Week 8: What Can Go Wrong and How To Fix It, Diagnostics and Solutions

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn, Jens Hainmueller, Erin Hartman and Kevin Quinn.

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- Today is setting the table for how we think about that problem. I think this is philosophically interesting and you may want to revisit at the end of the week.

Five Themes

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(1) Clearly define your goal.
(2) Examine your model.
(3) Diagnosis through treatment.
(4) Don't expect a free lunch.
(5) Re-examine defaults.

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- When thinking about whether an estimator is biased, always think 'biased with respect to what?'
- Predictive generalization is one unifying way to solve questions about what is best (particularly in machine learning).
- You may not know how to precisely define your goal yet (last few weeks are about causal goals). That's okay!

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- One way out of this we won't discuss is train-test splits.

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- We will talk about successes of this approach but also some catastrophic failures.

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  - data dependence vs. assumption dependence

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- But remember, different solutions are better for different settings—defaults are a function of their time and context.
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- The history of 20th century social science is defined by scarcity—data was hard to find, surveys were costly to field, computing resources were expensive and difficult to access. The methods we use are a consequence of that scarcity.
- But now we have abundance—new forms of data, cheap surveys, huge computation! It is changing the methods we consider.

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Regrettably we won't have time to cover two important areas: missing data and sensitivity analysis.

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Next Time: Non-normal Errors!

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- Fix $\mathbf{x}_i'$ and the distribution of errors are Normal.

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- The sample size (n) needed for approximation to hold depends on how far the errors are from Normal.
- Reasonable question: if we have enough data are non-normal errors even a problem?

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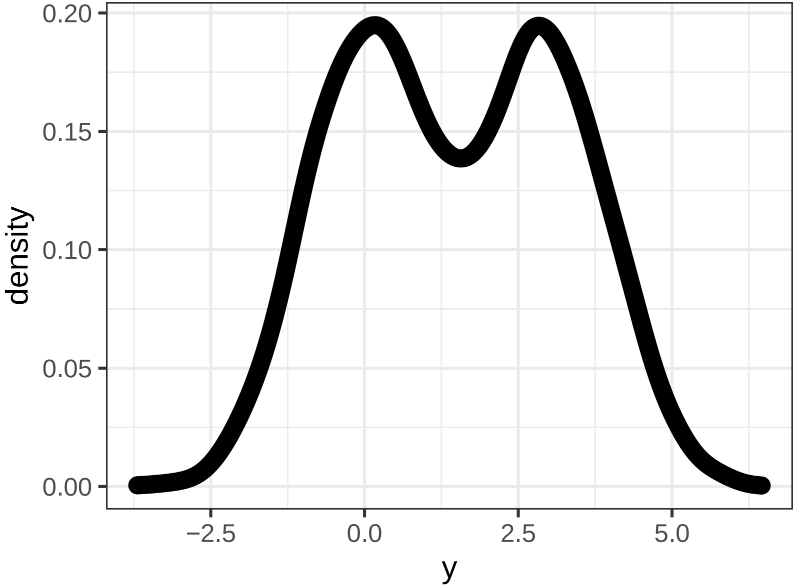
- The plausibility depends on the X chosen by the researcher.

Example: Is this a Violation?



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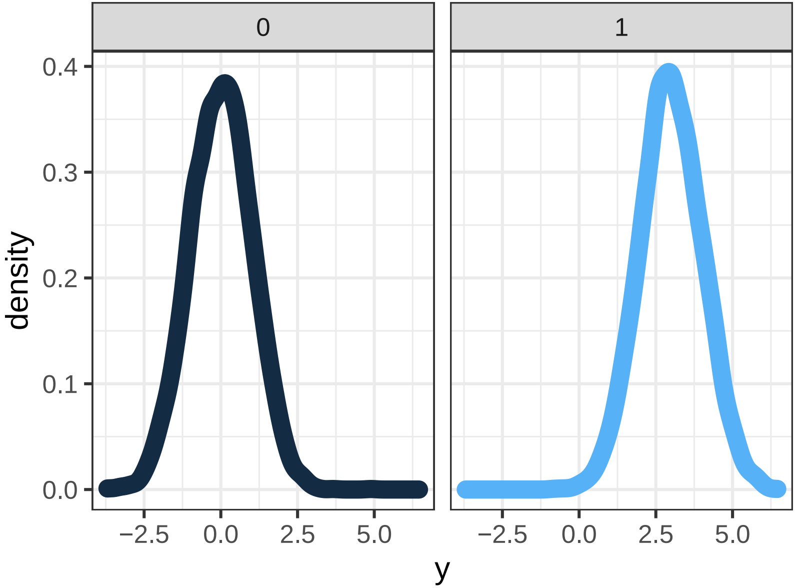

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To understand the relationship between residuals and errors, we need to derive the distribution of the residuals (which we will do over the next few slides).

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&\equiv \mathbf{y} - \mathbf{H}\mathbf{y} \\
&= (\mathbf{I} - \mathbf{H})\mathbf{y}
\end{aligned}$$

- $\mathbf{H} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'$ is the hat matrix because it puts the "hat" on $\mathbf{y}$:

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  it has a few nice properties:
  - $\mathbf{H}$ is an $n \times n$ symmetric matrix

Defining the Hat Matrix

- We want to figure out how the distribution of errors $\mathbf{u}$ relates to the distribution of residuals $\widehat{\mathbf{u}}$.
- To get there let's write $\widehat{\mathbf{u}}$ in terms of y, then we will be able to replace $\mathbf{y}$ with $\mathbf{X}\beta + \mathbf{u}$.

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  - $\mathbf{H}$ is an $n \times n$ symmetric matrix
  - $\mathbf{H}$ is idempotent: $\mathbf{H}\mathbf{H} = \mathbf{H}$

Relating the Residuals to the Errors

With the hat matrix, we are ready to relate the residuals to the errors.

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- Note that each residual is a function of all of the errors.

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The variance of the ith residual $\hat{u}_i$ is $V[\hat{u}_i] = \sigma^2(1 - h_{ii})$, where h_{ii} is the ith diagonal element of the matrix $\mathbf{H}$ (called the hat value).

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What if we could transform the residuals to address the two issues above?

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This produces standardized (or "internally studentized") residuals:

$$\hat{u}'_i = \frac{\hat{u}_i}{\hat{\sigma}\sqrt{1 - h_{ii}}}$$

where $\hat{\sigma} = \sqrt{\hat{\sigma}^2}$ and $\hat{\sigma}^2 = \frac{\hat{\mathbf{u}}'\hat{\mathbf{u}}}{n-(k+1)}$ is our usual estimate of the error variance.

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The standardized residuals are still not ideal, since the numerator and denominator of $\hat{u}_i'$ are not independent. This makes the distribution of $\hat{u}_i'$ nonstandard. If the distribution is non-standard, we can't easily check for violations.

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- Deviations from this t distribution of the residuals imply violation of Normality in the errors.

Example: Buchanan votes in Florida, 2000

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Wand et al. show that the ballot caused 2,000 Democratic voters to vote by mistake for Buchanan, a number more than enough to have tipped the vote in FL from Bush to Gore, thus giving him FL's 25 electoral votes and the presidency.

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The Butterfly Did It: The Aberrant Vote for Buchanan in Palm Beach County, Florida

JONATHAN N. WAND *Cornell University*
KENNETH W. SHOTTS *Northwestern University*
JASJEET S. SEKHON *Harvard University*
WALTER R. MEBANE, JR. *Cornell University*
MICHAEL C. HERRON *Northwestern University*
HENRY E. BRADY *University of California, Berkeley*

We show that the butterfly ballot used in Palm Beach County, Florida, in the 2000 presidential election caused more than 2,000 Democratic voters to vote by mistake for Reform candidate Pat Buchanan, a number larger than George W. Bush's certified margin of victory in Florida. We use multiple methods and several kinds of data to rule out alternative explanations for the votes Buchanan received in Palm Beach County. Among 3,053 U.S. counties where Buchanan was on the ballot, Palm Beach County has the most anomalous excess of votes for him. In Palm Beach County, Buchanan's proportion of the vote on election-day ballots is four times larger than his proportion on absentee (nonbutterfly) ballots, but Buchanan's proportion does not differ significantly between election-day and absentee ballots in any other Florida county. Unlike other Reform candidates in Palm Beach County, Buchanan tended to receive election-day votes in Democratic precincts and from individuals who voted for the Democratic U.S. Senate candidate. Robust estimation of overdispersed binomial regression models underpins much of the analysis.

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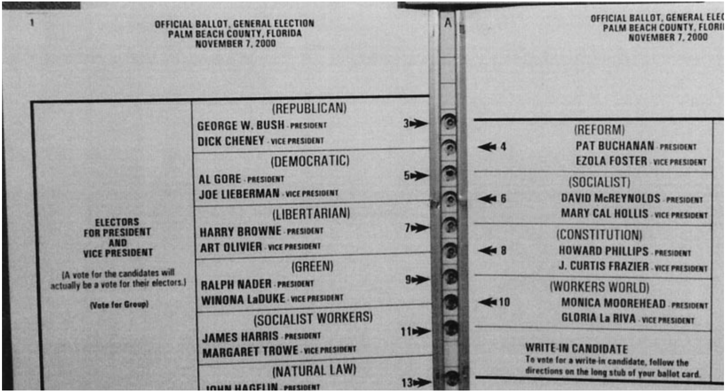
FIGURE 1. The Palm Beach County Bufferfly Ballot



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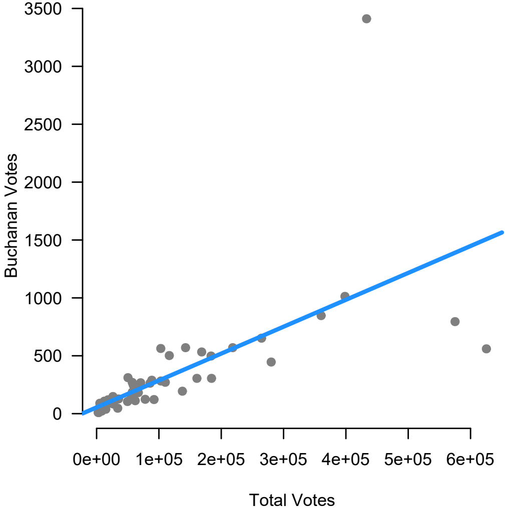

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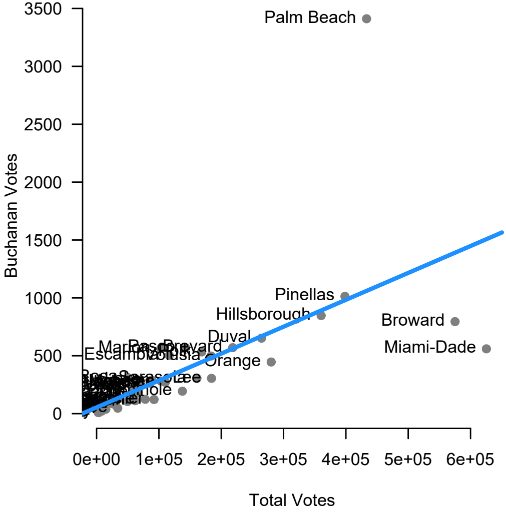

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- Now that our studentized residuals follow a known standard distribution, we can proceed with diagnostic analysis for the nonnormal errors.
- We examine data from the 2000 presidential election in Florida used in Wand et al. (2001).
- Our analysis takes place at the county level and we will regress the number of Buchanan votes in each county on the total number of votes in each county.

Buchanan Votes and Total Votes

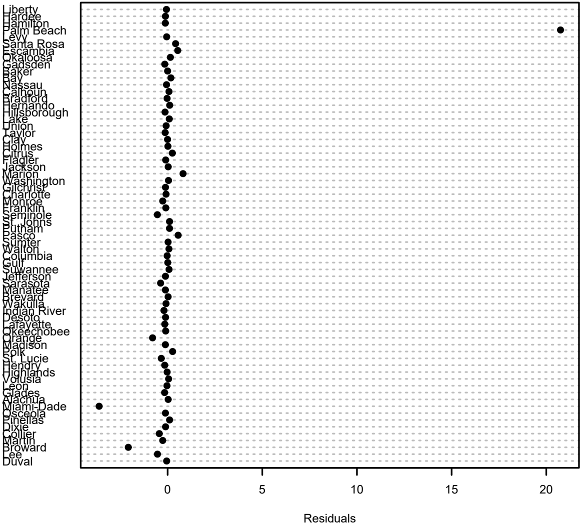
R Code

```
> mod1 <- lm(buchanan00~TotalVotes00,data=dta)
> summary(mod1)
Residuals:
    Min      1Q  Median      3Q     Max
-947.05  -41.74  -19.47   20.20 2350.54

Coefficients:
              Estimate Std. Error t value Pr(>|t|)
(Intercept)  5.423e+01  4.914e+01   1.104    0.274
TotalVotes00 2.323e-03  3.104e-04   7.483 2.42e-10 ***
---
Residual standard error: 332.7 on 65 degrees of freedom
Multiple R-squared: 0.4628,        Adjusted R-squared: 0.4545
F-statistic:    56 on 1 and 65 DF,  p-value: 2.417e-10

> residuals              <- resid(mod1)
> standardized_residuals <- rstandard(mod1)
> studentized_residuals  <- rstudent(mod1)
> dotchart(residuals,dta$name,cex=.7,xlab="Residuals")
```

Plotting the residuals



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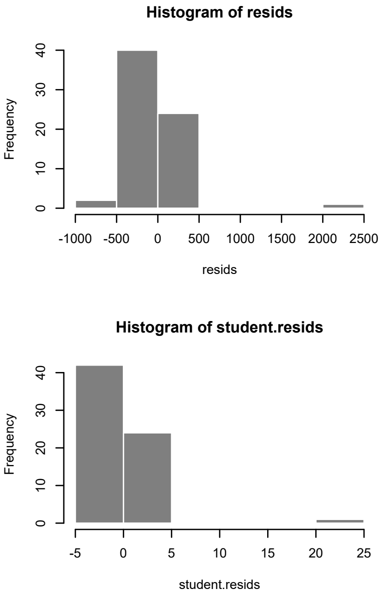

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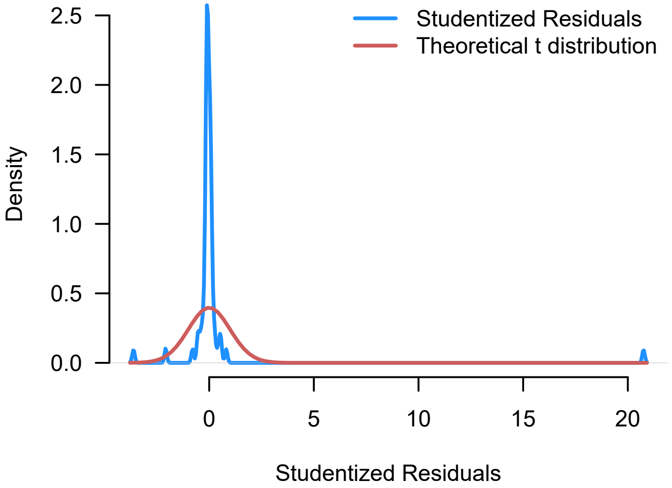

Quantile-Quantile plots

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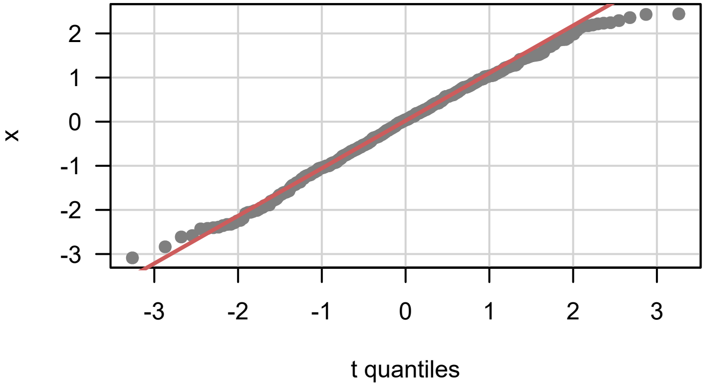
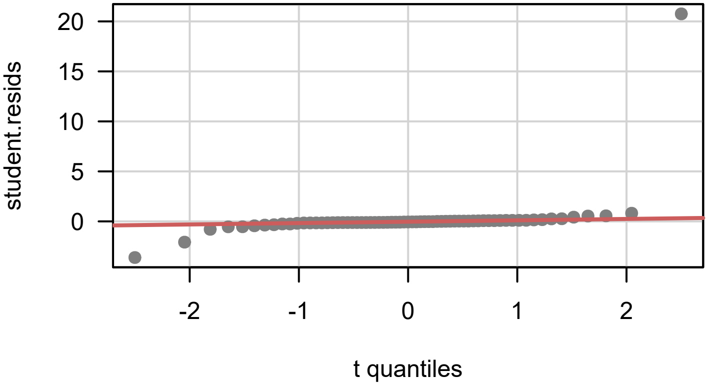
- If distributions are equal $\implies$ 45 degree line

Good QQ-plot



Buchanan QQ-plot



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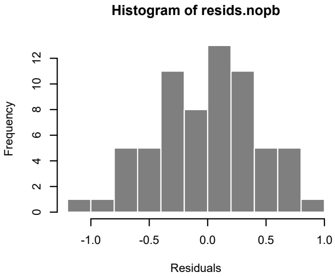
- Use estimators other than OLS that are robust to nonnormality (two videos from now!)

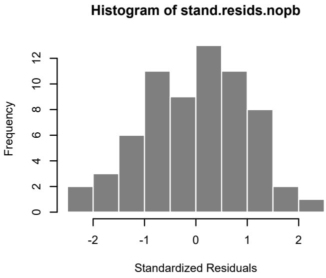
Buchanan Revisited

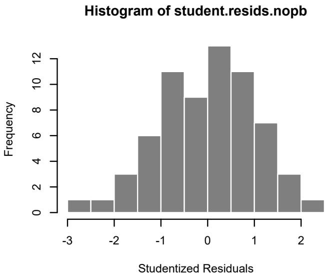
Let's delete Palm Beach and also use log transformations for both variables

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)    -2.48597    0.37889  -6.561 1.09e-08 ***
## log(edaytotal)  0.70311    0.03621  19.417  < 2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.4362 on 64 degrees of freedom
## Multiple R-squared:  0.8549, Adjusted R-squared:  0.8526
## F-statistic:   377 on 1 and 64 DF,  p-value: < 2.2e-16
```

Buchanan Revisited







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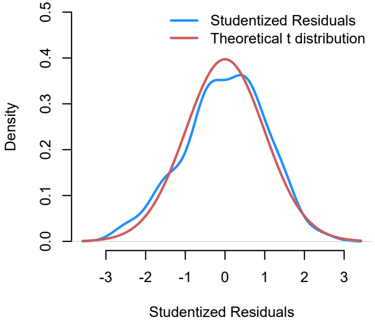

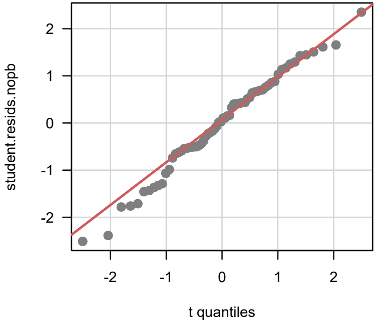

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- Remember the complexities of log transforms from Week 5!

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Next Time: Extreme Values

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This Week
  - diagnosing problems and troubleshooting the linear model
  - unusual and influential data $\rightarrow$ robust estimation
  - non-linearity $\rightarrow$ generalized additive models
  - unusual errors $\rightarrow$ sandwich SEs
- Next Week
  - frameworks for causal inference
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

The Trouble with Norway

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	Constant	x_1	x_2	$x_1 \cdot x_2$
Norway Obs Included	.814	-.192	-.278	.137
	(4.7)	(2.0)	(2.4)	(2.9)
Norway Obs Excluded	.641	-.068	-.138	.054
	(4.8)	(0.9)	(1.5)	(1.3)

Creative Curve Fitting with Norway

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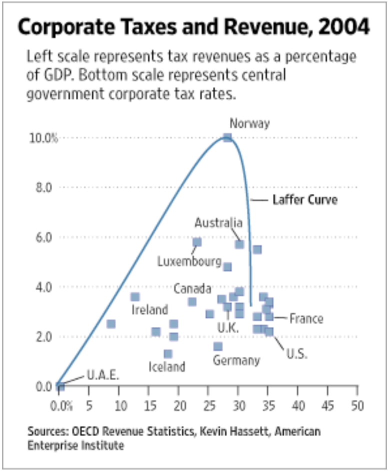

The Most Important Lesson: Check Your Data

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"Do not attempt to build a model on a set of poor data! In human surveys, one often finds 14-inch men, 1000-pound women, students with 'no' lungs, and so on. In manufacturing data, one can find 10,000 pounds of material in a 100 pound capacity barrel, and similar obvious errors.

All the planning, and training in the world will not eliminate these sorts of problems. In our decades of experience with 'messy data,' we have yet to find a large data set completely free of such quality problems."

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Carefully Examine the Data First!!

1. Examine summary statistics: `summary(data)`
2. Scatterplot matrix for densities and bivariate relationships: E.g. `scatterplotMatrix(data)` from `car` library.
3. Further conditional plots for multivariate data: E.g. `ggplot2`

Three Types of Extreme Values

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- Can be a violation of iid (not identically distributed)

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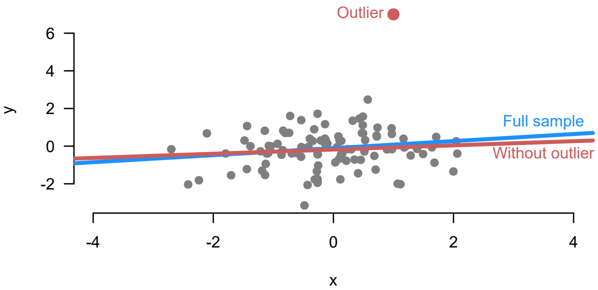

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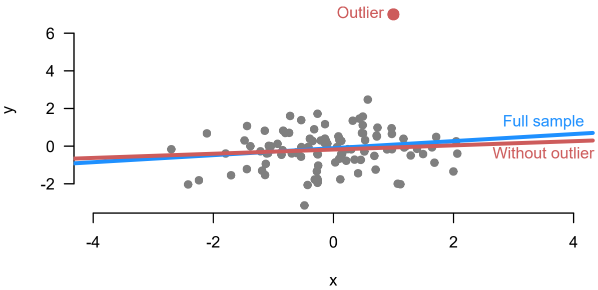

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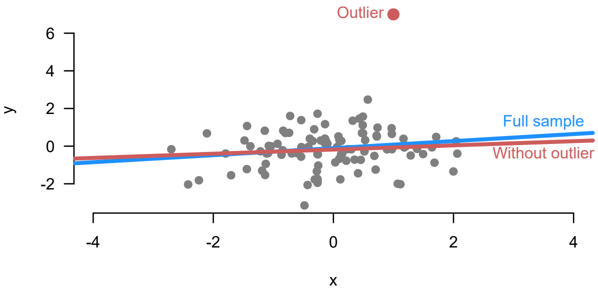

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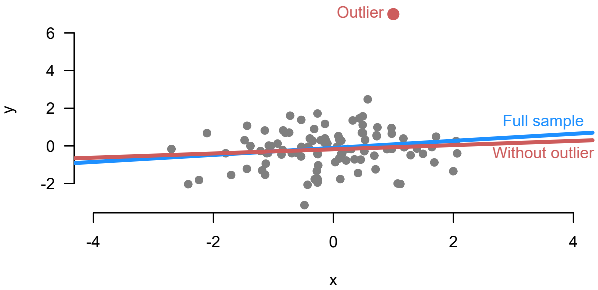

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- $\widehat{\sigma} > \widehat{\sigma}_{-i}$ because we drop the large residual from the outlier, and so $\widehat{u}_i' < \widehat{u}_i^*$

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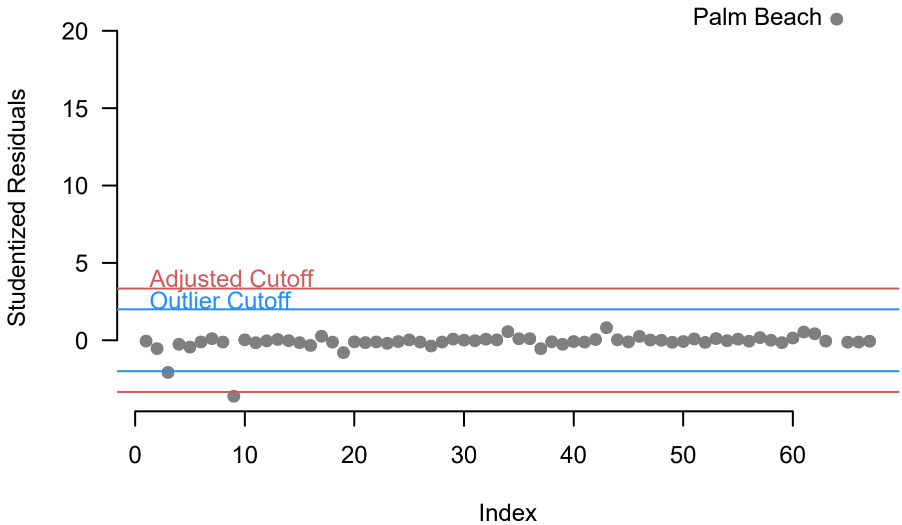
- People usually adjust cutoff for multiple testing

Buchanan outliers



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  - Use a method that is robust to outliers (robust regression)
- Key question is what is the goal? If you want to estimate the expectation of a distribution and a property of that distribution is extreme observations, that's just part of the story.

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- The ozone hole was detected in satellite data only when the raw data was reprocessed. When the software was rerun without the pre-processing flags, the ozone hole was seen as far back as 1976.

A Sociological Cautionary Tale

Comment on Herring, ASR*, April 2009*

AMERICAN SOCIOLOGICAL ASSOCIATION

American Sociological Review
2017, Vol. 82(4) 857–867

DOI: 10.1177/0003122417714422
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Does Diversity Pay? A Replication of Herring (2009)

Dragana Stojmenovska,[a] Thijs Bol,[a] and Thomas Leopold[a]

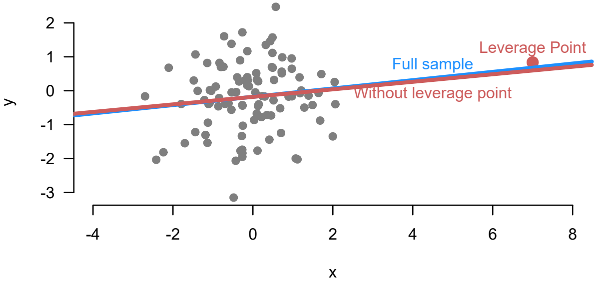
Abstract

In an influential article published in the *American Sociological Review* in 2009, Herring finds that diverse workforces are beneficial for business. His analysis supports seven out of eight hypotheses on the positive effects of gender and racial diversity on sales revenue, number of customers, perceived relative market share, and perceived relative profitability. This comment points out that Herring's analysis contains two errors. First, missing codes on the outcome variables are treated as substantive codes. Second, two control variables—company size and establishment size—are highly skewed, and this skew obscures their positive associations with the predictor and outcome variables. We replicate Herring's analysis correcting for both errors. The findings support only one of the original eight hypotheses, suggesting that diversity is nonconsequential, rather than beneficial, to business success.

A Sociological Cautionary Tale

In our correspondence with Herring, he did not offer a definitive explanation for these discrepancies, but indicated that he may have treated all codes other than "not applicable" (−999) as substantive codes. Given (1) the large difference between his sample size and the number of valid observations in the NOS, and (2) the large number of missing values due to reasons other than "not applicable"—in particular for sales revenue and number of customers—this coding error appears likely to account for much of the discrepancies. This means, for example, that 206 business organizations in which the sales revenue was unknown were treated as if they had sales of 88,888,888,888 US Dollars. Yet, even when we replicated this error (i.e., keeping all organizations with missing values other than −999 in our sample), we were unable to recover Herring's sample sizes, although the differences were smaller.

Leverage Point Definition



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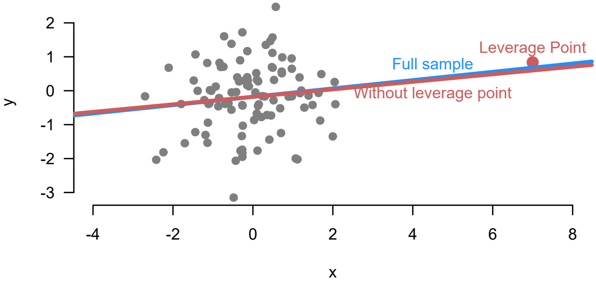

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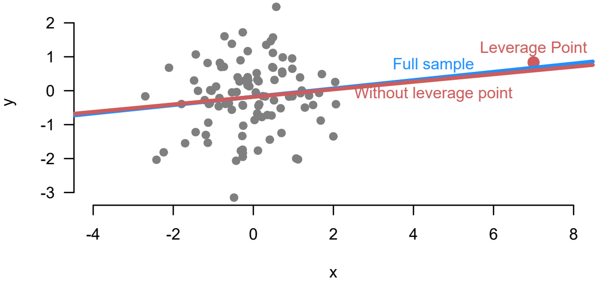

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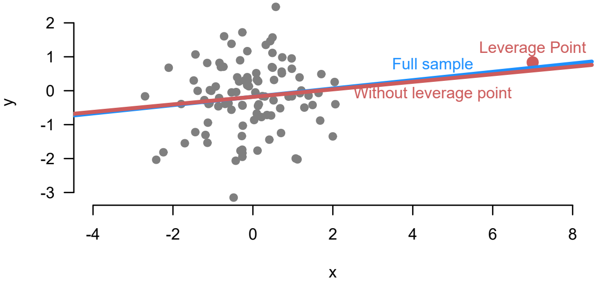

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Leverage Points: Hat values

To measure leverage in multivariate data we will go back to the hat matrix **H**:

$$\hat{\mathbf{y}} = \mathbf{X}\hat{\beta} = \mathbf{X}\left(\mathbf{X}'\mathbf{X}\right)^{-1}\mathbf{X}'\mathbf{y} = \mathbf{H}\mathbf{y}$$

H is $n \times n$, symmetric, and idempotent. It generates fitted values as follows:

$$\hat{y}_i = \mathbf{h}_i'\mathbf{y} = \begin{bmatrix} h_{i,1} & h_{i,2} & \cdots & h_{i,n} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \sum_{j=1}^{n} h_{i,j} y_j$$

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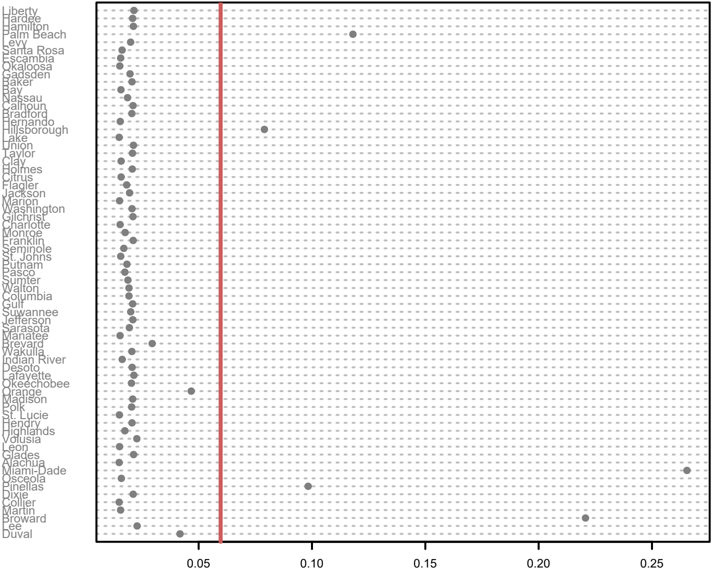
- Intuitively, the hat values measure how far a unit's vector of characteristics $\mathbf{x}_i$ is from the vector of means of **X**
- Rule of thumb: examine hat values greater than $2(k+1)/n$

Appendix: Facts about Hat Values

- $\sum_{i=1}^{n} h_i = k + 1$
- $1/n \geq h_i \geq 1$ for all i
- $\text{Var}[\widehat{u}_i] = (1 - h_i)\sigma^2$
- With a simple linear regression, we have

$$h_i = \frac{1}{n} + \frac{(X_i - \overline{X})^2}{\sum_{j=1}^{n}(X_j - \overline{X})^2}$$

Buchanan hats



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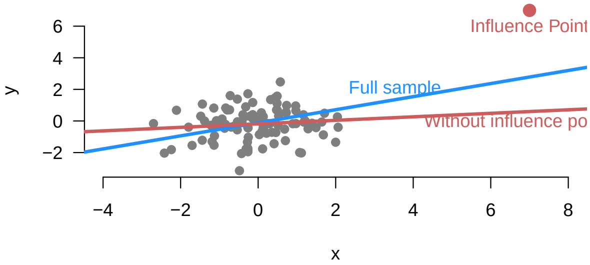

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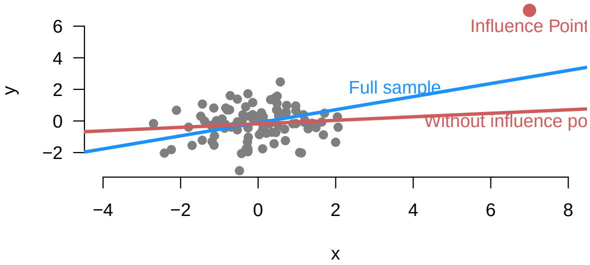

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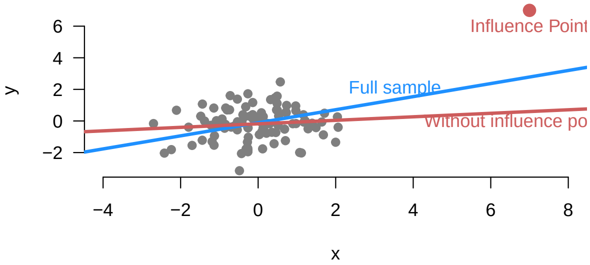

- An influence point is one that is both an outlier (extreme in Y) and a leverage point (extreme in X).
- Causes the regression line to move toward it.

Detecting Influence Points/Bad Leverage Points

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- D_{ij} is called the DFbeta, which measures the influence of observation i on the estimated coefficient for the jth explanatory variable.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

$$D^*_{ij} = \frac{\hat{\beta}_j - \hat{\beta}_{j(-i)}}{\hat{SE}_{-i}(\hat{\beta}_j)}$$

where D^*_{ij} is called DFbetaS.

Standardized Influence

To make comparisons across coefficients, it is helpful to scale D_{ij} by the estimated standard error of the coefficients:

$$D_{ij}^* = \frac{\hat{\beta}_j - \hat{\beta}_{j(-i)}}{\hat{SE}_{-i}(\hat{\beta}_j)}$$

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- In R: `dfbetas(model)`

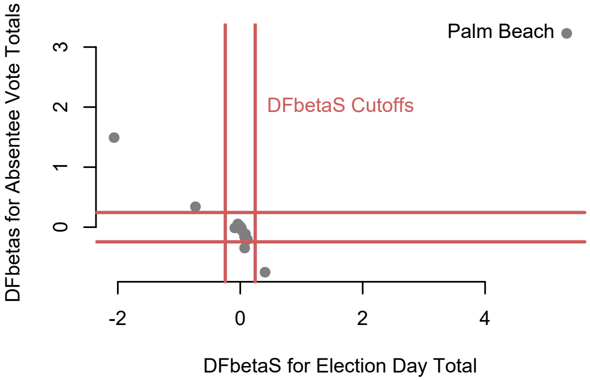
Buchanan influence

```
##
## Coefficients:
##                Estimate Std. Error t value Pr(>|t|)
## (Intercept)  -2.935e+01  5.520e+01  -0.532  0.59686
## edaytotal     1.100e-03  4.797e-04   2.293  0.02529 *
## absnbuchanan  6.895e+00  2.129e+00   3.238  0.00195 **
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 317.2 on 61 degrees of freedom
##   (3 observations deleted due to missingness)
## Multiple R-squared:  0.5361, Adjusted R-squared:  0.5209
## F-statistic: 35.24 on 2 and 61 DF,  p-value: 6.711e-11
```

Buchanan influence

```
##     (Intercept)     edaytotal  absnbuchanan
## 1  0.3454475146  0.4050504921 -0.7505222758
## 2 -0.0234266617 -0.0241000045 -0.0131672181
## 3  0.0650795039 -0.7319311820  0.3401669862
## 4 -0.0333980968  0.0133802934 -0.0087505576
## 5 -0.0397626659 -0.0073746223  0.0096551713
## 6 -0.0009277798  0.0001505476  0.0002210247
```

Buchanan influence



- Palm Beach county moves each of the coefficients by more than 3 standard errors!

Summarizing Influence across All Coefficients

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$$D_i = \underbrace{\frac{\hat{u}_i'^2}{k+1}}_{\text{outlyingness}} \times \underbrace{\frac{h_i}{1-h_i}}_{\text{leverage}}$$

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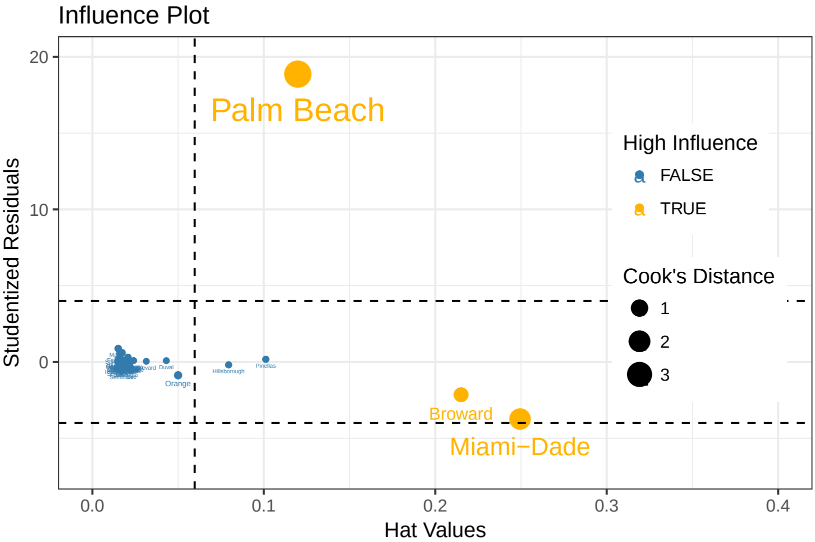
Influence Plot Buchanan



Courtesy of Erin Hartman

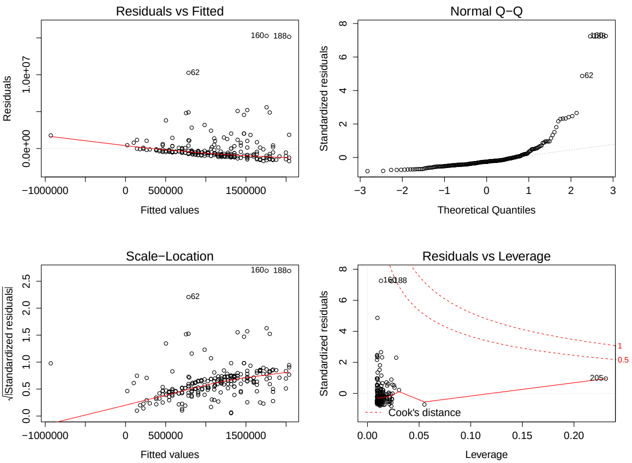
Code for Influence Plot

```
ggplot(fl_lm, aes(x = .hat, y = rstudent(fl_lm),
size = .cooksd,
col = .cooksd > 4/(nrow(fl_data) - 1 - 1),
label = fl_data$county)) +
geom_point() + geom_text(vjust = 2) +
xlab("Hat Values") + ylab("Studentized Residuals") +
geom_vline(xintercept = 2 * (fl_lm$rank - 1 + 1)/nrow(fl_data)
, linetype = 2) +
geom_hline(yintercept = c(-4, 4), linetype = 2) +
scale_color_manual("High Influence",
values = c("TRUE" = ucla_gold,
"FALSE" = ucla_blue)) +
scale_size("Cook's Distance") + theme_bw() +
theme(legend.position = c(0.9, 0.5)) + ylim(c(-7, 20)) +
xlim(c(0, 0.4)) + ggtitle("Influence Plot")
```

A Quick Function for Standard Diagnostic Plots

R Code

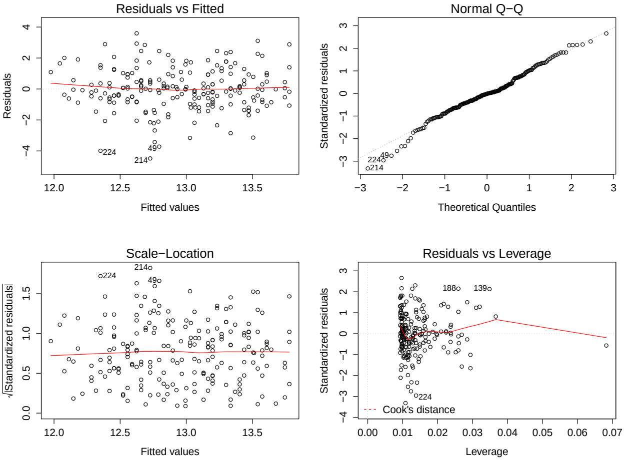
```
> par(mfrow=c(2,2))
> plot(mod1)
```



The Improved Model

R Code

```
> par(mfrow=c(2,2))
> plot(mod2)
```



‘Fun With Outliers’! (via FiveThirtyEight)

NOV. 9, 2018, AT 12:20 PM

Something Looks Weird In Broward County. Here’s What We Know About A Possible Florida Recount.

By Nathaniel Rakich

Filed under 2018 Election

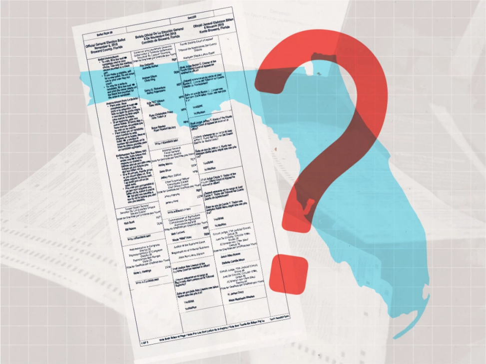


ILLUSTRATION BY FIVETHIRTYEIGHT

‘Fun With Outliers’! (via FiveThirtyEight)

The Florida U.S. Senate race is still too close to call. According to unofficial results on the Florida Department of State website at 11:45 a.m. Eastern on Friday, Nov. 9, Republican Gov. Rick Scott led Democratic Sen. Bill Nelson by 15,046 votes — or 0.18 percentage points. We’re watching that margin closely because if it stays about that small, it will trigger a recount. It’s already narrowed since election night, when Scott initially declared victory with a 56,000-vote lead.

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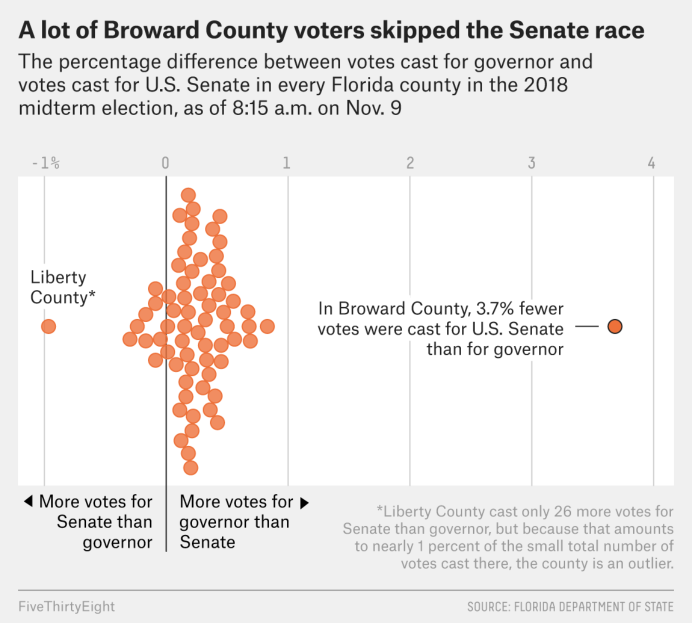

'Fun With Outliers'! (via FiveThirtyEight)

Broward County's undervote rate is *way* out of line with every other county in Florida, which exhibited, at most, a 0.8-percent difference. (There is one outlier — the sparsely populated Liberty County — where votes cast in the Senate race were 1 percent higher than in the governor race, but there we're talking about a difference of 26 votes, not more than 26,000, as is the case in Broward.)

To put in perspective what an eye-popping number of undervotes that is, more Broward County residents voted for the down-ballot constitutional offices of chief financial officer and state agriculture commissioner than U.S. Senate — an extremely high-profile election in which $181 million was spent. Generally, the higher the elected office, the less likely voters are to skip it on their ballots. Something sure does seem off in Broward County; we just don't know what yet.

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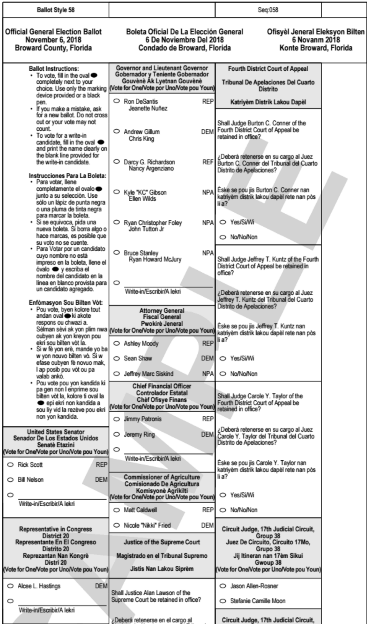
Ballot Style 58 | Seq:058

Official General Election Ballot
November 6, 2018
Broward County, Florida

Boleta Oficial De La Elección General
6 De Noviembre Del 2018
Condado de Broward, Florida

Ofisyèl Jeneral Eleksyon Bilten
6 Novanm 2018
Konte Broward, Florida

Ballot Instructions:

- To vote, fill in the oval ⬬ completely next to your choice. Use only the marking device provided or a black pen.
- If you make a mistake, ask for a new ballot. Do not cross out or your vote may not count.
- To vote for a write-in candidate, fill in the oval ⬬ and print the name clearly on the blank line provided for the write-in candidate.

Instrucciones Para La Boleta:

- Para votar, llene completamente el ovalo ⬬ junto a su selección. Use sólo un lápiz de punta negra o una pluma de tinta negra para marcar la boleta.
- Si se equivoca, pida una nueva boleta. Si borra algo o hace marcas, es posible que su voto no se cuente.
- Para Votar por un candidato cuyo nombre no está impreso en la boleta, llene el óvalo ⬬ y escriba el nombre del candidato en la línea en blanco provista para un candidato agregado.

Enfòmasyon Sou Bilten Vòt:

- Pou vote, byen kolore tout andan oval ⬬ ki akote respons ou chwazi a. Sèlman sèvi ak yon plim nwa oubyen ak yon kreyon pou ekri sou bilten vòt la.
- Si w fè yon erè, mande yo ba w yon nouvo bilten vò. Si w efase oubyen fè novuo mak, l ap posib pou vòt ou pa valab ankò.
- Pou vote pou yon kandida ki pa gen non l enprime sou bilten vòt la, kolore ti oval la ⬬ epi ekri non kandida a sou liy vid la rezève pou ekri non yon kandida.

United States Senator
Senador De Los Estados Unidos
Senatè Etazini
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Rick Scott — REP
- ○ Bill Nelson — DEM
- ○ ______ Write-in/Escribir/A lekri

Representative in Congress District 20
Representante En El Congreso Distrito 20
Reprezantan Nan Kongrè Distri 20
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Alcee L. Hastings — DEM
- ○ ______ Write-in/Escribir/A lekri

Governor and Lieutenant Governor
Gobernador y Teniente Gobernador
Gouvènè Ak Lyetnan Gouvènè
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Ron DeSantis / Jeanette Nuñez — REP
- ○ Andrew Gillum / Chris King — DEM
- ○ Darcy G. Richardson / Nancy Argenziano — REF
- ○ Kyle "KC" Gibson / Ellen Wilds — NPA
- ○ Ryan Christopher Foley / John Tutton Jr — NPA
- ○ Bruce Stanley / Ryan Howard McJury — NPA
- ○ ______ Write-in/Escribir/A lekri

Attorney General
Fiscal General
Pwokirè Jeneral
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Ashley Moody — REP
- ○ Sean Shaw — DEM
- ○ Jeffrey Marc Siskind — NPA

Chief Financial Officer
Controlador Estatal
Chèf Ofisye Finans
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Jimmy Patronis — REP
- ○ Jeremy Ring — DEM
- ○ ______ Write-in/Escribir/A lekri

Commissioner of Agriculture
Comisionado De Agricultura
Komisyonè Agrikilti
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Matt Caldwell — REP
- ○ Nicole "Nikki" Fried — DEM

Justice of the Supreme Court
Magistrado en el Tribunal Supremo
Jistis Nan Lakou Siprèm

Shall Justice Alan Lawson of the Supreme Court be retained in office?

¿Deberá retenerse en el cargo al

Fourth District Court of Appeal
Tribunal De Apelaciones Del Cuarto Distrito
Katriyèm Distrik Lakou Dapèl

Shall Judge Burton C. Conner of the Fourth District Court of Appeal be retained in office?

¿Deberá retenerse en su cargo al Juez Burton C. Conner del Tribunal del Cuarto Distrito de Apelaciones?

Èske se pou jis Burton C. Conner nan katriyèm distrik lakou dapèl rete nan pòs li a?

- ○ Yes/Sí/Wi
- ○ No/No/Non

Shall Judge Jeffrey T. Kuntz of the Fourth District Court of Appeal be retained in office?

¿Deberá retenerse en su cargo al Juez Jeffrey T. Kuntz del Tribunal del Cuarto Distrito de Apelaciones?

Èske se pou jis Jeffrey T. Kuntz nan katriyèm distrik lakou dapèl rete nan pòs li a?

- ○ Yes/Sí/Wi
- ○ No/No/Non

Shall Judge Carole Y. Taylor of the Fourth District Court of Appeal be retained in office?

¿Deberá retenerse en su cargo al Juez Carole Y. Taylor del Tribunal del Cuarto Distrito de Apelaciones?

Èske se pou jis Carole Y. Taylor nan katriyèm distrik lakou dapèl rete nan pòs li a?

- ○ Yes/Sí/Wi
- ○ No/No/Non

Circuit Judge, 17th Judicial Circuit, Group 38
Juez De Circuito, Circuito 17Mo, Grupo 38
Jij Itineran nan 17èm Sikui Gwoup 38
(Vote for One/Vote por Uno/Vote pou Youn)

- ○ Jason Allen-Rosner
- ○ Stefanie Camille Moon

Circuit Judge, 17th Judicial Circuit,

SAMPLE

'Fun With Outliers'! (via FiveThirtyEight)

Sun Sentinel reporters talked with a ballot expert, who said that some voters may not have noticed the Senate race (perhaps thinking it was just part of the ballot instructions) and started filling out their ballot with the governor race instead. That theory is supported by a data consultant who's worked for several political campaigns in Florida, who found that the parts of Broward County that fall in the 24th Congressional District did see higher levels of undervoting than other parts of the county. That might be because the 24th District was uncontested, which according to Florida law means that the congressional race did not appear on the ballot at all. As you can see in the sample ballot above, the congressional race would also appear in the lower-left corner on many ballots, along with the Senate race. In districts where there was no congressional race on the ballot, however, that corner would have looked even emptier, perhaps making it easier for voters to inadvertently skip over the Senate race.

'Fun With Outliers'! (via FiveThirtyEight)

One possible reason for the discrepancy is poor ballot design. Broward County ballots listed the U.S. Senate race first, right after the ballot instructions. But that pushed the U.S. Senate race to the far bottom left of the ballot, where voters may have skimmed over it, while the governor's race appears at the top of the ballot's center column, immediately to the right of the instructions.

'Fun With Outliers'! (via FiveThirtyEight)

An alternative explanation is that an error with the vote-tabulating machines in Broward County caused them to sometimes not read people's votes for U.S. Senate. If that's true, we would probably only find out if there is a manual recount. According to Florida law, any election that's within half a percentage point (as this one currently is) triggers a machine recount; then, after the machine recount, if the race is within a *quarter* of a percentage point, it goes to a much more complex manual recount — a.k.a. each ballot is recounted by hand. As long as the machine recount doesn't change the Senate results too much (barring a surprise in the remaining ballots in Broward and Palm Beach), it looks like that's where we're headed. In addition, Republican former Rep. Ron DeSantis and Democratic Tallahassee Mayor Andrew Gillum are separated by just 0.44 points in the governor's race, so that could go to a machine recount, too.

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- Outliers, Leverage and Influence Points

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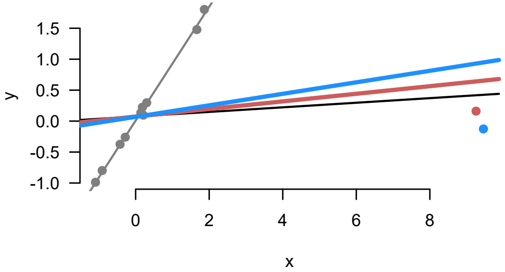
Next Time: Robust Regression

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This Week
  - diagnosing problems and troubleshooting the linear model
  - unusual and influential data $\rightarrow$ robust estimation
  - non-linearity $\rightarrow$ generalized additive models
  - unusual errors $\rightarrow$ sandwich SEs
- Next Week
  - frameworks for causal inference
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

Limitations of the Standard Tools



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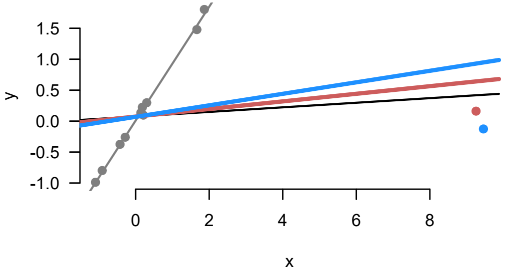

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- Red line drops the red influence point
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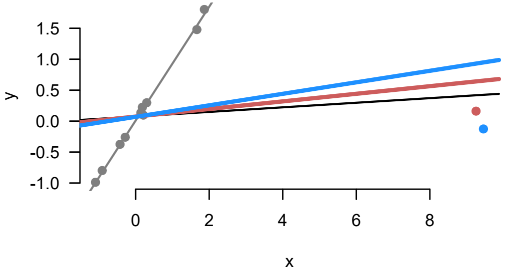

- What happens when there are two influence points?
- Red line drops the red influence point
- Blue line drops the blue influence point
- Neither of the "leave-one-out" approaches helps recover the line

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- How comforting should this be? Not very.
- The Linear point is an artificial restriction. It means the estimator has to be of the form $\hat{\beta} = \mathbf{W}y$ but why only use those?
- With normality assumption we get Best Unbiased Estimator (BUE) which is quite comforting when $n \gg p$ (number of observations much larger than number of variables).

This Point is Not Obvious

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"[Even without normally distributed errors] OLS coefficient estimators remain unbiased and efficient."
- Berry (1993)

Quotes from Rainey and Baissa (2015) presentation

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> *"[The Gauss-Markov theorem] justifies the use of the OLS method rather than using a variety of competing estimators"*
>
> *- Wooldridge (2013)*

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> *"We need not look for another linear unbiased estimator, for we will not find such an estimator whose variance is smaller than the OLS estimator"*
>
> *- Gujarati (2004)*

Quotes from Rainey and Baissa (2015) presentation

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> *"The Gauss-Markov theorem allows us to have considerable confidence in the least squares estimators."*
>
> *- Berry and Feldman (1993)*

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The Gauss-Markov theorem has convinced researchers in political science that as long as . . . the Gauss-Markov assumptions are met, the distribution of the errors is unimportant. But the distribution of the errors is crucial to a linear regression analysis. Deviations from normality, especially large deviations commonly found in regression models in political science, can devastate the performance of least squares compared to alternative estimators

- Baissa and Rainey (2018)

Robustly Estimating a Location

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- We can measure sensitivity with the influence function which measures change in estimator based on corruption in one datapoint.

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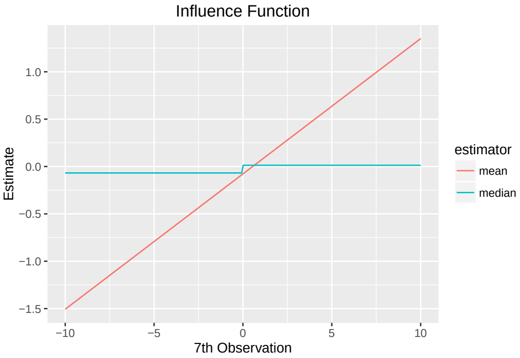

Example from Fox

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- The median has a breakdown point of 50% because half the data can be bad without causing the median to become completely unstuck.

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- Other objectives include the Huber objective and Tukey's biweight objective which have different properties.
- Calculating robust M estimators often requires an iterative procedure and a careful initialization.

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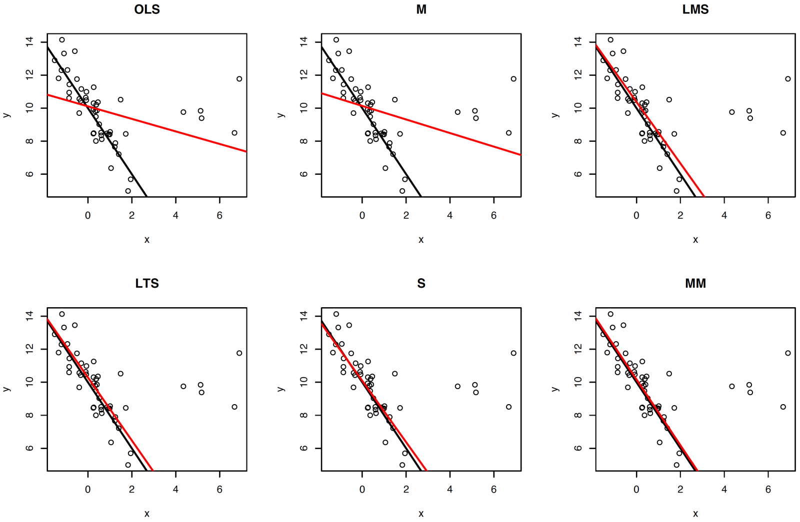
  - MM-estimator: what I recommend in practice (more in appendix)
- You can find an asymptotic covariance matrix for M-estimators but I would bootstrap it if possible as the asymptotics kick in slowly.

```
library(MASS)
set.seed(588)
n <- 50
x <- rnorm(n)
y <- 10 - 2*x + rnorm(n)
x[1:5] <- rnorm(5, mean=5)
y[1:5] <- 10 +  rnorm(5)
ols.out <- lm(y~x)
m.out <- rlm(y~x, method="M")
lms.out <- lqs(y~x, method="lms")
lts.out <- lqs(y~x, method="lts")
s.out <- lqs(y~x, method="S")
mm.out <- rlm(y~x, method="MM")
```

Simulation Results



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- In most cases I personally would start with OLS, do diagnostics and then consider a robust alternative. If I don't have time for diagnostics, maybe robust is better from the outset.
- See Baissa and Rainey (2018) "When BLUE is Not Best: Non-Normal Errors and the Linear Model" in *Political Science Research & Methods* for more on this topic.
- The Fox textbook Chapter 19 is also quite good on this and points out to the key references

We Covered

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- Appendix after these slides with some more formality on M-estimators.

Next Time: Nonlinearity

Appendix: Characterizing Estimator Robustness (formally)

Definition (Breakdown Point)

The breakdown point of an estimator is the smallest fraction of the data that can be changed an arbitrary amount to produce an arbitrarily large change in the estimate (Seber and Lee 2003, pg 82)

Definition (Influence Function)

Let $F_p = (1-p)F + p\delta_{\mathbf{z}_0}$ where F is a probability measure, $\delta_{\mathbf{z}_0}$ is the point mass at $\mathbf{z}_0 \in \mathbb{R}^k$, and $p \in (0,1)$.
Let $T(\cdot)$ be a statistical functional. The influence function of T is

$$IF(\mathbf{z}_0; T, F) = \lim_{p \downarrow 0} \frac{T(F_p) - T(F)}{p}$$

The influence function is a function of $\mathbf{z}_0$ given T and F. It describes how T changes with small amounts of contamination at $\mathbf{z}_0$ (Hampel, Rousseeuw, Ronchetti, and Stahel, (1986), p. 84).

Appendix: S Estimators

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An S-estimator for the regression model is defined as the values of $\hat{\boldsymbol{\beta}}_S$ and s that minimize s subject to the constraint:

$$\frac{1}{n}\sum_{i=1}^{n}\rho\left(\frac{y_i - \mathbf{x}_i'\hat{\boldsymbol{\beta}}_S}{s}\right) \geq K$$

where K is user-defined constant (typically set to 0.5) and $\rho : \mathbb{R} \to [0,1]$ is a function with the following properties (Davies, 1990, p. 1653):

1. $\rho(0) = 1$
2. $\rho(u) = \rho(-u), u \in \mathbb{R}$
3. $\rho : \mathbb{R}_+ \to [0,1]$ is nonincreasing, continuous at 0, and continuous on the left
4. for some $c > 0$, $\rho(u) > 0$ if $|u| < c$ and $\rho(u) = 0$ if $|u| > c$

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Good properties, but costly to compute (usually impossible to compute exactly).

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This Week
  - diagnosing problems and troubleshooting the linear model
  - unusual and influential data $\rightarrow$ robust estimation
  - non-linearity $\rightarrow$ generalized additive models
  - unusual errors $\rightarrow$ sandwich SEs
- Next Week
  - frameworks for causal inference
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression $\rightarrow$ causal inference

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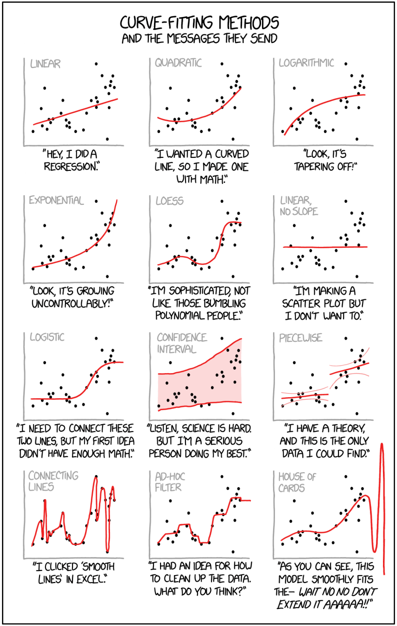
- Many forms of machine learning are best thought of as nonparametric regressions in higher dimensions.
- We can often see poor fits of the conditional expectation function in the residuals, but let's instead just do diagnosis by treatment and look at some of these other approaches to modeling.



Thanks XKCD for having a comic for everything!

Bias-Variance Tradeoff

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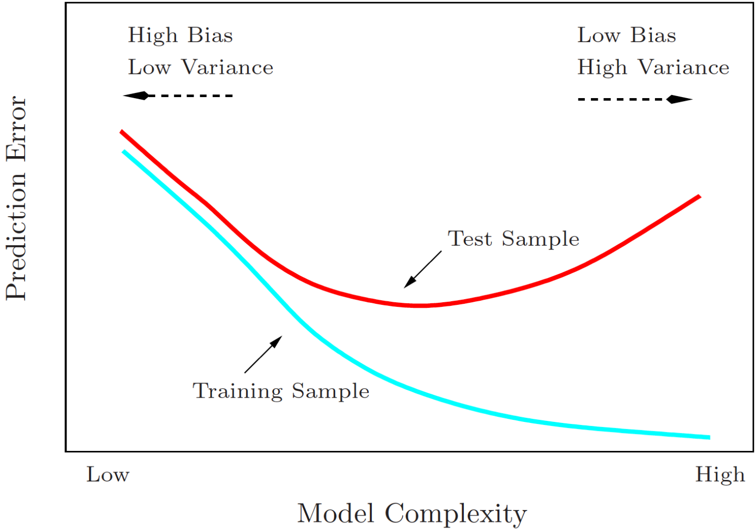
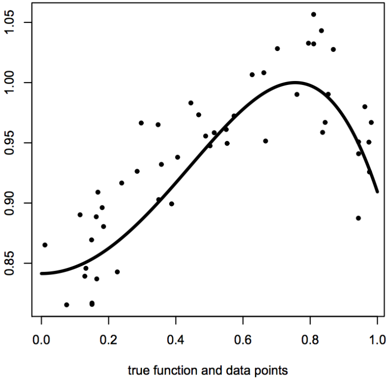


Example Synthetic Problem

$$y = \sin(1 + x^2) + \epsilon$$



This section adapted from slides by Radford Neal.

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- The model is now:

$$y = f(x, \beta) + \epsilon$$

$$f(x, \beta) = \beta_0 + \sum_{j=1}^{m-1} \beta_j \phi_j(x) = \beta^T \phi(x)$$

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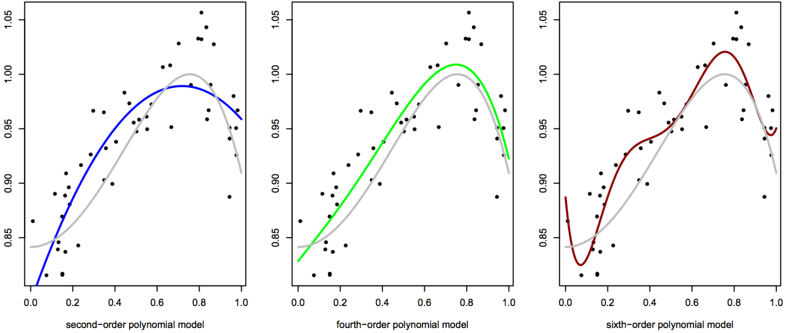

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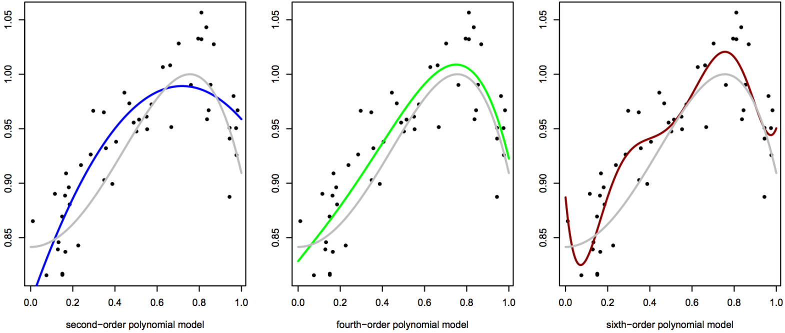

It appears that the last model is too complex and is overfitting a bit.

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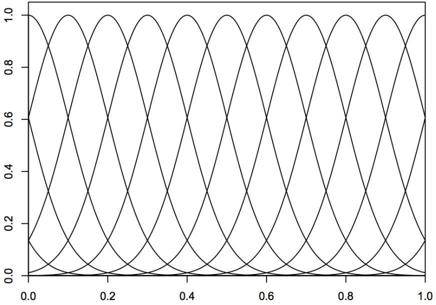
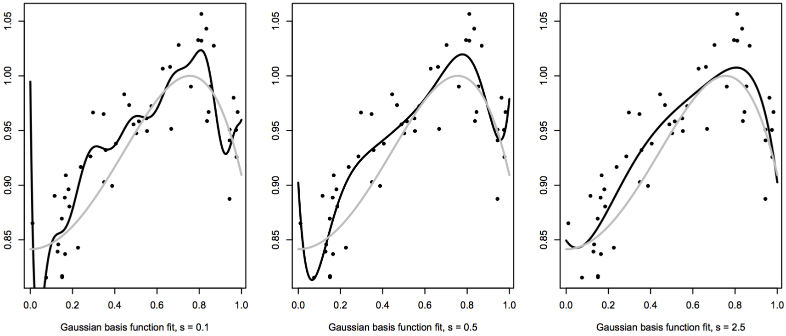


Gaussian basis functions, s = 0.1

Gaussian Basis Fits



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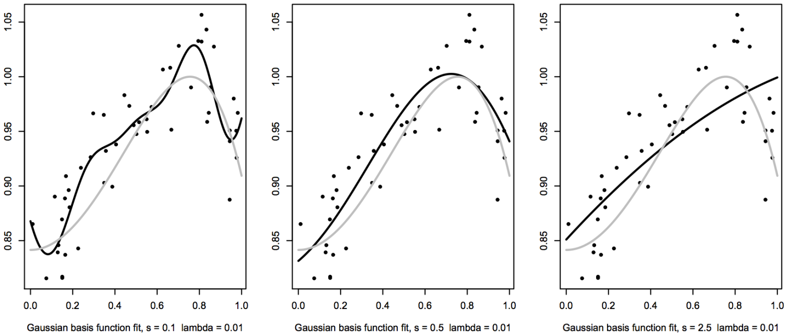
- The trick in general is how to set λ

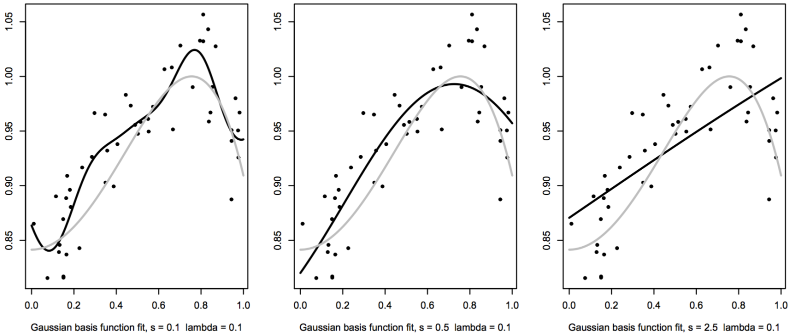
Results

Here are the results with $\lambda = 0.01$:



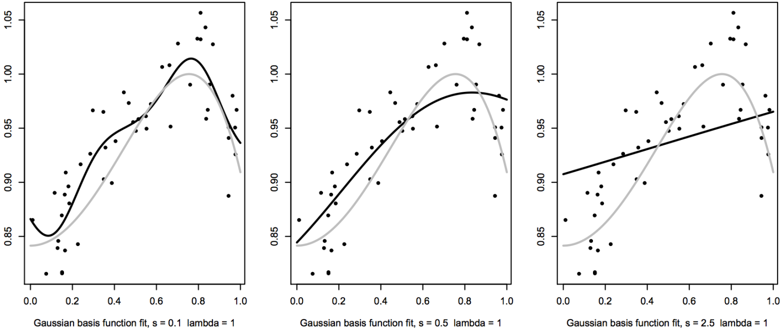
Results

Here are the results with $\lambda = 0.1$:



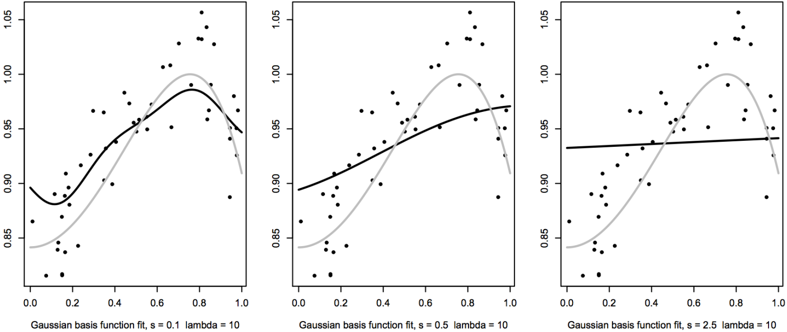
Results

Here are the results with $\lambda = 1$:



Results

Here are the results with $\lambda = 10$:



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For GAMs, we maintain additivity, but instead of imposing termwise linearity we allow flexible functional forms for each explanatory variable, where $s_1(\cdot), s_2(\cdot),$ and $s_3(\cdot)$ are smooth functions that are estimated from the data:

Generalized Additive Models (GAM)

Recall the linear model,

$$y_i = \beta_0 + x_{1i}\beta_1 + x_{2i}\beta_2 + x_{3i}\beta_3 + u_i$$

For GAMs, we maintain additivity, but instead of imposing termwise linearity we allow flexible functional forms for each explanatory variable, where $s_1(\cdot), s_2(\cdot),$ and $s_3(\cdot)$ are smooth functions that are estimated from the data:

$$y_i = \beta_0 + s_1(x_{1i}) + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

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- $s_j(\cdot)$ are usually estimated with locally weighted regression smoothers or cubic smoothing splines (but many approaches are possible)
- They do NOT give you a set of regression parameters $\hat{\beta}$. Instead one obtains a graphical summary of how $E[Y|X_, X_2, ..., X_k]$ varies with X_1 (estimates of $s_j(\cdot)$ at every value of $X_{i,j}$)

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- Theory and estimation are somewhat involved, but they are easy to use:
  - ```
    gam.out <- gam(y~s(x1)+s(x2)+x3)
    plot(gam.out)
    ```
  - Multiple functions but I recommend `mgcv` package
    ```

Generalized Additive Models (GAM)

The GAM approach can be extended to allow interactions ($s_{12}(\cdot)$) between explanatory variables, but this eats up degrees of freedom so you need a lot of data.

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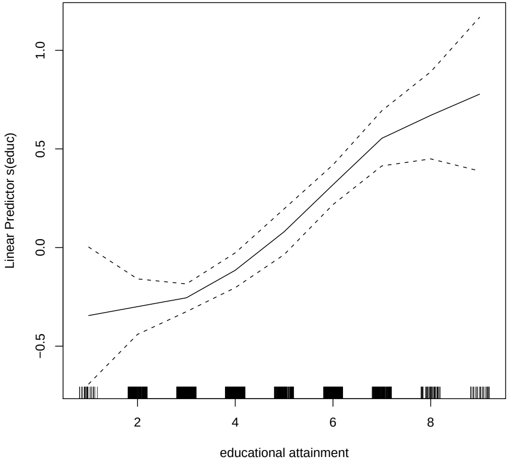
It can also be used for hybrid models where we model some variables as parametrically and other with a flexible function:

$$y_i = \beta_0 + \beta_1 x_{1i} + s_2(x_{2i}) + s_3(x_{3i}) + u_i$$

GAM Fit to Attitudes Toward Immigration



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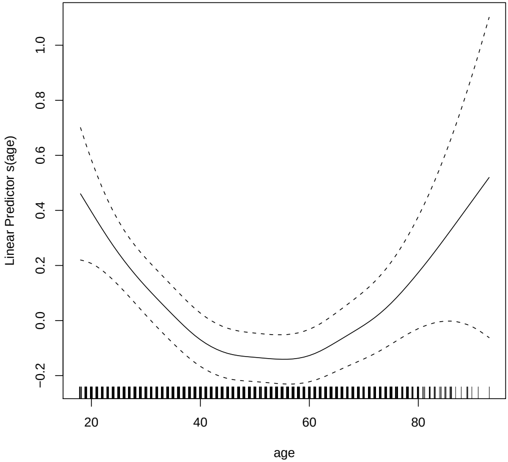

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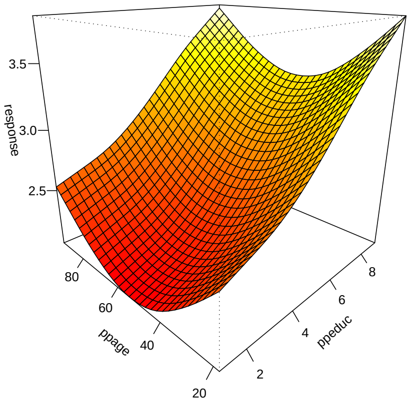

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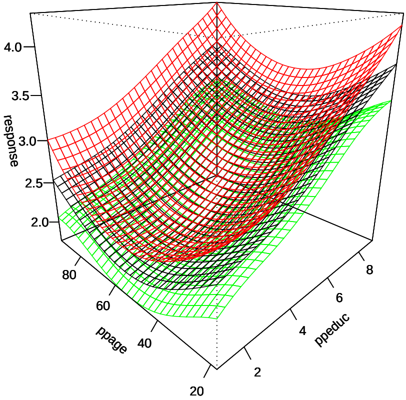

red/green are +/− 2 s.e.

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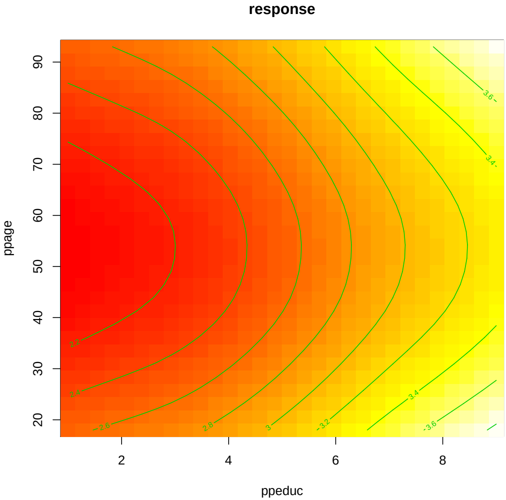

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- However, be wary of the global properties of transformations and polynomials
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- NB: it is okay if you didn't follow all of this today! GAMs are tricky.

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Next Time: Clustering

Where We've Been and Where We're Going...

- Last Week
  - multiple regression
- This Week
  - diagnosing problems and troubleshooting the linear model
  - unusual and influential data → robust estimation
  - non-linearity → generalized additive models
  - unusual errors → sandwich SEs
- Next Week
  - frameworks for causal inference
- Long Run
  - probability → inference → regression → causal inference

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  - $\rightsquigarrow$ violation of homoskedasticity
- Called clustering or clustered dependence

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- Ignoring clustering is "cheating": units not independent

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- Variance of the composite error is σ^2:

$$\begin{aligned} \text{Var}[\varepsilon_{ij}] &= \text{Var}[v_j + u_{ij}] \\ &= \text{Var}[v_j] + \text{Var}[u_{ij}] \\ &= \rho\sigma^2 + (1-\rho)\sigma^2 = \sigma^2 \end{aligned}$$

Lack of Independence

- Covariance between two units i and s in the same cluster is $\rho\sigma^2$:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sj}] = \rho\sigma^2$$

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- Zero covariance of two units i and s in different clusters j and k:

$$\text{Cov}[\varepsilon_{ij}, \varepsilon_{sk}] = 0$$

Example Covariance Matrix

$$\boldsymbol{\varepsilon} = \left[\begin{array}{cccccc} \varepsilon_{1,1} & \varepsilon_{2,1} & \varepsilon_{3,1} & \varepsilon_{4,2} & \varepsilon_{5,2} & \varepsilon_{6,2} \end{array} \right]'$$

$$\text{Var}[\varepsilon] = \boldsymbol{\Sigma} = \begin{bmatrix} \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho & 0 & 0 & 0 \\ \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 & \sigma^2 \cdot \rho \\ 0 & 0 & 0 & \sigma^2 \cdot \rho & \sigma^2 \cdot \rho & \sigma^2 \end{bmatrix}$$

Appendix: Example 6 Units, 2 Clusters

$\boldsymbol{\varepsilon} = [\ \varepsilon_{1,1}\quad \varepsilon_{2,1}\quad \varepsilon_{3,1}\quad \varepsilon_{4,2}\quad \varepsilon_{5,2}\quad \varepsilon_{6,2}\ ]'$

$$V[\boldsymbol{\varepsilon}] = \boldsymbol{\Sigma} = \begin{bmatrix} V[\varepsilon_{1,1}] & Cov[\varepsilon_{2,1}, \varepsilon_{1,1}] & Cov[\varepsilon_{3,1}, \varepsilon_{1,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{2,1}] & V[\varepsilon_{2,1}] & Cov[\varepsilon_{3,1}, \varepsilon_{2,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{3,1}] & Cov[\varepsilon_{2,1}, \varepsilon_{3,1}] & V[\varepsilon_{3,1}] & . & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{4,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{4,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{4,2}] & V[\varepsilon_{4,2}] & . & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{5,2}] & Cov[\varepsilon_{4,2}, \varepsilon_{5,2}] & V[\varepsilon_{5,2}] & . \\ Cov[\varepsilon_{1,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{2,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{3,1}, \varepsilon_{6,2}] & Cov[\varepsilon_{4,2}, \varepsilon_{6,2}] & Cov[\varepsilon_{5,2}, \varepsilon_{6,2}] & V[\varepsilon_{6,2}] \end{bmatrix}$$

$$= \begin{bmatrix} \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho & 0 & 0 & 0 \\ \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma^2 & \sigma^2\cdot\rho & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2 & \sigma^2\cdot\rho \\ 0 & 0 & 0 & \sigma^2\cdot\rho & \sigma^2\cdot\rho & \sigma^2 \end{bmatrix}$$

which can be verified as follows:

- $V[\varepsilon_{ij}] = V[v_j + u_{ij}] = V[v_j] + V[u_{ij}] = \rho\sigma^2 + (1-\rho)\sigma^2 = \sigma^2$
- $Cov[\varepsilon_{ij}, \varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] - E[\varepsilon_{ij}]E[\varepsilon_{lj}] = E[\varepsilon_{ij}\varepsilon_{lj}] = E[(v_j + u_{ij})(v_j + u_{lj})]$
  $= E[{v_j}^2] + E[v_j u_{ij}] + E[v_j u_{lj}] + E[u_{ij} u_{lj}]$
  $= E[{v_j}^2] + E[v_j]E[u_{ij}] + E[v_j]E[u_{lj}] + E[u_{ij}]E[u_{lj}]$
  $= E[{v_j}^2] = V[v_j] + (E[v_j])^2 = V[v_j] = \rho\sigma^2$
- $Cov[\varepsilon_{ij}, \varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] - E[\varepsilon_{ij}]E[\varepsilon_{lk}] = E[\varepsilon_{ij}\varepsilon_{lk}] = E[(v_j + u_{ij})(v_k + u_{lk})]$
  $= E[v_j v_k] + E[v_j u_{lk}] + E[v_k u_{ij}] + E[u_{ij} u_{lk}]$
  $= E[v_j]E[v_k] + E[v_j]E[u_{lk}] + E[v_k]E[u_{ij}] + E[u_{ij}]E[u_{lk}] = 0$

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- Under this clustered dependence, we can write this as:

$$\text{Var}[\hat{\boldsymbol{\beta}}|\mathbf{X}] = \left(\mathbf{X}'\mathbf{X}\right)^{-1}\left(\sum_{j=1}^{m}\mathbf{X}_j'\Sigma_j\mathbf{X}_j\right)\left(\mathbf{X}'\mathbf{X}\right)^{-1}$$

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3. The within cluster correlation is then estimated as: $\hat{\rho} = \frac{\widehat{\sigma}^2 - \widehat{\tilde{\sigma}}^2}{\widehat{\sigma}^2}$

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- There are multiple implementations in `R` including `multiwayvcov:cluster.vcov` and `sandwich::vcovCL`

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- There are numerous alternative clustered standard error variants out there.

Concluding Thoughts on Diagnostics

Residuals are important. Look at them.

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Next week: Frameworks for Causal Inference!