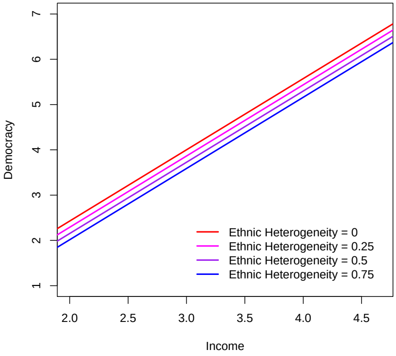

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  Predicted difference is thus: 1.8 or $(3.5 - 2.5)\widehat{\beta}_1 + (.06 - .5)\widehat{\beta}_2$

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- NB: if you want to sound like a machine learning person you can call it a one-hot encoding.

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- For this we use a single dummy variable which is coded like:

$$D_i = \begin{cases} 1 & \text{if country } i \text{ has a Majoritarian Electoral System} \\ 0 & \text{if country } i \text{ has a Proportional Electoral System} \end{cases}$$

Dummy Variables for Multiple Categories

- More generally, let's say X measures which of m categories each unit i belongs to. E.g. the type of electoral system or region of country i is given by:
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$$D_m = 1 - (D_1 + \cdots + D_{m-1})$$

- The omitted category is our baseline case (also called a reference category) against which we compare the conditional means of Y for the other $m - 1$ categories.

Example: Regions of the World

- Consider the case of our "polytomous" variable world region with $m = 5$:

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Category	D_1	D_2	D_3	D_4
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Our regression equation is:

$$Y = \beta_0 + \beta_1 D_1 + \beta_2 D_2 + \beta_3 D_3 + \beta_4 D_4 + u$$

Example: GDP per capita on Regions

R Code

```
> summary(lm(REALGDPCAP ~ Asia + Africa + LatAmerica + Oecd, data = D))

~~~~~~


Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   4452.7      783.4   5.684 2.07e-07 ***
Asia           148.9     1149.8   0.129   0.8973
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---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,       Adjusted R-squared: 0.6951
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What does β_0 mean?

$$\beta_0 = E[GDP|D_j = 0 \text{ for all } j] = E[GDP|Transition]$$

So the mean for the baseline category shows up as the intercept.

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$$\beta_{Africa} = E[GDP|Africa] - E[GDP|Transition]$$

The difference in means between the baseline and that category.

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Do Latin America economies have higher or lower average GDP than Asian economies?

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$$\beta_{LatAmerica} - \beta_{Asia} = E[GDP|LatAmerica] - E[GDP|Asia] = -420$$

Dealing with a Categorical Variable in R

- In fact, R automatically expands an m-category variable into an $m-1$ dummy variables:

R Code

```
> head(D$Region)
[1] LatAmerica Oecd       Oecd       LatAmerica Asia       LatAmerica
Levels: Africa Asia LatAmerica Oecd Transition

> summary(lm(REALGDPCAP ~ Region, data = D))

~~~~~
Coefficients:
                 Estimate Std. Error t value Pr(>|t|)
(Intercept)        1899.9      914.9   2.077   0.0410 *
RegionAsia         2701.7     1243.0   2.173   0.0327 *
RegionLatAmerica   2281.5     1112.3   2.051   0.0435 *
RegionOecd        12224.2     1112.3  10.990   <2e-16 ***
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Dealing with a Categorical Variable in R

- You can change the baseline category by the `relevel()` function:

R Code

```
> D$Region <- relevel(D$Region, ref="Transition")
> summary(lm(REALGDPCAP ~ Region, data = D))

~~~~~

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- In this setting, linearity holds by construction because we are estimating a single mean for every combination of X_i variables.

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- Two binary variables, X_{1i} for marriage status and X_{2i} for having children.

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$$E[Y_i|X_{1i} = 0, X_{2i} = 0]$$
$$E[Y_i|X_{1i} = 1, X_{2i} = 0]$$
$$E[Y_i|X_{1i} = 0, X_{2i} = 1]$$
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$$E[Y_i|X_{1i} = 0, X_{2i} = 0] = \alpha$$
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- We can write the CEF as follows:

$$E[Y_i|X_{1i}, X_{2i}] = \alpha + \beta X_{1i} + \gamma X_{2i} + \delta(X_{1i}X_{2i})$$

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E[Y_i|X_{1i} = 0, X_{2i} = 0] &= \alpha \\
E[Y_i|X_{1i} = 1, X_{2i} = 0] &= \alpha + \beta \\
E[Y_i|X_{1i} = 0, X_{2i} = 1] &= \alpha + \gamma \\
E[Y_i|X_{1i} = 1, X_{2i} = 1] &= \alpha + \beta + \gamma + \delta
\end{aligned}
$$

- We can write the CEF as follows:

$$E[Y_i|X_{1i}, X_{2i}] = \alpha + \beta X_{1i} + \gamma X_{2i} + \delta(X_{1i}X_{2i})$$

Saturated model example

$$E[Y_i|X_{1i}, X_{2i}] = \alpha + \beta X_{1i} + \gamma X_{2i} + \delta(X_{1i}X_{2i})$$

- Basically, each value of the CEF is being estimated separately.

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- Basically, each value of the CEF is being estimated separately.
  - $\rightsquigarrow$ within-strata estimation.
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- Requires a set of dummies for each categorical variable plus all interactions.
- i.e. a series of dummies for each unique combination of X_i.

Saturated model example

- Ebonya Washington (AER) data from *AER* paper "Female socialization: how daughters affect their legislator fathers"

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```r
girls <- foreign::read.dta("girls.dta")
head(girls[, c("name", "totchi", "aauw")])
```

```
##                     name totchi aauw
## 1   ABERCROMBIE, NEIL       0  100
## 2   ACKERMAN, GARY L.       3   88
## 3 ADERHOLT, ROBERT B.       0    0
## 4     ALLEN, THOMAS H.      2  100
## 5  ANDREWS, ROBERT E.       2  100
## 6          ARCHER, W.R.     7    0
```

Linear model

```r
summary(lm(aauw ~ totchi, data = girls))
```

```
##
## Coefficients:
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)    61.31       1.81   33.81   <2e-16 ***
## totchi         -5.33       0.62   -8.59   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 42 on 1733 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.0408, Adjusted R-squared:  0.0403
## F-statistic: 73.8 on 1 and 1733 DF,  p-value: <2e-16
```

Saturated model

```r
summary(lm(aauw ~ as.factor(totchi), data = girls))
```

```
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## (Intercept)             56.41       2.76   20.42  < 2e-16 ***
## as.factor(totchi)1       5.45       4.11    1.33   0.1851
## as.factor(totchi)2      -3.80       3.27   -1.16   0.2454
## as.factor(totchi)3     -13.65       3.45   -3.95  8.1e-05 ***
## as.factor(totchi)4     -19.31       4.01   -4.82  1.6e-06 ***
## as.factor(totchi)5     -15.46       4.85   -3.19   0.0015 **
## as.factor(totchi)6     -33.59      10.42   -3.22   0.0013 **
## as.factor(totchi)7     -17.13      11.41   -1.50   0.1336
## as.factor(totchi)8     -55.33      12.28   -4.51  7.0e-06 ***
## as.factor(totchi)9     -50.41      24.08   -2.09   0.0364 *
## as.factor(totchi)10    -53.41      20.90   -2.56   0.0107 *
## as.factor(totchi)12    -56.41      41.53   -1.36   0.1745
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 41 on 1723 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.0506, Adjusted R-squared:  0.0446
## F-statistic: 8.36 on 11 and 1723 DF,  p-value: 1.84e-14
```

Saturated model minus the constant

```r
summary(lm(aauw ~ as.factor(totchi) - 1, data = girls))
```

```
##
## Coefficients:
##                      Estimate Std. Error t value Pr(>|t|)
## as.factor(totchi)0     56.41       2.76   20.42   <2e-16 ***
## as.factor(totchi)1     61.86       3.05   20.31   <2e-16 ***
## as.factor(totchi)2     52.62       1.75   30.13   <2e-16 ***
## as.factor(totchi)3     42.76       2.07   20.62   <2e-16 ***
## as.factor(totchi)4     37.11       2.90   12.79   <2e-16 ***
## as.factor(totchi)5     40.95       3.99   10.27   <2e-16 ***
## as.factor(totchi)6     22.82      10.05    2.27   0.0233 *
## as.factor(totchi)7     39.29      11.07    3.55   0.0004 ***
## as.factor(totchi)8      1.08      11.96    0.09   0.9278
## as.factor(totchi)9      6.00      23.92    0.25   0.8020
## as.factor(totchi)10     3.00      20.72    0.14   0.8849
## as.factor(totchi)12     0.00      41.43    0.00   1.0000
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 41 on 1723 degrees of freedom
##   (5 observations deleted due to missingness)
## Multiple R-squared:  0.587,  Adjusted R-squared:  0.584
## F-statistic:  204 on 12 and 1723 DF,  p-value: <2e-16
```

Compare to within-strata means

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```
c1 <- coef(lm(aauw ~ as.factor(totchi) - 1, data = girls))
c2 <- with(girls, tapply(aauw, totchi, mean, na.rm = TRUE))
rbind(c1, c2)
```

```
##     0  1  2  3  4  5  6  7   8 9 10 12
## c1 56 62 53 43 37 41 23 39 1.1 6  3  0
## c2 56 62 53 43 37 41 23 39 1.1 6  3  0
```

We Covered

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- How to add a binary variable.

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Next Time: Estimation and Inference!

Where We've Been and Where We're Going...

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- Last Week
  - mechanics of OLS with one variable
  - properties of OLS
- This Week
  - adding a second variable
  - new mechanics
  - omitted variable bias
  - multicollinearity
  - interactions
- Next Week
  - multiple regression
- Long Run
  - probability → inference → regression → causal inference

Fitted values and Residuals

- Where do we get our hats?

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$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i$$

- Residuals for $i = 1, \dots, n$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i$$

Least Squares is Still Least Squares

- How do we estimate $\widehat{\beta}_0$, $\widehat{\beta}_1$, and $\widehat{\beta}_2$?

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- Minimize the sum of the squared residuals,

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Plan is conceptually the same as before

1. Take the partial derivatives of S with respect to b_0, b_1, b_2.
2. Set each of the partial derivatives to 0 to obtain the first order conditions.
3. Substitute $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ for b_0, b_1, b_2 and solve for $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ to obtain the OLS estimator.

Take partial derivatives

$(\widehat{\beta}_0, \widehat{\beta}_1, \widehat{\beta}_2) = \arg\min_{b_0,b_1,b_2} \sum_{i=1}^{n}(Y_i - b_0 - b_1X_i - b_2Z_i)^2$
After some calculus and algebra we can show that:

$$\frac{\partial S}{\partial b_0} = \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i)$$

$$\frac{\partial S}{\partial b_1} = \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i)$$

$$\frac{\partial S}{\partial b_2} = \sum_{i=1}^{n} z_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i)$$

First Order Conditions

Setting the partial derivatives equal to zero leads to a system of 3 linear equations in 3 unknowns: $\hat{\beta}_0, \hat{\beta}_1$ and $\hat{\beta}_2$

$$
\begin{aligned}
\frac{\partial S}{\partial b_0} &= \sum_{i=1}^{n}(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
\frac{\partial S}{\partial b_1} &= \sum_{i=1}^{n} x_i(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i - \hat{\beta}_2 z_i) = 0 \\
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When will this linear system have a unique solution?

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When will this linear system have a unique solution?

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  - x is not a linear function of z (or vice versa)

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  - x is not a linear function of z (or vice versa)
- Typically called no perfect collinearity

The OLS Estimator

After lots of algebra, the OLS estimator for $(\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2)$ can be written as

$$\begin{aligned}
\hat{\beta}_0 &= \bar{y} - \hat{\beta}_1 \bar{x} - \hat{\beta}_2 \bar{z} \\
\hat{\beta}_1 &= \frac{Cov(x,y)Var(z) - Cov(z,y)Cov(x,z)}{Var(x)Var(z) - Cov(x,z)^2} \\
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This requirement fails if:

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This requirement fails if:

1. If x or z is a constant ($\Rightarrow Var(x)Var(z) = Cov(x,z) = 0$)
2. One explanatory variable is an exact linear function of another ($\Rightarrow \; Cor(x,z) = 1 \; \Rightarrow \; Var(x)Var(z) = Cov(x,z)^2$)

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3. No perfect collinearity
4. Zero conditional mean error
$$E[u_i | X_i, Z_i] = 0$$

New assumption

Assumption 3: No perfect collinearity

(1) No explanatory variable is constant in the sample and (2) there are no exactly linear relationships among the explanatory variables.

- Two components

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- Notice how this is linear (equation of a line) and there is no error, so it is deterministic.
- What's the correlation between Z_i and X_i? 1!

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- These two variables are perfectly collinear.

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- No! Because while Z_i is a deterministic function of X_i, it is not a linear function of X_i.

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```
##
## Coefficients: (1 not defined because of singularities)
##             Estimate Std. Error t value Pr(>|t|)
## (Intercept)  8.71638    0.08991  96.941  < 2e-16 ***
## africa      -1.36119    0.16306  -8.348 4.87e-14 ***
## nonafrica         NA         NA      NA       NA
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9125 on 146 degrees of freedom
##   (15 observations deleted due to missingness)
## Multiple R-squared:  0.3231, Adjusted R-squared:  0.3184
## F-statistic: 69.68 on 1 and 146 DF,  p-value: 4.87e-14
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```
## (Intercept)    meantemp  meantemp.f
##  10.8454999  -0.1206948          NA
```

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- Under assumptions 1-6, we have the following small change to our small-n sampling distribution:

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- $\rightsquigarrow$ small adjustments to the critical values and the t-values for our hypothesis tests and confidence intervals.

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- Estimate of $\widehat{\beta}_1$ will be the same as running:
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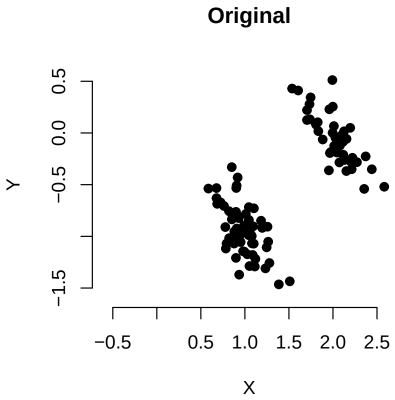
A Visual of Partialling Out



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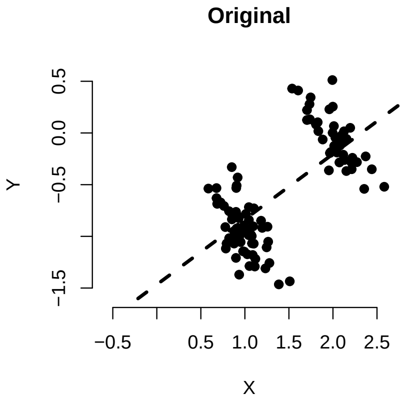

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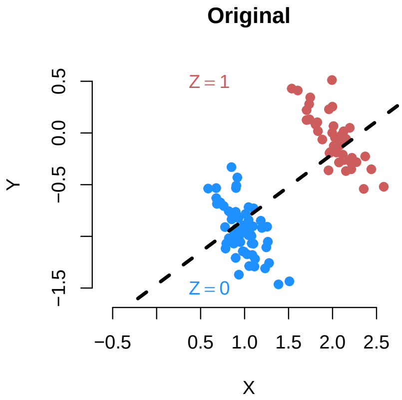

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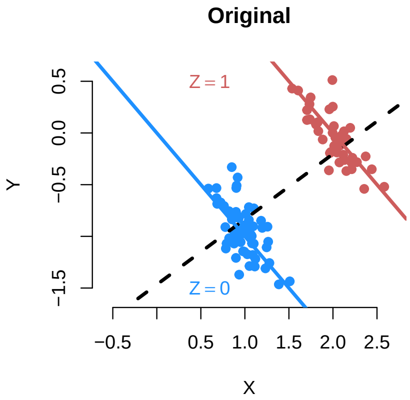

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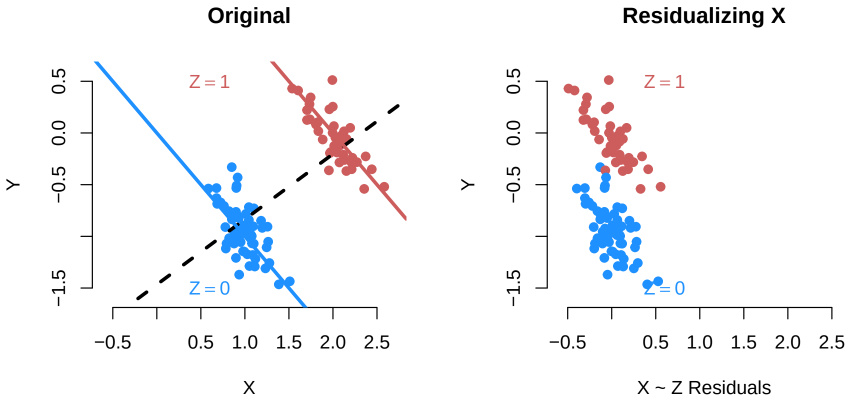

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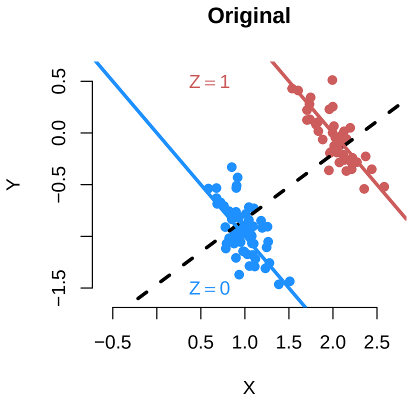

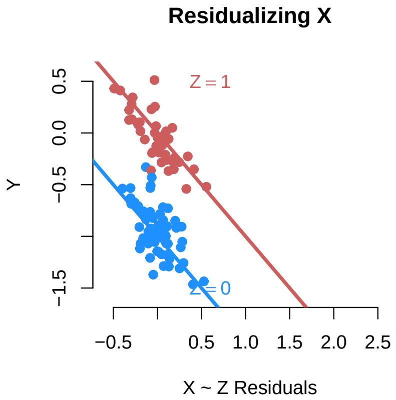

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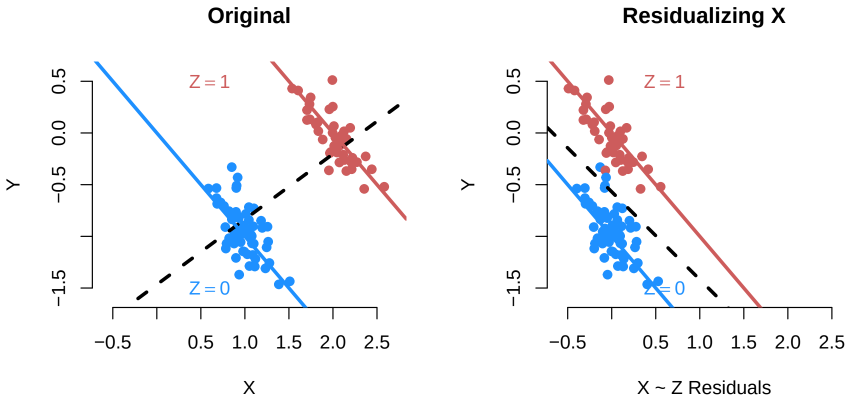

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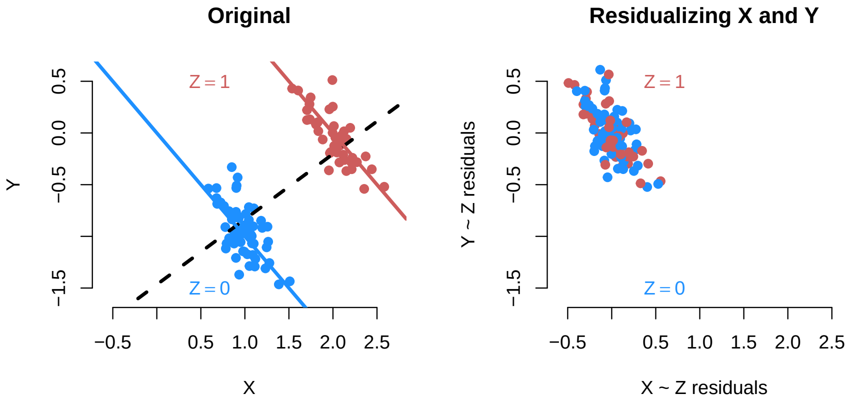

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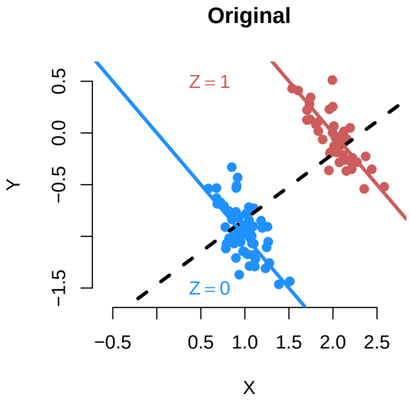

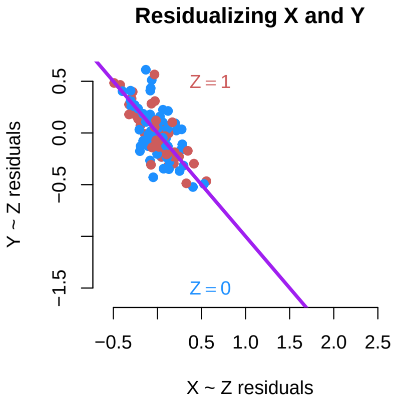

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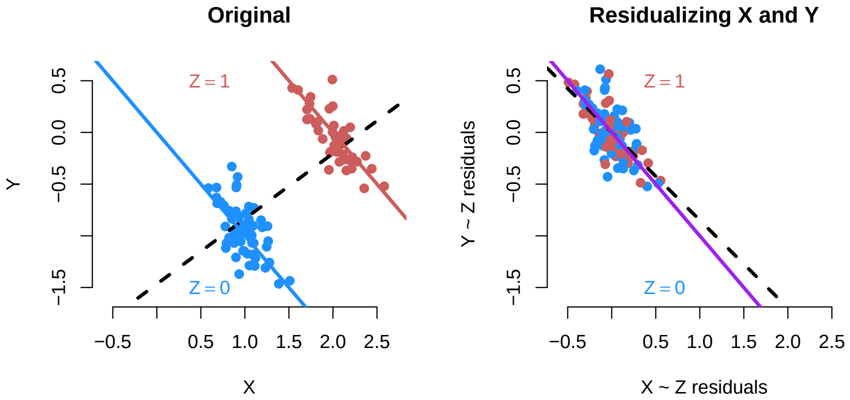

Origin of the Partial Out Recipe

Assume $Y = \beta_0 + \beta_1 X + \beta_2 Z + u$. Another way to write the OLS estimator is:

$$\hat{\beta}_1 = \frac{\sum_i^n \hat{r}_{xz,i}\, y_i}{\sum_i^n \hat{r}_{xz,i}^2}$$

where $\hat{r}_{xz,i}$ are the residuals from the regression of X on Z:

$$X = \lambda + \delta Z + r_{xz}$$

In other words, both of these regressions yield identical estimates $\hat{\beta}_1$:

$$y = \hat{\gamma}_0 + \hat{\beta}_1 \hat{r}_{xz} \quad \text{and} \quad y = \hat{\beta}_0 + \hat{\beta}_1 x + \hat{\beta}_2 z$$

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- Can use same equation with k explanatory variables; $\hat{r}_{xz}$ will then come from a regression of X on all the other explanatory variables.

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- This set up will also be the basis of diagnostic plots that we will cover in a couple of weeks. It allows us to visualize the conditional relationship.
- Finally, it forms the foundation of a number of machine learning strategies including double machine learning by breaking down the regression problem.

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Next Time: Omitted Variables and Multicollinearity

Where We've Been and Where We're Going...

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- Last Week
  - mechanics of OLS with one variable
  - properties of OLS
- This Week
  - adding a second variable
  - new mechanics
  - omitted variable bias
  - multicollinearity
  - interactions
- Next Week
  - multiple regression
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Remember This?

- **Identification / Data Description**
  - Variation in X
- **Unbiasedness / Consistency**
  - Variation in X
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
- **Gauss-Markov (BLUE) / Asymptotic Inference (z and χ^2)**
  - Variation in X
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
  - Homoskedasticity
- **Classical LM (BUE) / Small-Sample Inference (t and F)**
  - Variation in X
  - Random Sampling
  - Linearity in Parameters
  - Zero Conditional Mean
  - Homoskedasticity
  - Normality of Errors

Unbiasedness revisited

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- OLS estimates from the misspecified model:

$$\widehat{Y}_i = \hat{\alpha}_0 + \hat{\alpha}_1 X_i$$

Omitted Variable Bias: Simple Case

True Population Model:

$$\text{Voted Republican} = \beta_0 + \beta_1 \text{Watch Fox News} + \beta_2 \text{Strong Republican} + u$$

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Underspecified Model that we use:

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Expected Behavior: $\hat{\alpha}_1$ is upward biased for β_1 since being a strong Republican is positively correlated with both watching Fox News and voting Republican. We have $E[\hat{\alpha}_1] > \beta_1$.

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True Population Model:

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Under-specified Model that we use:

$$\widehat{\text{Survival}} = \hat{\alpha}_0 + \hat{\alpha}_1 \text{Hospitalized}$$

Omitted Variable Bias: Simple Case

True Population Model:

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Expected Behavior: The negative coefficient $\hat{\alpha}_1$ is downward biased compared to the true β_1 so $E[\hat{\alpha}_1] < \beta_1$. Being hospitalized is negatively correlated with health, and health is positively correlated with survival.

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$\hat{\alpha}_1 = \hat{\beta}_1$ when $\tilde{\delta} = 0$ or $\hat{\beta}_2 = 0$.

Omitted Variable Bias: Simple Case

We take expectations to see what the bias will be:

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So

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- imbalance is how the expectation of the unobserved X_2 varies across levels of X_1.

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	$\text{cov}(X_1, X_2) > 0$	$\text{cov}(X_1, X_2) < 0$	$\text{cov}(X_1, X_2) = 0$
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- The Omitted Variable Bias formula provides the foundation for many forms of sensitivity analysis.
- In the more general case with more than two covariates the bias is more difficult to discern. It depends on all the pairwise correlations.

Sampling variance for simple linear regression

- Under simple linear regression, we found that the distribution of the slope was the following:

$$\text{var}(\widehat{\beta}_1) = \frac{\sigma_u^2}{\sum_{i=1}^{n}(X_i - \overline{X})^2}$$

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- Regression with an additional independent variable:

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- What happens with perfect collinearity? $R_1^2 = 1$ and the variances are infinite.

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- Given the symmetry, it will also increase $\text{var}(\widehat{\beta}_2)$ as well.

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- Low variation in an independent variable (here, $\widehat{r}_{xz,i}$) $\rightsquigarrow$ high SEs
- Basically, there is less residual variation left in X_i after "partialling out" the effect of Z_i

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In R, `vif()` in the `car` package.

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- Or maybe linear regression is not the right tool

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Next Time: Interactions

Where We've Been and Where We're Going...

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- Last Week
  - mechanics of OLS with one variable
  - properties of OLS
- This Week
  - adding a second variable
  - new mechanics
  - omitted variable bias
  - multicollinearity
  - interactions
- Next Week
  - multiple regression
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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  - two or more continuous variables
- Interactions often confuses researchers and mistakes in use and interpretation occur frequently (even in top journals)

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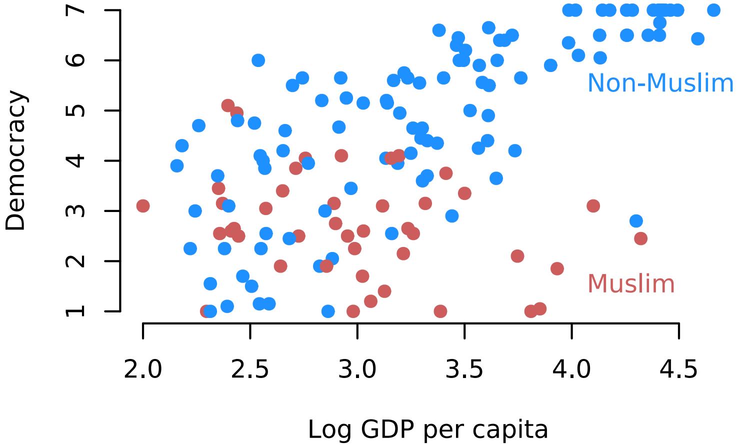
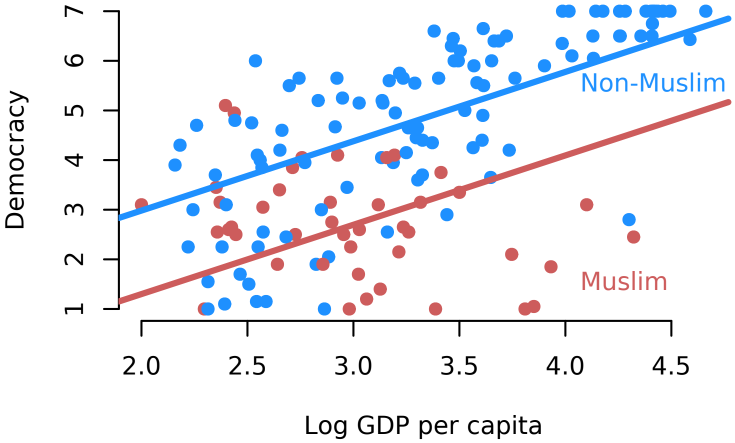
Let's See the Data



Fish argues that Muslim countries are less likely to be democratic no matter their economic development

Controlling for Religion Additively



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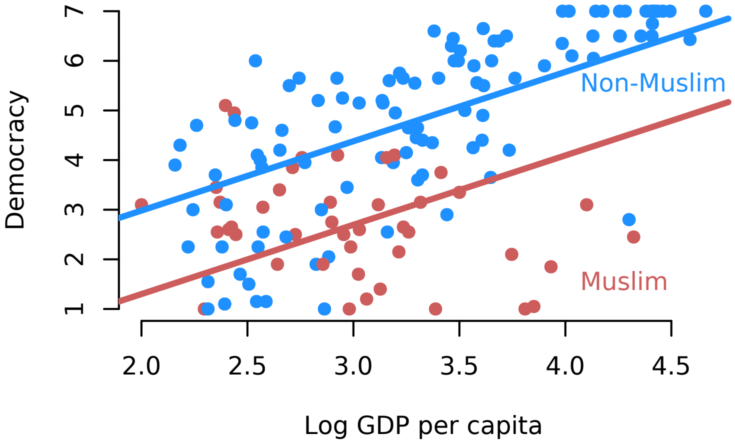

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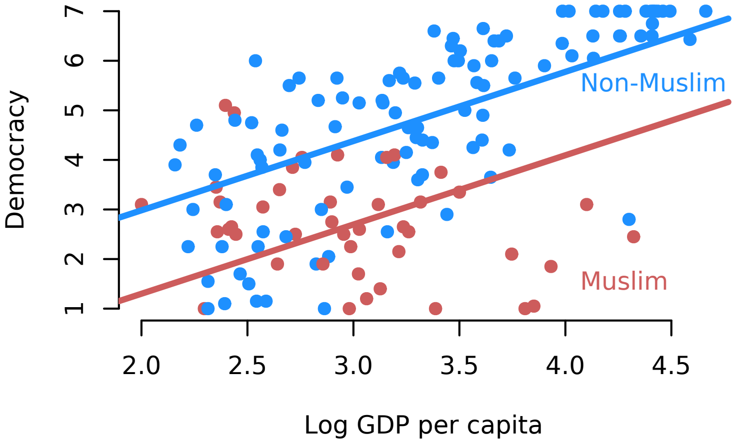

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Can we allow for different slopes for each group?

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- We can add another covariate to the baseline model that allows the effect of income to vary by Muslim status.
- This covariate is called an interaction term and it is the product of the two marginal variables of interest: $income_i \times muslim_i$
- Here is the model with the interaction term:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

Two Lines in One Regression

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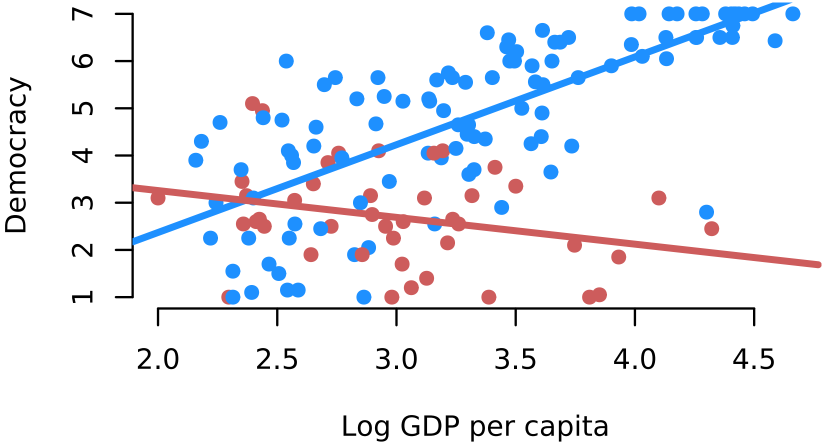
Example Interpretation of the Coefficients

	Intercept for X_i	Slope for X_i
Non-Muslim country ($Z_i = 0$)	$\widehat{\beta}_0$	$\widehat{\beta}_1$
Muslim country ($Z_i = 1$)	$\widehat{\beta}_0 + \widehat{\beta}_2$	$\widehat{\beta}_1 + \widehat{\beta}_3$



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Lower Order Terms

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- Imagine we omitted the lower order term for `muslim`:

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + 0 \times Z_i + \widehat{\beta}_3 X_i Z_i$$

Omitting Lower Order Terms

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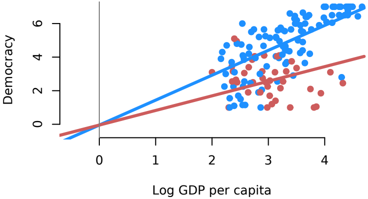

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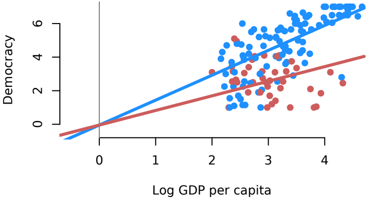

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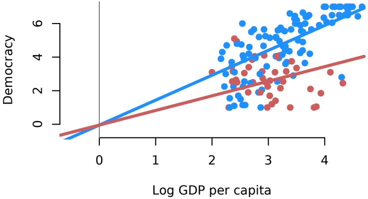

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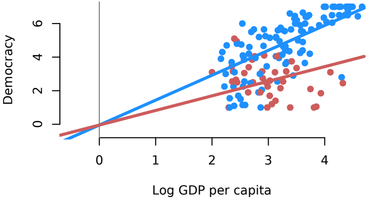

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- Very rarely justified, but for some reason, people keep doing it (as you will see in your problem set).

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- And include it in the regression:

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Interpretation

- With a continuous Z_i, we can have more than two values that it can take on:

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$Z_i = 5$	$\widehat{\beta}_0 + \widehat{\beta}_2 \times 5$	$\widehat{\beta}_1 + \widehat{\beta}_3 \times 5$

General Interpretation

$$\widehat{Y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 X_i + \widehat{\beta}_2 Z_i + \widehat{\beta}_3 X_i Z_i$$

- The coefficient $\widehat{\beta}_1$ measures how the predicted outcome varies in X_i when $Z_i = 0$.

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- The coefficient $\widehat{\beta}_3$ is the change in the effect of Z_i given a one-unit change in X_i:
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Interaction effects are particularly susceptible to model dependence. We are making two assumptions for the estimated effects to be meaningful:

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2. Common support (variation in X throughout the range of Z)

We will talk about checking these assumptions in a few weeks.

PA

How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice

Jens Hainmueller[1], Jonathan Mummolo[2] and Yiqing Xu[3]

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[2] *Assistant Professor of Politics and Public Affairs, Princeton University, Department of Politics, Woodrow Wilson School of Public and International Affairs, Princeton, NJ 08544, USA. Email: jmummolo@princeton.edu*
[3] *Assistant Professor of Political Science, University of California, San Diego, Department of Political Science, La Jolla, CA 92093, USA. Email: yiqingxu@ucsd.edu*

Abstract

Multiplicative interaction models are widely used in social science to examine whether the relationship between an outcome and an independent variable changes with a moderating variable. Current empirical practice tends to overlook two important problems. First, these models assume a linear interaction effect that changes at a constant rate with the moderator. Second, estimates of the conditional effects of the independent variable can be misleading if there is a lack of common support of the moderator. Replicating 46 interaction effects from 22 recent publications in five top political science journals, we find that these core assumptions often fail in practice, suggesting that a large portion of findings across all political science subfields based on interaction models are fragile and model dependent. We propose a checklist of simple diagnostics to assess the validity of these assumptions and offer flexible estimation strategies that allow for nonlinear interaction effects and safeguard against excessive extrapolation. These statistical routines are available in both R and STATA.

Keywords: misspecification, linear regression, local regression, interaction models, marginal effects

Example: Common Support

Chapman 2009 analysis
example and reanalysis from Hainmueller, Mummolo, Xu 2019

"The interaction term shows a strong negative and statically significant coefficient; suggesting that when UN authorization occurs and the interaction term is 'switched on' positive movement in the similarity score (towards more similar) reduces rallies. Rallies with UN authorization are only larger than average when the pivotal member is ideologically distant from the United States. This provides strong support for the informational rationale for IO legitimacy… Clearly, the effect of authorization on rallies decreases as similarity increases: foreign policy actions that receive authorization from a less conservative institution receive similar rallies to those that do not receive authorization from an IO."

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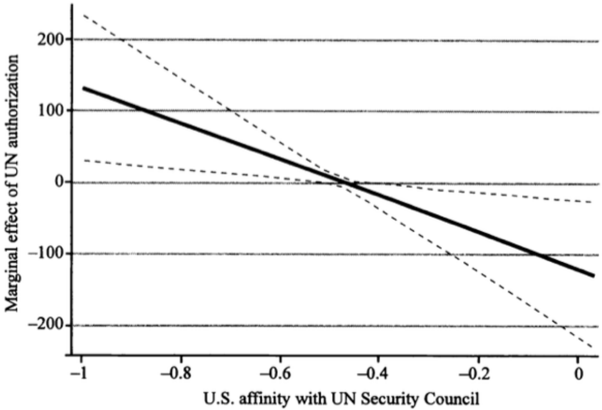

Note: Dashed lines give 95 percent confidence interval.

Example: Common Support

Chapman 2009 analysis

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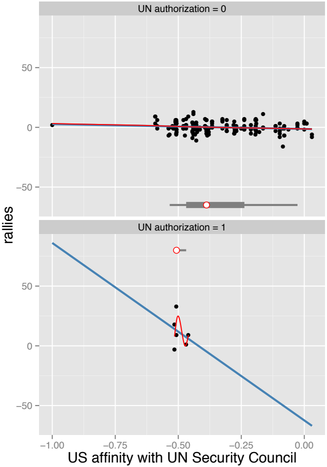

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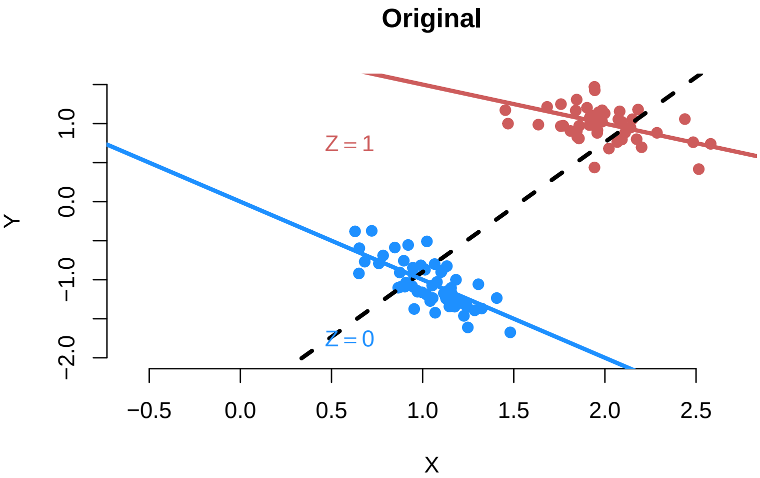

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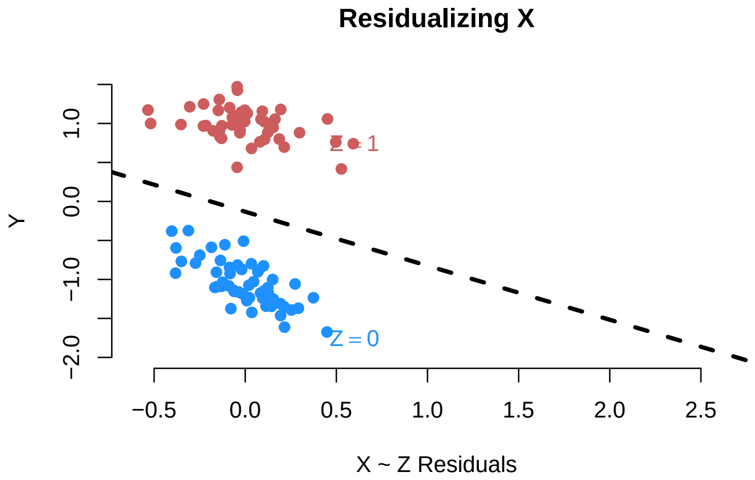

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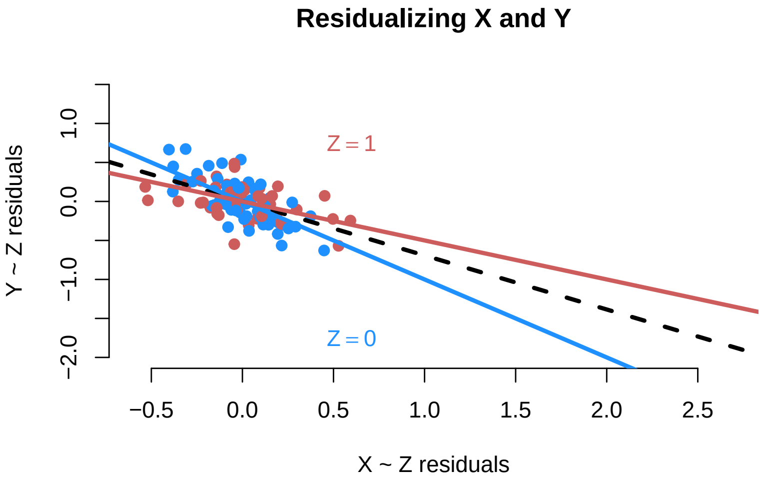

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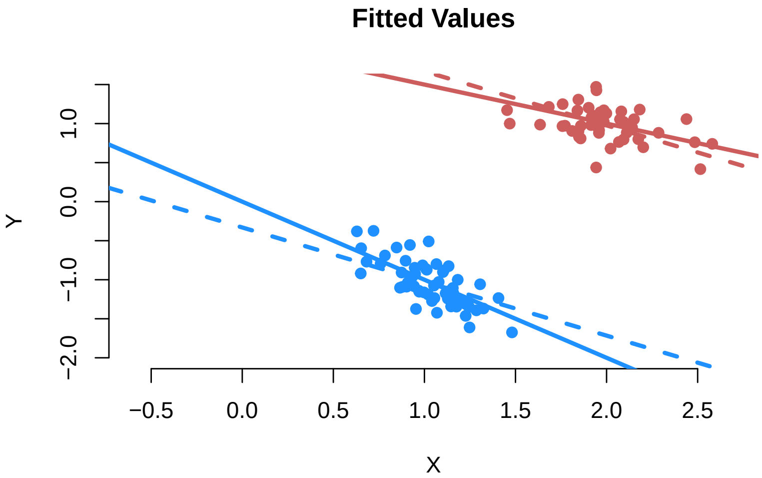

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Further Reading: Brambor, Clark, and Golder. 2006. Understanding Interaction Models: Improving Empirical Analyses. *Political Analysis*.

Hainmueller, Mummolo, Xu. 2019. How Much Should We Trust Estimates from Multiplicative Interaction Models? Simple Tools to Improve Empirical Practice. *Political Analysis*.

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$$Y = \beta_0 + (\beta_1 + \beta_2) X_1 + \beta_3 X_1 X_1 + u$$
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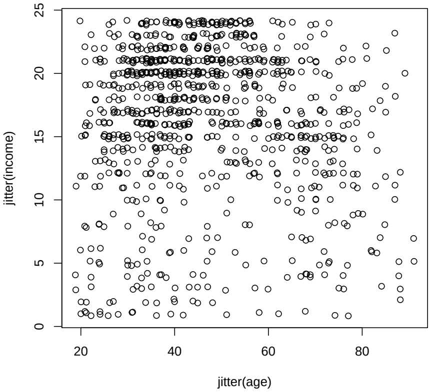
- A third order polynomial is given by: $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + \beta_3 X_1^3 + u$

Polynomial Example: Income and Age

- Let's look at data from the U.S. and examine the relationship between Y: income and X: age



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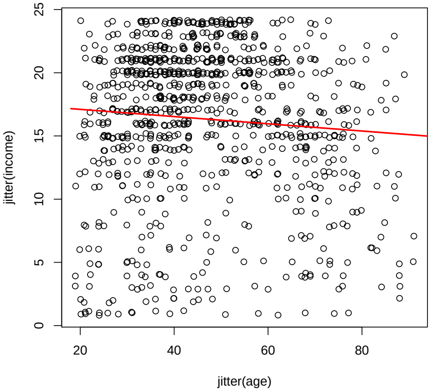

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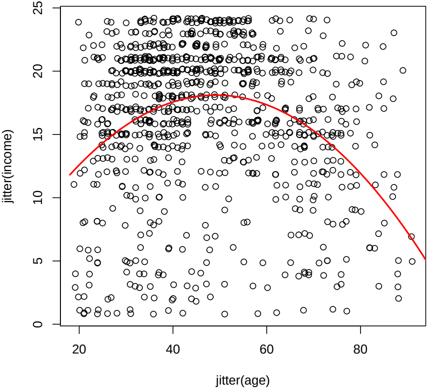
- A second order polynomial in age fits the data a lot better:
  $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$



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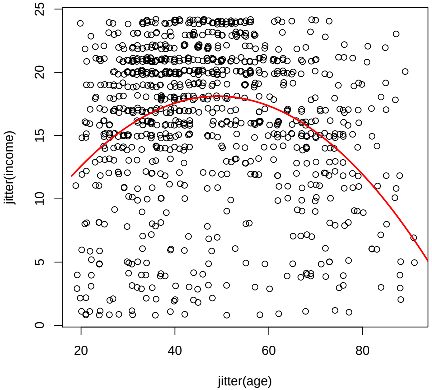

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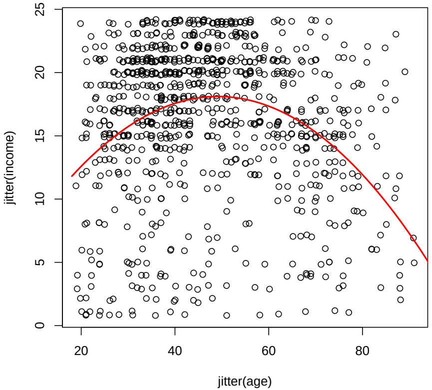

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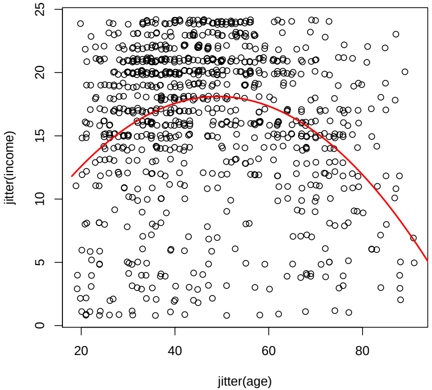
- $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_1^2 + u$
- Is β_1 the marginal effect of age on income?
- No! The marginal effect of age depends on the level of age: $\frac{\partial Y}{\partial X_1} = \widehat{\beta}_1 + 2\,\widehat{\beta}_2\,X_1$
  Here the effect of age changes monotonically from positive to negative with income.



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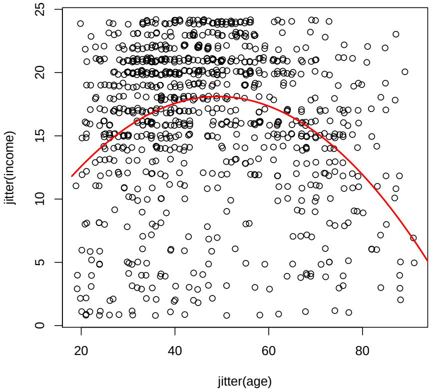

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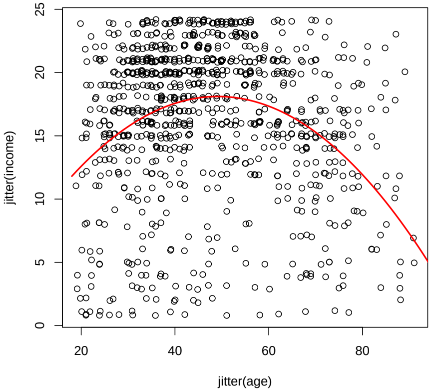
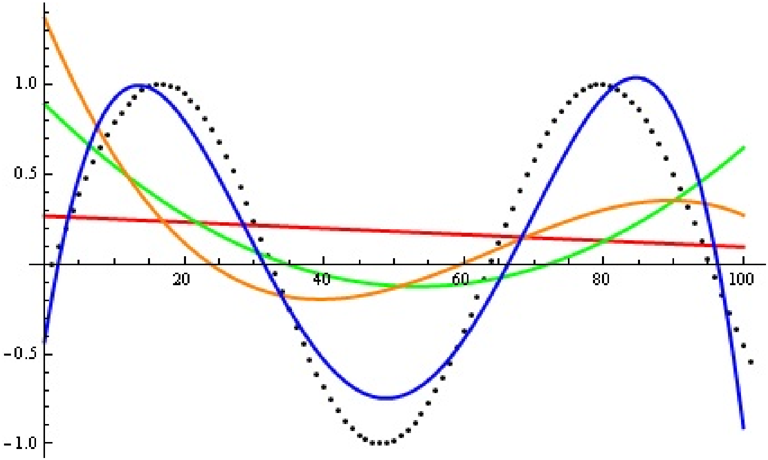
- If $\beta_2 > 0$ we get a U-shape, and if $\beta_2 < 0$ we get an inverted U-shape.
- Maximum/Minimum occurs at $|\frac{\beta_1}{2\beta_2}|$. Here turning point is at $X_1 = 50$.



Higher Order Polynomials



Approximating data generated with a sine function. Red line is a first degree polynomial, green line is second degree, orange line is third degree and blue is fourth degree

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- Getting uncertainty on this quantity is a great use case for the bootstrap!

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Next week: Linear Regression in its Full Glory!