Week 7: Multiple Regression

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Justin Grimmer, Jens Hainmueller and Erin Hartman.

Where We've Been and Where We're Going...

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  - regression with two variables
  - omitted variables, multicollinearity, interactions

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- Long Run
  - probability → inference → regression → causal inference

1. Matrix Form of Regression
   - Estimation
   - Fun With(out) Weights
2. OLS Classical Inference in Matrix Form
   - Unbiasedness
   - Classical Standard Errors
3. Agnostic Inference
4. Standard Hypothesis Tests
   - t-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

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- We can write this as:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \beta_0 + \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \beta_1 + \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \beta_2 + \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}$$

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- Outcome is a linear combination of the the $\mathbf{x}$, $\mathbf{z}$, and $\mathbf{u}$ vectors

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- We can also write this at the individual level, where $\mathbf{x}_i'$ is the ith row of $\mathbf{X}$:

$$y_i = \mathbf{x}_i' \boldsymbol{\beta} + u_i$$

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- We've learned about matrix multiplication, but what about matrix "division"?

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- Need a matrix version of this: $\frac{1}{a}$.

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- "Denominator" $\mathbf{X}'\mathbf{X}$ is approximately composed of the sample variances and covariances of variables within $\mathbf{X}$

Intuition for the OLS in Matrix Form

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What's the intuition here?

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- Thus, we have something like:

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Disclaimer: the final equation is exactly true for all non-intercept coefficients if you remove the intercept from $\mathbf{X}$ such that $\hat{\boldsymbol{\beta}}_{-0} = \text{Var}(\mathbf{X}_{-0})^{-1}\text{Cov}(\mathbf{X}_{-0}, \mathbf{y})$. The numerator and denominator are the variances and covariances if $\mathbf{X}$ and $\mathbf{y}$ are demeaned and normalized by the sample size minus 1.

Fun Without Weights

Fun Without Weights

The Robust Beauty of Improper Linear Models in Decision Making

ROBYN M. DAWES *University of Oregon*

ABSTRACT: *Proper linear models are those in which predictor variables are given weights in such a way that the resulting linear composite optimally predicts some criterion of interest; examples of proper linear models are standard regression analysis, discriminant function analysis, and ridge regression analysis. Research summarized in Paul Meehl's book on clinical versus statistical prediction—and a plethora of research stimulated in part by that book—all indicates that when a numerical criterion variable (e.g., graduate grade point average) is to be predicted from numerical predictor variables, proper linear models outperform clinical intuition. Improper linear models are those in which the weights of the predictor variables are obtained by some nonoptimal method; for example, they may be obtained on the basis of intuition, derived from simulating a clinical judge's predictions, or set to be equal. This article presents evidence that even such improper linear models are superior to clinical intuition when predicting a numerical criterion from numerical predictors. In fact, unit (i.e., equal) weighting is quite robust for making such predictions. The article discusses, in some detail, the application of unit weights to decide what bullet the Denver Police Department should use. Finally, the article considers commonly raised technical, psychological, and ethical resistances to using linear models to make important social decisions and presents arguments that could weaken these resistances.*

A *proper linear model* is one in which the weights given to the predictor variables are chosen in such a way as to optimize the relationship between the prediction and the criterion. Simple regression analysis is the most common example of a proper linear model; the predictor variables are weighted in such a way as to maximize the correlation between the subsequent weighted composite and the actual criterion. Discriminant function analysis is another example of a proper linear model; weights are given to the predictor variables in such a way that the resulting linear composites maximize the discrepancy between two or more groups. Ridge regression analysis, another example (Darlington, 1978; Marquardt & Snee, 1975), attempts to assign weights in such a way that the linear composites correlate maximally with the criterion of interest in a new set of data.

Thus, there are many types of proper linear models and they have been used in a variety of contexts. One example (Dawes, 1971) was presented in this Journal; it involved the prediction of faculty ratings of graduate students. All gradu-

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- Proper linear model is one where predictor variables are given optimized weights in some way (for example through regression).
- Dawes argues that even improper linear models (those where weights are set by hand or set to be equal), outperform clinical intuition.

Example: Graduate Admissions

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- Standardized and equally weighted improper linear model, correlated at .48.

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- Einhorn (1972) study of doctors coding biopsies of patients with Hodgkin's disease and then rated severity. Their rating of severity was essentially uncorrelated with survival times, but the variables they coded predicted outcomes using a regression model.

Other Examples

TABLE 1

Correlations Between Predictions and Criterion Values

Example	Average validity of judge	Average validity of judge model	Average validity of random model	Validity of equal weighting model	Cross-validity of regression analysis	Validity of optimal linear model
Prediction of neurosis vs. psychosis	.28	.31	.30	.34	.46	.46
Illinois students' predictions of GPA	.33	.50	.51	.60	.57	.69
Oregon students' predictions of GPA	.37	.43	.51	.60	.57	.69
Prediction of later faculty ratings at Oregon	.19	.25	.39	.48	.38	.54
Yntema & Torgerson's (1961) experiment	.84	.89	.84	.97	—	.97

Note. GPA = grade point average.

Column descriptions:

- C1) average of human judges
- C2) model based on human judges
- C3) randomly chosen weights preserving signs
- C4) equal weighting
- C5) cross-validated weights
- C6) unattainable optimal linear model

Common pattern: c2, c3, c4, c5, c6 > c1

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- People are good at picking out relevant information, but terrible at integrating it.
- The difficulty arises in part because people in general lack a strong reference to the distribution of the predictors.
- Linear models are robust to deviations from the optimal weights (see also Waller 2008 on "Fungible Weights in Multiple Regression")

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- The broader research project suggests that a big part of what quantitative models are doing predictively, is focusing human talent in the right place.
- This all applies because predictors well chosen and the sample size is small (so it is hard to learn much from the data).
- Dawes (1979) is an intellectual basis to support algorithmic decision making. Roughly, if simple models are better than experts, than with lots of data, complicated model could be much better than experts.

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Next Time: Classical Inference and Properties

Where We've Been and Where We're Going...

- Last Week
  - regression with two variables
  - omitted variables, multicollinearity, interactions
- This Week
  - matrix form of linear regression
  - inference and hypothesis tests
- Next Week
  - diagnostics
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Matrix Form of Regression
   - Estimation
   - Fun With(out) Weights
2. OLS Classical Inference in Matrix Form
   - Unbiasedness
   - Classical Standard Errors
3. Agnostic Inference
4. Standard Hypothesis Tests
   - t-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

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1. Linearity: $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{u}$
2. Random/iid sample: $(y_i, \mathbf{x}_i')$ are a iid sample from the population.
3. No perfect collinearity: $\mathbf{X}$ is an $n \times (k+1)$ matrix with rank $k+1$
4. Zero conditional mean: $E[\mathbf{u}|\mathbf{X}] = \mathbf{0}$
5. Homoskedasticity: $\text{var}(\mathbf{u}|\mathbf{X}) = \sigma_u^2 \mathbf{I}_n$
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- If $\mathbf{X}$ has rank $k+1$, then $(\mathbf{X}'\mathbf{X})$ is invertible (see linear algebra book for proof)

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- If $\mathbf{X}$ has rank $k+1$, then $(\mathbf{X}'\mathbf{X})$ is invertible (see linear algebra book for proof)
- Just like variation in X led us to be able to divide by the variance in simple OLS

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$$E[\mathbf{u}|\mathbf{X}] = \begin{bmatrix} E[u_1|\mathbf{X}] \\ E[u_2|\mathbf{X}] \\ \vdots \\ E[u_n|\mathbf{X}] \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \mathbf{0}$$

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So, yes!

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Now we know the sampling distribution is centered on β we want to derive the variance of the sampling distribution conditional on X.

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$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

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- To really understand this we need to know what $\text{var}(\mathbf{u}|\mathbf{X})$ is in full generality.
- The variance of a vector is actually a matrix:

$$\text{var}[\mathbf{u}] = \Sigma_u = \begin{bmatrix} \text{var}(u_1) & \text{cov}(u_1, u_2) & \dots & \text{cov}(u_1, u_n) \\ \text{cov}(u_2, u_1) & \text{var}(u_2) & \dots & \text{cov}(u_2, u_n) \\ \vdots & & \ddots & \\ \text{cov}(u_n, u_1) & \text{cov}(u_n, u_2) & \dots & \text{var}(u_n) \end{bmatrix}$$

- This matrix is always symmetric since $\text{cov}(u_i, u_j) = \text{cov}(u_j, u_i)$ by definition.

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- In less matrix notation:
  - $\text{var}(u_i) = \sigma_u^2$ for all i (constant variance)
  - $\text{cov}(u_i, u_j) = 0$ for all $i \neq j$ (implied by iid)

Rule: Variance of Linear Function of Random Vector

Recall that for a linear transformation of a random variable X we have $V[aX + b] = a^2 V[X]$ with constants a and b.

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Definition (Variance of Linear Transformation of Random Vector)

Let $f(\mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{B}$ be a linear transformation of a random vector $\mathbf{u}$ with non-random vectors or matrices $\mathbf{A}$ and $\mathbf{B}$. Then the variance of the transformation is given by:

$$V[f(\mathbf{u})] = V[\mathbf{A}\mathbf{u} + \mathbf{B}] = \mathbf{A}V[\mathbf{u}]\mathbf{A}'$$

Conditional Variance of $\hat{\beta}$

$\hat{\boldsymbol{\beta}} = \boldsymbol{\beta} + (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}$ and $E[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \boldsymbol{\beta} + E[(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{u}|\mathbf{X}] = \boldsymbol{\beta}$ so the OLS estimator is a linear function of the errors. Thus:

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This gives the $(k+1) \times (k+1)$ variance-covariance matrix of $\hat{\beta}$.

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This gives the $(k+1)\times(k+1)$ variance-covariance matrix of $\hat{\boldsymbol{\beta}}$.

To estimate $V[\hat{\boldsymbol{\beta}}|\mathbf{X}]$, we replace σ^2 with its unbiased estimator $\hat{\sigma}^2$, which is now written using matrix notation as:

$$\hat{\sigma}^2 = \frac{\sum_i \hat{u}_i^2}{n-(k+1)} = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$$

Sampling Variance for $\hat{\beta}$

Under assumptions 1-5, the variance-covariance matrix of the OLS estimators is given by:

$$V[\hat{\beta}|\mathbf{X}] = \sigma^2 \left(\mathbf{X}'\mathbf{X}\right)^{-1} =$$

	$\widehat{\beta}_0$	$\widehat{\beta}_1$	$\widehat{\beta}_2$	$\cdots$	$\widehat{\beta}_k$
$\widehat{\beta}_0$	$V[\widehat{\beta}_0]$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k]$
$\widehat{\beta}_1$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_1]$	$V[\widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_k]$
$\widehat{\beta}_2$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_2]$	$\text{Cov}[\widehat{\beta}_1, \widehat{\beta}_2]$	$V[\widehat{\beta}_2]$	$\cdots$	$\text{Cov}[\widehat{\beta}_2, \widehat{\beta}_k]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\widehat{\beta}_k$	$\text{Cov}[\widehat{\beta}_0, \widehat{\beta}_k]$	$\text{Cov}[\widehat{\beta}_k, \widehat{\beta}_1]$	$\text{Cov}[\widehat{\beta}_k, \widehat{\beta}_2]$	$\cdots$	$V[\widehat{\beta}_k]$

Recall that standard errors are the square root of the diagonals of this matrix.

Overview of Inference in the General Setting

- Under assumption 1-5 in large samples:

$$\frac{\widehat{\beta}_j - \beta_j}{\widehat{SE}[\widehat{\beta}_j]} \sim N(0, 1)$$

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- Thus, confidence intervals and hypothesis tests proceed in essentially the same way.

Properties of the OLS Estimator: Summary

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Theorem

Under Assumptions 1–6, the $(k+1) \times 1$ vector of OLS estimators $\hat{\boldsymbol{\beta}}$, conditional on $\mathbf{X}$, follows a multivariate normal distribution with mean $\boldsymbol{\beta}$ and variance-covariance matrix $\sigma^2 (\mathbf{X}'\mathbf{X})^{-1}$:

$$\hat{\boldsymbol{\beta}} | \mathbf{X} \sim \mathcal{N}\left(\boldsymbol{\beta}, \sigma^2 (\mathbf{X}'\mathbf{X})^{-1}\right)$$

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- *Each element of $\hat{\boldsymbol{\beta}}$ (i.e. $\hat{\beta}_0, ..., \hat{\beta}_{k+1}$) is normally distributed, and $\hat{\boldsymbol{\beta}}$ is an unbiased estimator of $\boldsymbol{\beta}$ as $E[\hat{\boldsymbol{\beta}}] = \boldsymbol{\beta}$*

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Properties of the OLS Estimator: Summary

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Under Assumptions 1–6, the $(k+1)\times 1$ vector of OLS estimators $\hat{\boldsymbol{\beta}}$, conditional on $\mathbf{X}$, follows a *multivariate normal distribution* *with mean $\boldsymbol{\beta}$ and variance-covariance matrix $\sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$:*

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- *Variances and covariances are given by $V[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \sigma^2\left(\mathbf{X}'\mathbf{X}\right)^{-1}$*
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$$\widehat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$$

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$$\widehat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n - (k+1)}$$

- *With a large sample, $\hat{\boldsymbol{\beta}}$ approximately follows the same distribution under Assumptions 1–5 only, i.e., without assuming the normality of $\mathbf{u}$.*

Implications of the Variance-Covariance Matrix

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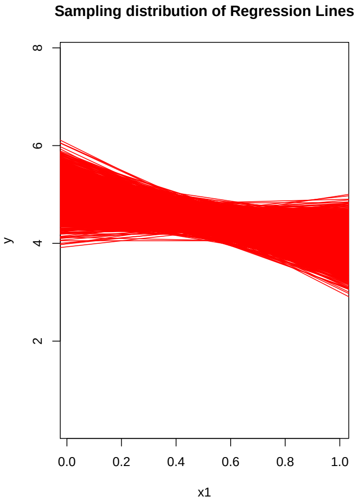
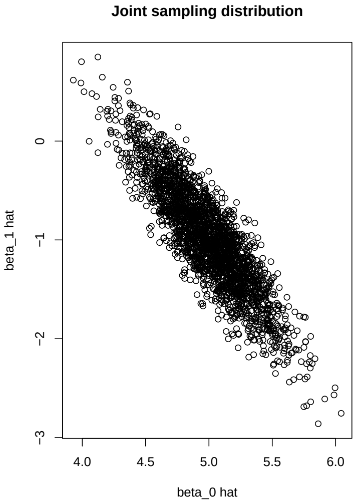
- Note that the sampling distribution is a joint distribution because it involves multiple random variables.
- This is because the sampling distribution of the terms in $\hat{\boldsymbol{\beta}}$ are correlated.
- In a practical sense, this means that our uncertainty about coefficients is correlated across variables.

Multivariate Normal: Simulation

$Y = \beta_0 + \beta_1 X_1 + u$ with $u \sim N(0, \sigma_u^2 = 4)$ and $\beta_0 = 5$, $\beta_1 = -1$, and $n = 100$:





Marginals of Multivariate Normal RVs are Normal

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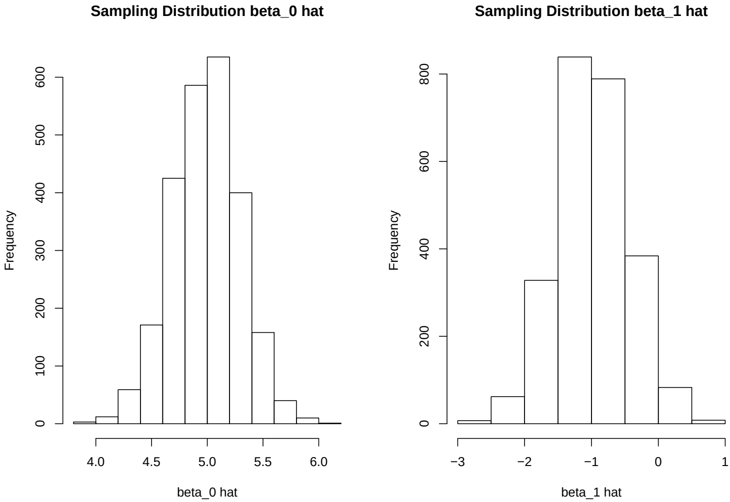

Matrix Notation Overview

	Old notation (for univariate regression)	Matrix notation
Linear model	$Y_i = \beta_0 + \beta_1 X_i + u$	$y = X\beta + u$
Coefficient	$\hat{\beta}_1 = \frac{\sum_{i=1}^n (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^n (X_i - \bar{X})^2}$	$\hat{\beta} = (X'X)^{-1} X' y$
Homoskedasticity assumption	$Var[u \mid X] = \sigma_u^2$	$Var[u \mid X] = \sigma_u^2 I_n$
Variance of coefficient	$\frac{\sigma_u^2}{\sum_{i=1}^n (X_i - \bar{X})^2}$	$\sigma_u^2 (X'X)^{-1}$
Error variance	$\frac{\sum_{i=1}^n \hat{u}_i^2}{n-2}$	$\hat{\sigma}_u^2 = \frac{\hat{u}'\hat{u}}{n-k-1}$, $\hat{u}'\hat{u} = (y - X\hat{\beta})'(y - X\hat{\beta})$
SS_{tot}	$\sum_{i=1}^n (Y_i - \bar{Y}_i)^2$	$(y - \bar{y})'(y - \bar{y})$
SS_{res}	$\sum_{i=1}^n (Y_i - \hat{Y}_i)^2$	$\hat{u}'\hat{u}$

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- Unbiasedness
- Classical Standard Errors

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Next Time: Agnostic Inference

Where We've Been and Where We're Going...

- Last Week
  - regression with two variables
  - omitted variables, multicollinearity, interactions
- This Week
  - matrix form of linear regression
  - inference and hypothesis tests
- Next Week
  - diagnostics
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Matrix Form of Regression
   - Estimation
   - Fun With(out) Weights
2. OLS Classical Inference in Matrix Form
   - Unbiasedness
   - Classical Standard Errors
3. Agnostic Inference
4. Standard Hypothesis Tests
   - t-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

Agnostic Perspective on the OLS estimator

- We know the population value of β is:

$$\beta = E\left[\mathbf{X}'\mathbf{X}\right]^{-1} E\left[\mathbf{X}'\mathbf{y}\right]$$

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- Plug-in principle $\rightsquigarrow$ replace population expectation with sample versions:

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Asymptotic OLS inference

- With this representation, we can write the OLS estimator as follows:

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- No need for assumptions A1 (linearity), A4 (conditional mean zero errors) or A5 (homoskedasticity) needed! Just IID (A2), no perfect collinearity (A3) and asymptotics.

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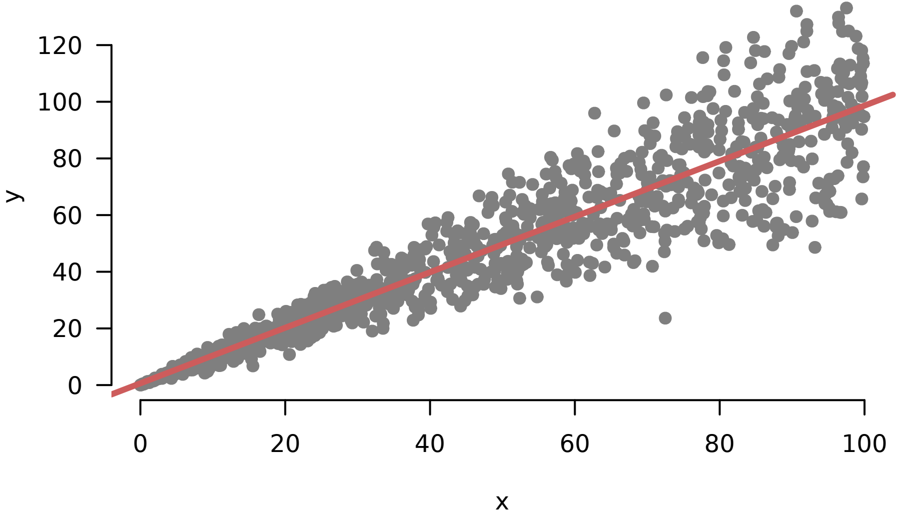
- Replace σ^2 with estimate $\widehat{\sigma}^2$ will give us our estimate of the covariance matrix

What Does This Rule Out?



Non-constant Error Variance

- Homoskedastic:

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- Idea: If we can consistently estimate the components of $\mathbf{\Sigma}$, we could directly use this expression by replacing $\mathbf{\Sigma}$ with its estimate, $\hat{\mathbf{\Sigma}}$.

White's Heteroskedasticity Consistent Estimator

Suppose we have heteroskedasticity of unknown form (but zero covariance):

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The estimate based on the above is called the heteroskedasticity consistent (HC) or robust standard errors. This also coincides with the agnostic standard errors!

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Core intuition: while $\widehat{\mathbf{\Sigma}}$ is an $n \times n$ matrix, $\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}$ is a $(k+1) \times (k+1)$ matrix.

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- There are various small sample corrections to improve performance when sample size is small. The most common variant (sometimes labeled HC1) is:

$$V[\hat{\beta}|\mathbf{X}] = \frac{n}{n-k-1}\cdot\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\widehat{\mathbf{\Sigma}}\mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

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- Robust SEs converge to same point as the bootstrap.
- This is a general framework (more to come in Week 8).

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Next Time: Hypothesis Tests

Where We've Been and Where We're Going...

- Last Week
  - regression with two variables
  - omitted variables, multicollinearity, interactions
- This Week
  - matrix form of linear regression
  - inference and hypothesis tests
- Next Week
  - diagnostics
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

1. Matrix Form of Regression
   - Estimation
   - Fun With(out) Weights
2. OLS Classical Inference in Matrix Form
   - Unbiasedness
   - Classical Standard Errors
3. Agnostic Inference
4. Standard Hypothesis Tests
   - *t*-Tests
   - Adjusted R^2
   - F Tests for Joint Significance

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- We model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

Hypothesis Testing in R

Model the intended Pinochet vote as a linear function of gender, education, and age of respondents.

R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

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where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.

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where $\mathbf{A}_{(j,j)}$ is the (j,j) element of matrix $\mathbf{A}$.
That is, take the variance-covariance matrix of $\hat{\boldsymbol{\beta}}$ and square root the diagonal element corresponding to j.

Getting the Standard Errors

R Code

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R Code

```
> V <- vcov(fit)
> V
              (Intercept)           fem          educ           age
(Intercept)  2.642311e-03 -3.455498e-04 -5.270913e-04 -3.357119e-05
fem         -3.455498e-04  5.623170e-04  2.249973e-05  8.285291e-07
educ        -5.270913e-04  2.249973e-05  1.922354e-04  3.411049e-06
age         -3.357119e-05  8.285291e-07  3.411049e-06  6.914098e-07

> sqrt(diag(V))
 (Intercept)          fem         educ          age
0.0514034097 0.0237132251 0.0138648980 0.0008315105
```

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4. Finally, either declare that we reject H_0 or not, or report the p-value.

Confidence Intervals

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So we also know the probability that the value of our test statistics falls into a given interval:

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$$\left[\widehat{\beta}_j - t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\, ,\, \widehat{\beta}_j + t_{\alpha/2}\hat{SE}[\hat{\beta}_j]\right]$$

and thus can construct the confidence intervals as usual using:

$$\hat{\beta}_j \pm t_{\alpha/2} \cdot \hat{SE}[\hat{\beta}_j]$$

Confidence Intervals in R

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Coefficients:
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---
```

R Code

```
> confint(fit)
                   2.5 %      97.5 %
(Intercept)  0.303407780  0.50504909
fem          0.089493169  0.18251357
educ        -0.087954435 -0.03356629
age          0.002147755  0.00540954
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> fit <- lm(REALGDPCAP ~ Region, data = D)
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Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4452.7      783.4   5.684 2.07e-07 ***
RegionAfrica     -2552.8     1204.5  -2.119   0.0372 *
RegionAsia         148.9     1149.8   0.129   0.8973
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against the alternative of

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RegionOecd        9671.3     1007.0   9.604 5.74e-15 ***
```

- $\hat{\beta}_{Asia}$ and $\hat{\beta}_{LAm}$ are close. So we may want to test the null hypothesis:

$$H_0: \ \beta_{LAm} = \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} = 0$$

against the alternative of

$$H_1: \ \beta_{LAm} \neq \beta_{Asia} \ \Leftrightarrow \ \beta_{LAm} - \beta_{Asia} \neq 0$$

- What would be an appropriate test statistic for this hypothesis?

Testing Hypothesis About a Linear Combination of β_j

R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> summary(fit)

Coefficients:
                Estimate Std. Error t value Pr(>|t|)
(Intercept)       4452.7      783.4   5.684 2.07e-07 ***
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RegionAsia         148.9     1149.8   0.129   0.8973
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- Let's consider a t-value:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})}$$

We will reject H_0 if T is sufficiently different from zero.

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- Note that unlike the test of a single hypothesis, both $\hat{\beta}_{LAm}$ and $\hat{\beta}_{Asia}$ are random variables, hence the denominator.

Testing Hypothesis About a Linear Combination of β_j

- Our test statistic:

$$T = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \ \sim \ t_{n-k-1}$$

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which we can calculate from the estimated covariance matrix of $\hat{\boldsymbol{\beta}}$.

- Since the estimates of the coefficients are correlated, we need the covariance term.

Example: GDP per capita on Regions

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R Code

```
> fit <- lm(REALGDPCAP ~ Region, data = D)
> V <- vcov(fit)
> V
                (Intercept) RegionAfrica RegionAsia RegionLatAmerica
(Intercept)        613769.9    -613769.9  -613769.9        -613769.9
RegionAfrica      -613769.9    1450728.8   613769.9         613769.9
RegionAsia        -613769.9     613769.9  1321965.9         613769.9
RegionLatAmerica  -613769.9     613769.9   613769.9        1014054.6
RegionOecd        -613769.9     613769.9   613769.9         613769.9
                 RegionOecd
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RegionLatAmerica   613769.9
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R Code

```
> se <- sqrt(V[4,4] + V[3,3] - 2*V[3,4])
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[1] 1052.844
>
> tstat <- (coef(fit)[4] - coef(fit)[3])/se
> tstat
RegionLatAmerica
      -0.3990977
```

$$t = \frac{\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia}}{\hat{SE}(\widehat{\beta}_{LAm} - \widehat{\beta}_{Asia})} \quad \text{where}$$

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Plugging in we get $t \approx -0.40$. So what do we conclude?

We cannot reject the null that the difference in average GDP resulted from chance.

Aside: Adjusted R^2

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R Code

```
> fit <- lm(vote1 ~ fem + educ + age, data = d)
> summary(fit)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4042284  0.0514034   7.864 6.57e-15 ***
fem          0.1360034  0.0237132   5.735 1.15e-08 ***
educ        -0.0607604  0.0138649  -4.382 1.25e-05 ***
age          0.0037786  0.0008315   4.544 5.90e-06 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4875 on 1699 degrees of freedom
Multiple R-squared: 0.05112,       Adjusted R-squared: 0.04945
F-statistic: 30.51 on 3 and 1699 DF,  p-value: < 2.2e-16
```

Aside: Adjusted R^2

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where SS_{res} are the sum of squared residuals and the SS_{tot} are the sum of the squared deviations from the mean.

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- So, how do we penalize more complex models? Adjusted R^2
- This makes R^2 more 'comparable' across models with different numbers of variables, but the next section will show you an even better way to approach that problem in a testing framework.
- Still since people report it, the next slide derives adjusted R^2 (but we are going to skip it),

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$$R^2 = 1 - \frac{\text{SS}_{\text{res}}/n}{\text{SS}_{\text{tot}}/n}$$
$$= 1 - \frac{\tilde{V}(\text{SS}_{\text{res}})}{\tilde{V}(\text{SS}_{\text{tot}})}$$

where $\tilde{V}$ is a biased estimator of the population variance.

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- Key idea: rewrite R^2 in terms of variances

$$
\begin{aligned}
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$$\hat{V}(\text{SS}_{\text{res}}) = \text{SS}_{\text{res}}/(n-k-1)$$
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$$R^2_{adj} = R^2 - \underbrace{(1-R^2)\frac{k-1}{n-k}}_{\text{model complexity penalty}}$$

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- Adjusted R^2 will always be smaller than R^2 and can sometimes be negative!

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- Stay tuned for more in Week 8!

F Test for Joint Significance of Coefficients

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- F tests allows us to to test joint hypothesis

The χ^2 Distribution

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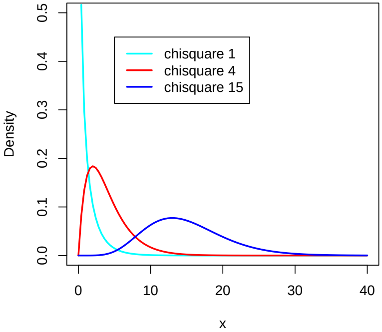

Properties: $X > 0$, $E[X] = n$ and $V[X] = 2n$. In R: `dchisq()`, `pchisq()`, `rchisq()`

The F distribution

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The F distribution arises as a ratio of two independent chi-squared distributed random variables:

$$F = \frac{X_1/df_1}{X_2/df_2} \sim \mathcal{F}_{df_1,df_2}$$

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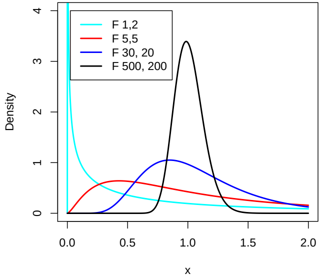

In R: `df()`, `pf()`, `rf()`

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The F statistic can be calculated by the following procedure:

1. Fit the Unrestricted Model (UR) which *does not* impose H_0:

$$Vote = \beta_0 + \gamma_1 FEM + \beta_1 EDUC + \gamma_2 (FEM * EDUC) + \beta_2 AGE + \gamma_3 (FEM * AGE) + u$$

F Test against $H_0: \gamma_1 = \gamma_2 = \gamma_3 = 0.$

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   $$\frac{\text{increase in prediction error}}{\text{original prediction error}}$$

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- Under Assumptions 1–6, $F_0 \sim \mathcal{F}_{q,n-k-1}$ regardless of the sample size.

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- Under Assumptions 1–5, $qF_0 \overset{a.}{\sim} \chi^2_q$ as $n \to \infty$ (see next section).

Unrestricted Model (UR)

R Code

```
> fit.UR <- lm(vote1 ~ fem + educ + age + fem:age + fem:educ, data = Chile)
> summary(fit.UR)
~~~~~
Coefficients:
             Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
educ        -0.038571   0.019578  -1.970 0.048988 *
age          0.005482   0.001114   4.921 9.44e-07 ***
fem:age     -0.003779   0.001673  -2.259 0.024010 *
fem:educ    -0.044484   0.027697  -1.606 0.108431
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,        Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Restricted Model (R)

R Code

```
> fit.R <- lm(vote1 ~ educ + age, data = Chile)
> summary(fit.R)
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.4878039  0.0497550   9.804  < 2e-16 ***
educ        -0.0662022  0.0139615  -4.742 2.30e-06 ***
age          0.0035783  0.0008385   4.267 2.09e-05 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 0.4921 on 1700 degrees of freedom
Multiple R-squared:  0.03275,       Adjusted R-squared:  0.03161
F-statistic: 28.78 on 2 and 1700 DF,  p-value: 5.097e-13
```

F Test in R

R Code

```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

> DFdenom <- df.residual(fit.UR)  # = 1703
> DFnum <- 3

> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

F Test in R

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```
> SSR.UR <- sum(resid(fit.UR)^2)  # = 402
> SSR.R <- sum(resid(fit.R)^2)    # = 411

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> F <- ((SSR.R - SSR.UR)/DFnum) / (SSR.UR/DFdenom)
> F
[1] 13.01581

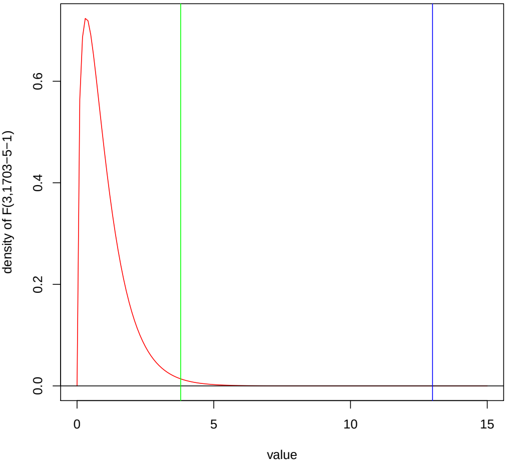
> qf(0.99, DFnum, DFdenom)
[1] 3.793171
```

Given above, what do we conclude?

$F_0 = 13$ is greater than the critical value for a .01 level test. So we *reject* the null hypothesis.

Null Distribution, Critical Value, and Test Statistic

Note that the F statistic is always positive, so we only look at the right tail of the reference F (or χ^2 in a large sample) distribution.



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- Have any of you used an F-test like this in your research?
- This is called the omnibus test and is routinely reported by statistical software.

Omnibus Test in R

R Code

```
> summary(fit.UR)
~~~~~
Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  0.293130   0.069242   4.233 2.42e-05 ***
fem          0.368975   0.098883   3.731 0.000197 ***
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Residual standard error: 0.487 on 1697 degrees of freedom
Multiple R-squared: 0.05451,        Adjusted R-squared: 0.05172
F-statistic: 19.57 on 5 and 1697 DF,  p-value: < 2.2e-16
```

Omnibus Test in R with Random Noise

R Code

```
> set.seed(08540)
> p <- 10; x <- matrix(rnorm(p*1000), nrow=1000)
> y <- rnorm(1000); summary(lm(y~x))
~~~~~

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept) -0.0115475  0.0320874  -0.360   0.7190
x1          -0.0019803  0.0333524  -0.059   0.9527
x2           0.0666275  0.0314087   2.121   0.0341 *
x3          -0.0008594  0.0321270  -0.027   0.9787
x4           0.0051185  0.0333678   0.153   0.8781
x5           0.0136656  0.0322592   0.424   0.6719
x6           0.0102115  0.0332045   0.308   0.7585
x7          -0.0103903  0.0307639  -0.338   0.7356
x8          -0.0401722  0.0318317  -1.262   0.2072
x9           0.0553019  0.0315548   1.753   0.0800 .
x10          0.0410906  0.0319742   1.285   0.1991
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 1.011 on 989 degrees of freedom
Multiple R-squared:  0.01129,       Adjusted R-squared:  0.001294
F-statistic: 1.129 on 10 and 989 DF,  p-value: 0.3364
```

F Test Examples II

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + ... + \beta_k X_k + u$$

Next, let's consider:

$$H_0 : \beta_1 = \beta_2 = \beta_3$$

F Test Examples II

The F test can be used to test various joint hypotheses which involve multiple linear restrictions. Consider the regression model:

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- What question are we asking?

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  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?

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- What question are we asking?
  $\rightarrow$ Are the coefficients X_1, X_2 and X_3 different from each other?
- How many restrictions?
  $\rightarrow$ Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)

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- How do we fit the restricted model?

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- What question are we asking?
  → Are the coefficients X_1, X_2 and X_3 different from each other?
- How many restrictions?
  → Two ($\beta_1 - \beta_2 = 0$ and $\beta_2 - \beta_3 = 0$)
- How do we fit the restricted model?
  → The null hypothesis implies that the model can be written as:

  $$Y = \beta_0 + \beta_1 (X_1 + X_2 + X_3) + ... + \beta_k X_k + u$$

  So we create a new variable $X^* = X_1 + X_2 + X_3$ and fit:

  $$Y = \beta_0 + \beta_1 X^* + ... + \beta_k X_k + u$$

Testing Equality of Coefficients in R

R Code

```
> fit.UR2 <- lm(REALGDPCAP ~ Asia + LatAmerica + Transit + Oecd, data = D)
> summary(fit.UR2)
~~~~~
Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)   1899.9      914.9   2.077   0.0410 *
Asia          2701.7     1243.0   2.173   0.0327 *
LatAmerica    2281.5     1112.3   2.051   0.0435 *
Transit       2552.8     1204.5   2.119   0.0372 *
Oecd         12224.2     1112.3  10.990   <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1  1

Residual standard error: 3034 on 80 degrees of freedom
Multiple R-squared: 0.7096,        Adjusted R-squared: 0.6951
F-statistic: 48.88 on 4 and 80 DF,  p-value: < 2.2e-16
```

Are the coefficients on *Asia*, *LatAmerica* and *Transit* statistically significantly different?

Testing Equality of Coefficients in R

R Code

```
> D$Xstar <- D$Asia + D$LatAmerica + D$Transit
> fit.R2 <- lm(REALGDPCAP ~ Xstar + Oecd, data = D)

> SSR.UR2 <- sum(resid(fit.UR2)^2)
> SSR.R2 <- sum(resid(fit.R2)^2)

> DFdenom <- df.residual(fit.UR2)

> F <- ((SSR.R2 - SSR.UR2)/2) / (SSR.UR2/DFdenom)
> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

Testing Equality of Coefficients in R

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> F
[1] 0.08786129

> pf(F, 2, DFdenom, lower.tail = F)
[1] 0.9159762
```

So, what do we conclude?

The three coefficients are statistically indistinguishable from each other, with the p-value of 0.916.

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

t Test vs. F Test

Consider the hypothesis test of

$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

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- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.

t Test vs. F Test

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$$H_0: \ \beta_1 = \beta_2 \quad \text{vs.} \quad H_1: \ \beta_1 \neq \beta_2$$

What ways have we learned to conduct this test?

- Option 1: Compute $T = (\hat{\beta}_1 - \hat{\beta}_2)/\hat{SE}(\hat{\beta}_1 - \hat{\beta}_2)$ and do the t test.
- Option 2: Create $X^* = X_1 + X_2$, fit the restricted model, compute $F = (SSR_R - SSR_{UR}) \big/ (SSR_R/(n-k-1))$ and do the F test.

t Test vs. F Test

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It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \Longleftrightarrow \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

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It turns out these two tests give identical results. This is because

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- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.

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It turns out these two tests give identical results. This is because

$$X \ \sim \ t_{n-k-1} \quad \Longleftrightarrow \quad X^2 \ \sim \ \mathcal{F}_{1,n-k-1}$$

- So, for testing a single hypothesis it does not matter whether one does a t test or an F test.
- Usually, the t test is used for single hypotheses and the F test is used for joint hypotheses.

Some More Notes on F Tests

- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_{R})/q}{(1 - R^2_{UR})/(n - k - 1)}$$

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- The F-value can also be calculated from R^2:

$$F = \frac{(R^2_{UR} - R^2_R)/q}{(1 - R^2_{UR})/(n-k-1)}$$

- F tests only work for testing nested models, i.e. the restricted model must be a special case of the unrestricted model.

  For example F tests cannot be used to test

$$Y = \beta_0 + \beta_1 X_1 \qquad\qquad + \beta_3 X_3 + u$$

against

$$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \qquad + u$$

Some More Notes on F Tests

Joint significance does not necessarily imply the significance of individual coefficients, or vice versa:

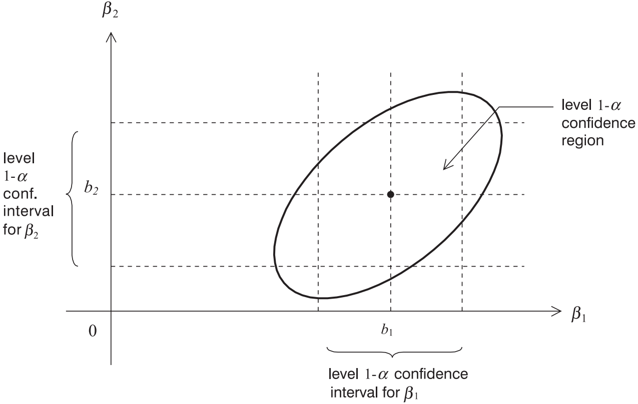

Figure 1.5: t- versus F-Tests

Image Credit: Hayashi (2011) *Econometrics*

Goal Check: Understand `lm()` Output

```
Call:
lm(formula = sr ~ pop15, data = LifeCycleSavings)

Residuals:
   Min     1Q Median     3Q    Max
-8.637 -2.374  0.349  2.022 11.155

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept) 17.49660    2.27972   7.675 6.85e-10 ***
pop15       -0.22302    0.06291  -3.545 0.000887 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1

Residual standard error: 4.03 on 48 degrees of freedom
Multiple R-squared:  0.2075,Adjusted R-squared:  0.191
F-statistic: 12.57 on 1 and 48 DF,  p-value: 0.0008866
```

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- Much of the hypothesis test infrastructure ports over nicely, plus there are new joint tests we can use.

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- We showed how to estimate the coefficients and get the variance covariance matrix.
- You can also calculate robust standard errors which provide a plug and play replacement.
- Much of the hypothesis test infrastructure ports over nicely, plus there are new joint tests we can use.

Next week: Troubleshooting the Linear Model!