Week 3: Learning from Random Samples

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Princeton

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[1]These slides are heavily influenced by Matt Blackwell, Adam Glynn, Justin Grimmer and Jens Hainmueller. Some illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

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- Moving forward this is going to be very important. Why? Because we are going to want to estimate the population conditional expectation in regression.

Primary Goals for This Week

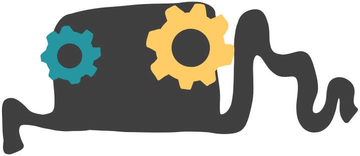
We want to be able to interpret the numbers in this table (and a couple of numbers that can be derived from these numbers).

Table 1. Mean Level of Anger Toward A Black Family Moving in Next Door, by Region (Whites Only)

Region	Experimental Condition: Baseline	Experimental Condition: Black Family	Estimated Percent Angry
Non-South	2.28[a]	2.24	0
	(.07)	(.05)	
	425[b]	461	
South	1.95	2.37	42
	(.06)	(.08)	
	139	136	

[a]Standard error of the estimate.
[b]Number of cases.

An Overview



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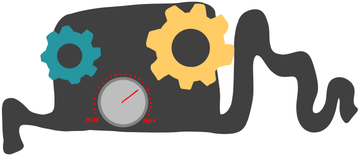

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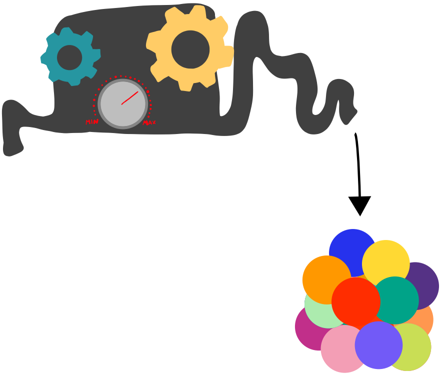

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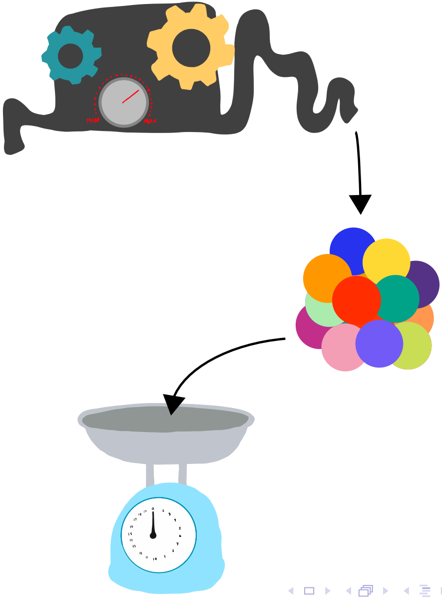

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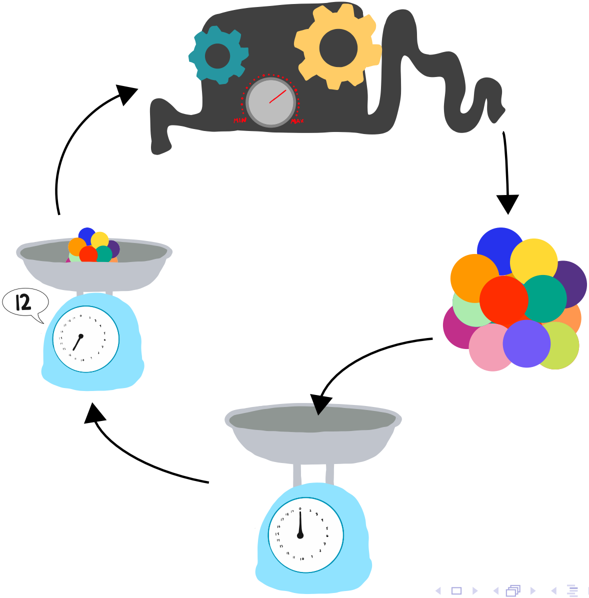

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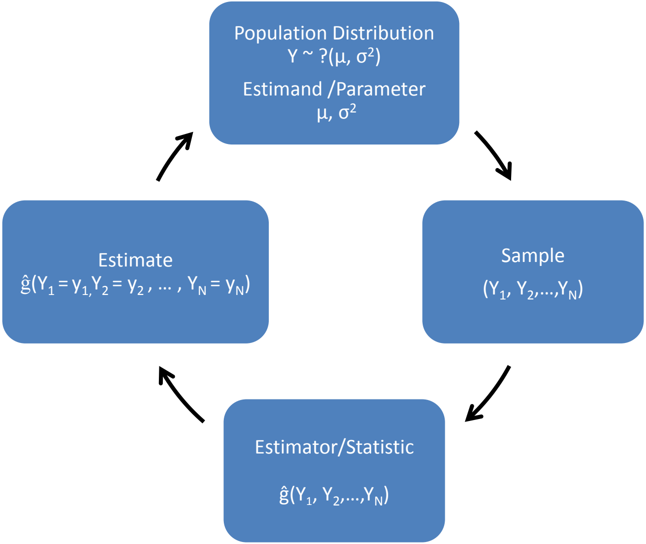

Population Distribution
Y ~ ?(μ, σ2)
Estimand /Parameter
μ, σ2
Sample
(Y1, Y2,…,YN)
Estimator/Statistic
ĝ(Y1, Y2,…,YN)
Estimate
ĝ(Y1 = y1, Y2 = y2 , … , YN = yN)

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- e.g. the distribution of votes for Hillary Clinton in the population of registered voters in the United States. This is an example of a finite population.
- Sometimes the population will be more abstract, such as the population of all possible television ads. This is an example of an infinite population.
- With either a finite or infinite population our main goal in inference is to learn about the population distribution or particular aspects of that distribution, like the mean or variance, which we call a population parameter (or just parameter).

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- Instead, we will often make a parametric assumption and assume that the formula for f is known up to some unknown parameters.
- Thus, f has two parts: the known part which is the formula for the pmf/pdf (sometimes called the parametric model and comes from the distributional assumptions) and the unknown part, which are the parameters, θ.

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- Probability tells us what types of samples we should expect for different values of θ.
- For some problems, such as estimating the mean of a distribution, we actually won't need to specify a parametric model for the distribution allowing us to take an agnostic view of statistics.

Using Random Samples to Estimate Population Parameters

Let's look at possible estimators $\widehat{\mu}$ for the (unobserved) mean income in the population μ.

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Our estimators, $\widehat{\mu}$, are functions of $Y_1, \ldots, Y_n$ and will therefore be random variables with their own probability distributions.

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  This occurs whenever we are interested in making causal inferences.

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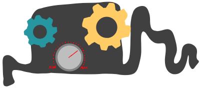

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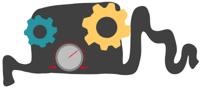

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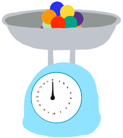

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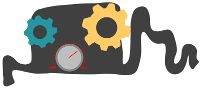

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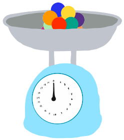
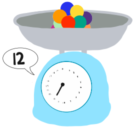


- Estimates are particular values of estimators that are realized in a given sample (e.g. sample mean): $\frac{1}{n}\sum_{i=1}^{n} y_i$



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- Even without a full probability model we can estimate particular properties of a distribution such as the mean $E[Y_i] = \mu$ or the variance $V[Y_i] = \sigma^2$
- An estimator $\hat{\theta}$ of some parameter θ, is a function of the sample $\hat{\theta} = h(Y_1, \ldots, Y_n)$ and thus is a random variable.

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2. Calculate the sample mean.
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4. Plot the sampling distribution of the sample means (maybe as a histogram).

Repeated Sampling Procedure

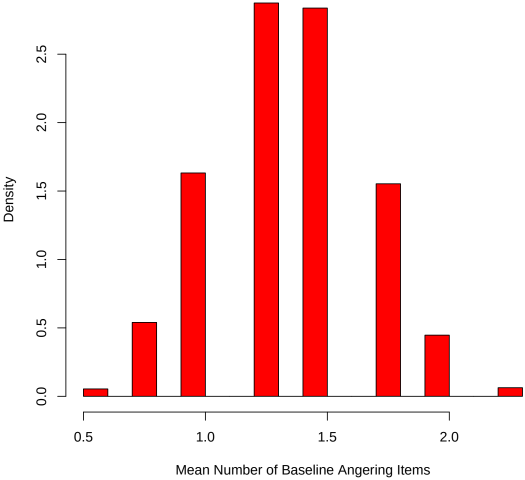
```
# population data

ypop <- c(rep(0,0),rep(1,17),rep(2,10),rep(3,4))

# simulate the sampling distribution of the sample mean

SamDistMeans <- replicate(10000, mean(sample(ypop,size=4,repla
```

Sampling Distribution of the Sample Mean



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2. Calculate the sample standard deviation.
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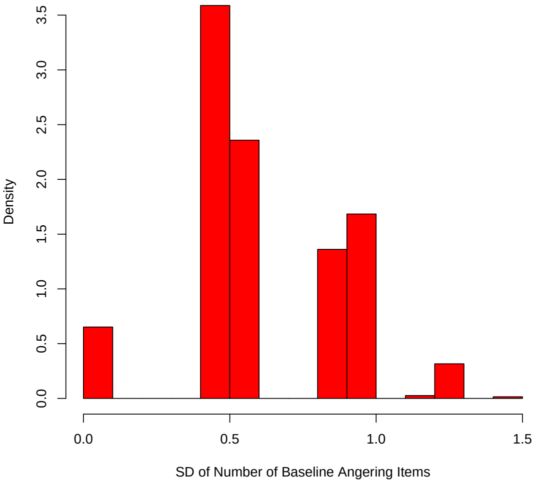
4. Plot the sampling distribution of the sample standard deviations (maybe as a histogram).

Sampling Distribution of the Sample Standard Deviation



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Two Points of Potential Confusion:

- Each sampling distribution has its own standard deviation, and therefore its own standard error. (.35 for mean, .30 for sd)
- Some people refer to an estimated standard error as the standard error.

Bootstrapped Sampling Distributions

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Because we will not have actual sampling distributions, we cannot actually calculate standard errors (we can only approximate them).

Southern Responses to Baseline List

# of angering items	0	1	2	3
# of responses	2	37	66	34

```
ysam <- c(rep(0,2),rep(1,37),rep(2,66),rep(3,34))

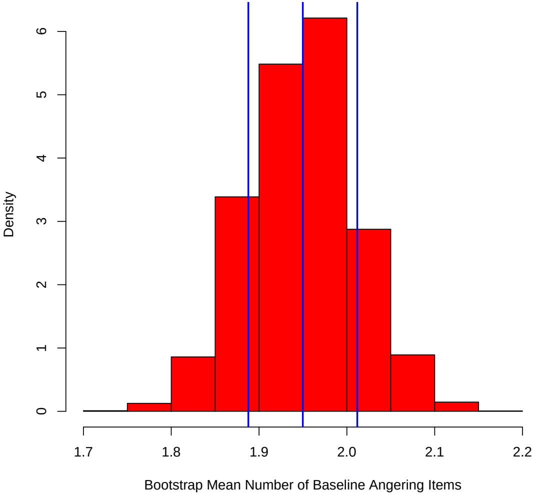
mean(ysam) # sample mean

# Bootstrapping
BootMeans <- replicate(
10000,
mean(sample(ysam,size=139,replace=TRUE))
)

sd(BootMeans) # estimated standard error
```

```
> ysam <- c(rep(0,2),rep(1,37),rep(2,66),rep(3,34))
>
> mean(ysam) # sample mean
[1] 1.949640
>
> BootMeans <- replicate(
+  10000,
+  mean(sample(ysam,size=n,replace=TRUE))
+  )
>
> sd(BootMeans) # estimated standard error
[1] 0.06234988
```

Bootstrapped Sampling Distribution of the Sample Mean



Independence for Random Variables

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Recall:

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x, y) = f_X(x) f_Y(y)$$

for all x and y. We write this as $Y \perp\!\!\!\perp X$.

Independence for Random Variables

Recall:

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

for all x and y. We write this as $Y \perp\!\!\!\perp X$.

Independence implies

$$f_{Y|X}(y|x) = f_Y(y)$$

and thus

$$E[Y|X=x] = E[Y]$$

Notation for Sampling Distributions

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We say that $X_1, X_2, \ldots X_n$ are identically and independently distributed from a population distribution with a mean ($E[X_1] = \mu$) and a variance ($V[X_1] = \sigma^2$).

Then we write $X_1, X_2, \ldots X_n \sim_{i.i.d} \quad ?(\mu, \sigma^2)$

Describing the Sampling Distribution for the Mean

We would like a full description of the sampling distribution for the mean, but it will be useful to separate this description into three parts.

If we assume that $X_1, \dots X_n \sim_{i.i.d} \; ?(\mu, \sigma^2)$, then we would like to identify the following things about $\overline{X}_n$.

- $E[\overline{X}_n]$
- $V[\overline{X}_n]$
- ?

Expectation of $\overline{X}_n$

Again, let $X_1, X_2, \ldots X_n$ be identically and independently distributed from a population distribution with a mean ($E[X_1] = \mu$) and a variance ($V[X_1] = \sigma^2$). Using the properties of expectation, calculate

$$E[\overline{X}_n] = E[\frac{1}{n}\sum_{i=1} X_i]$$
$$= ?$$

Variance of $\overline{X}_n$

Again, let $X_1, X_2, \ldots X_n$ be identically and independently distributed from a population distribution with a mean ($E[X_1] = \mu$) and a variance ($V[X_1] = \sigma^2$). Using the properties of variances, calculate

$$V[\overline{X}_n] = V[\frac{1}{n}\sum_{i=1}^{n} X_i]$$
$$= ?$$

Question:

What is the standard deviation of $\overline{X}_n$, also known as the standard error of $\overline{X}_n$?

A) $\frac{\sigma}{n}$
B) $\frac{\sigma}{n-1}$
C) $\frac{\sigma}{\sqrt{n}}$
D) $\frac{\sigma}{\sqrt{n-1}}$

What about the "?"

If $X_1, \ldots, X_n \sim_{i.i.d.} N(\mu, \sigma^2)$, then

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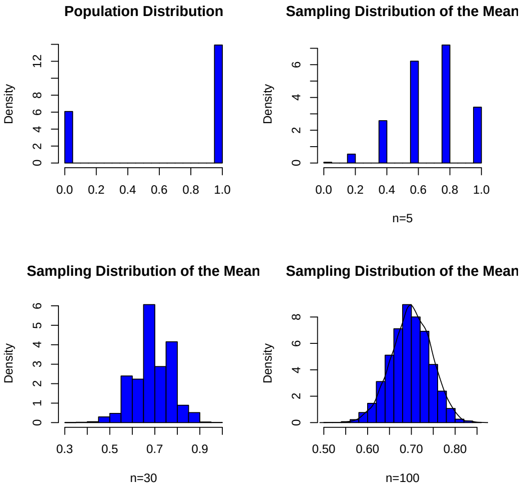
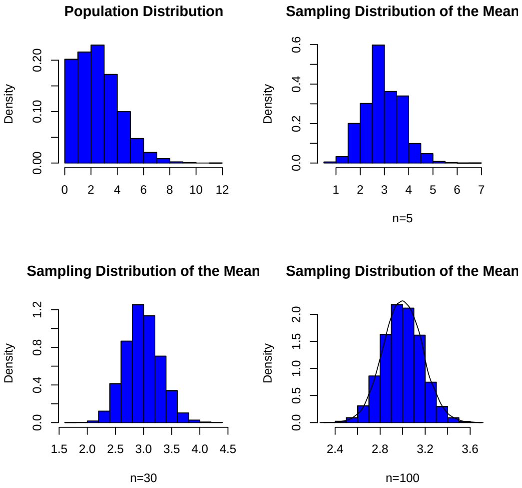
What if $X_1, \ldots, X_n$ are not normally distributed?

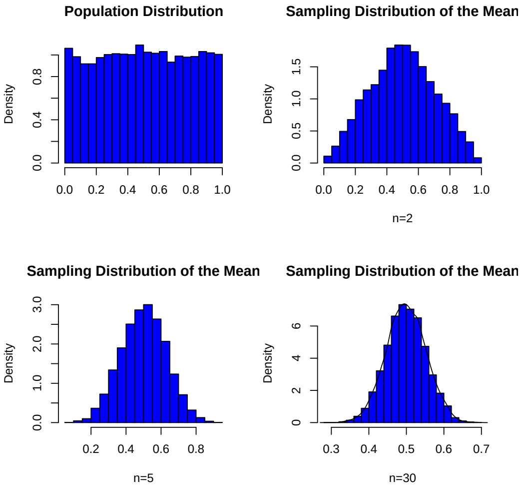
Bernoulli (Coin Flip) Distribution



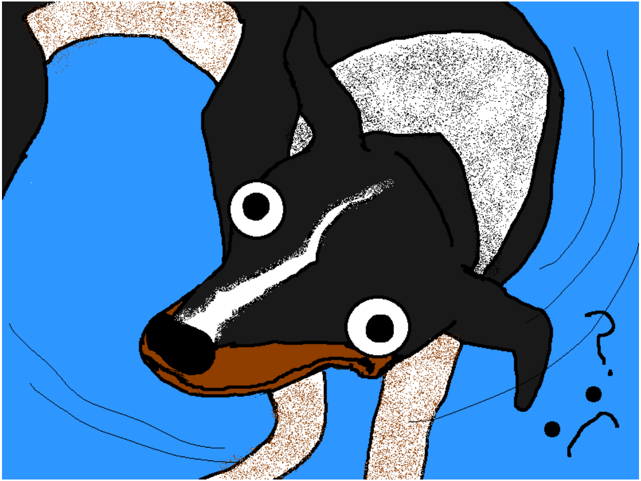
Poisson (Count) Distribution



Uniform Distribution



Why would this be true?



Images from *Hyperbole and a Half* by Allie Brosh.

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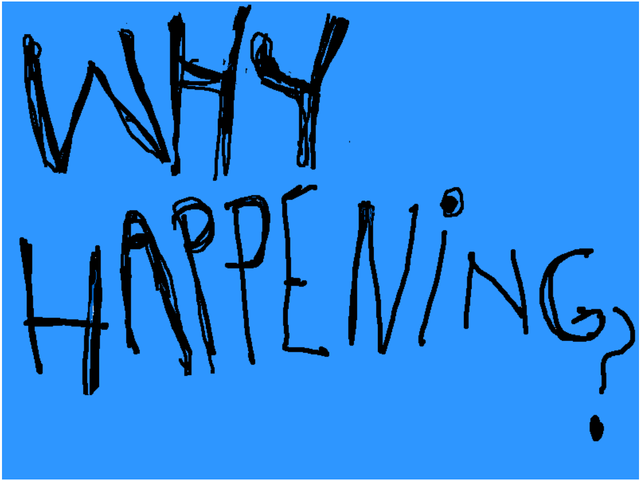

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The Central Limit Theorem

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If $X_1, \ldots, X_n \sim_{i.i.d.} ?(\mu, \sigma^2)$ and n is large, then

$$\overline{X}_n \sim_{approx} N(\mu, \frac{\sigma^2}{n})$$

The Central Limit Theorem: What are we glossing over?

To understand the Central Limit Theorem mathematically we need a few basic definitions in place first.

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Definition (Convergence in Probability)

A sequence $X_1, ..., X_n$ of random variables converges in probability towards a real number a if, for all accuracy levels $\varepsilon > 0$,

$$\lim_{n\to\infty} \Pr\left(|X_n - a| \geq \varepsilon\right) = 0$$

We write this as

$$X_n \overset{p}{\to} a \quad \text{or} \quad \operatorname*{plim}_{n\to\infty} X_n = a.$$

The Central Limit Theorem: What are we glossing over?

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Definition (Law of Large Numbers)

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d. random variables, each with finite mean μ. Then for all $\varepsilon > 0$,

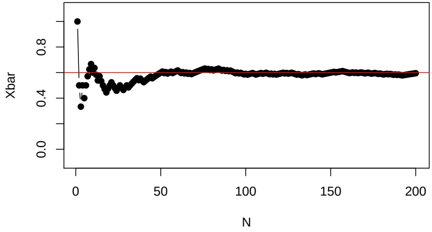
$$\overline{X}_n \xrightarrow{p} \mu \quad \text{as} \quad n \to \infty$$

or equivalently,

$$\lim_{n \to \infty} \Pr\left(|\overline{X}_n - \mu| \geq \varepsilon \right) = 0$$

where $\overline{X}_n$ is the sample mean.

Example: Mean of N independent tosses of a coin:



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Definition (Convergence in Distribution)

Consider a sequence of random variables $X_1, ..., X_n$, each with CDFs $F_1, ..., F_n$. The sequence is said to converge in distribution to a limiting random variable X with CDF F if

$$\lim_{n \to \infty} F_n(x) = F(x),$$

for every point x at which F is continuous. We write this as

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- Convergence in probability is a special case of convergence in distribution in which the distribution converges to a degenerate distribution (i.e. a probability distribution which only takes a single value).

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Definition (Lindeberg-Lévy Central Limit Theorem)

Let $X_1, ..., X_n$ a sequence of i.i.d. random variables each with mean μ and variance $\sigma^2 < \infty$. Then, for *any* population distribution of X,

$$\sqrt{n}(\overline{X}_n - \mu) \xrightarrow{d} \mathcal{N}(0, \sigma^2).$$

- As n grows, the $\sqrt{n}$-scaled sample mean converges to a normal random variable.

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- CLT also implies that the standardized sample mean converges to a standard normal random variable:

$$Z_n \equiv \frac{\overline{X}_n - E\left[\overline{X}_n\right]}{\sqrt{V\left[\overline{X}_n\right]}} = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, 1).$$

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- Note that CLT holds for a random sample from *any* population distribution (with finite mean and variance) — what a convenient result!

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C) Both statements are true.

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We use a statistic (e.g., $\overline{X}$) to estimate a parameter, and we will denote this with a hat (e.g. $\hat{\mu}$). A statistic is a function of the sample.

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- Estimating one number is typically easier than estimating many (or an infinite number of) numbers.

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- The question of interest may be answerable with single characteristic of the distribution (e.g., if $E[Y] - E[X]$ identifies the proportion angered by the sensitive item, then it may be sufficient to estimate $E[Y]$ and $E[X]$)

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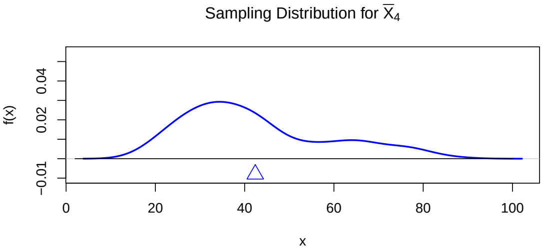
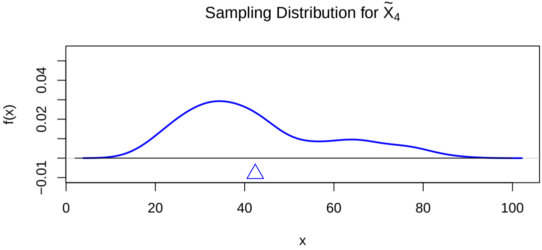
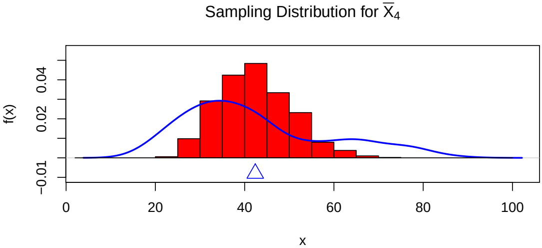
Clearly, one of these estimators is better than the other, but how can we define "better"?

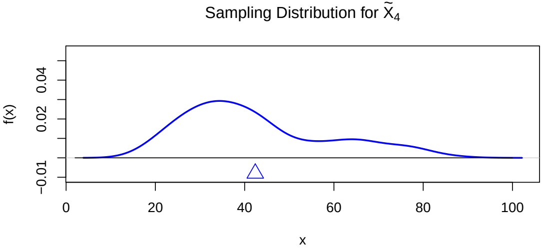
Age population distribution in blue





Age population distribution in blue, sampling distributions in red





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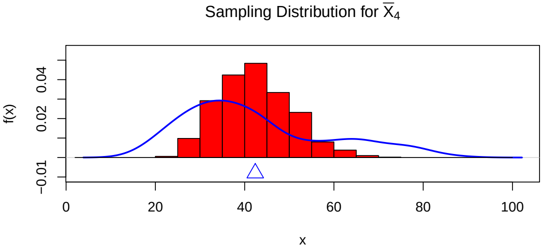

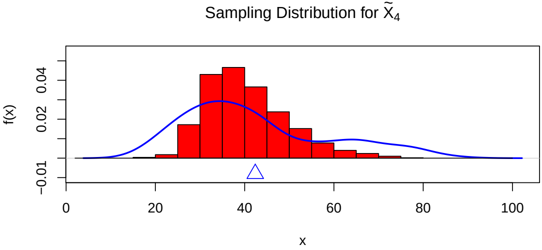

Methods of Finding Estimators

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When estimating simple features of a distribution we can use the plug-in principle, the idea that you write down the feature of the distribution you are interested in and estimate with the sample analog. Formally this is using the Empirical CDF to estimate features of the population.

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- We'd like an estimator that has a known sampling distribution (approximately) when the sample size is large.

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Asymptotic Properties (kick in when n is large):

- Consistency: As our sample size grows to infinity, does the sampling distribution of our estimator converge to the true parameter value?
- Asymptotic Normality: As our sample size grows large, does the sampling distribution of our estimator approach a normal distribution?

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An estimator is unbiased iff:

$$\text{Bias}(\hat{\mu}) = 0$$

Example: Estimators for Population Mean

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Candidate estimators:

1. $\hat{\mu}_1 = Y_1$ (the first observation)
2. $\hat{\mu}_2 = \frac{1}{2}(Y_1 + Y_n)$ (average of the first and last observation)
3. $\hat{\mu}_3 = 42$
4. $\hat{\mu}_4 = \overline{Y}_n$ (the sample average)

How do we choose between these estimators?

Bias of Example Estimators

Which of these estimators are unbiased?

1. $E[Y_1 - \mu] =$
2. $E[\frac{1}{2}(Y_1 + Y_n) - \mu] =$
3. $E[42 - \mu] =$
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- Estimators 1,2, and 4 are unbiased because they get the right answer on average.

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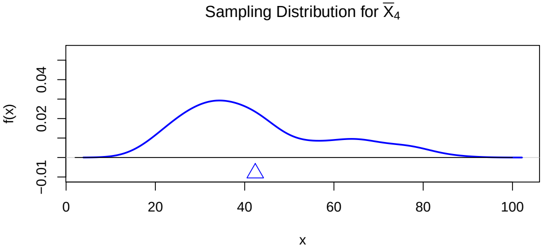
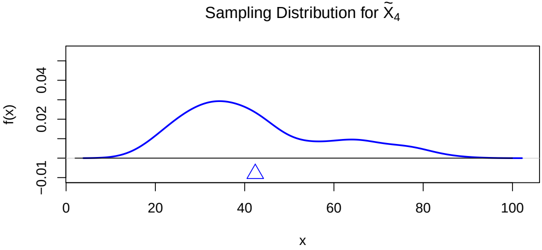
4. $E[\overline{Y}_n - \mu] = \frac{1}{n}\sum_1^n E[Y_i] - \mu = \mu - \mu = 0$

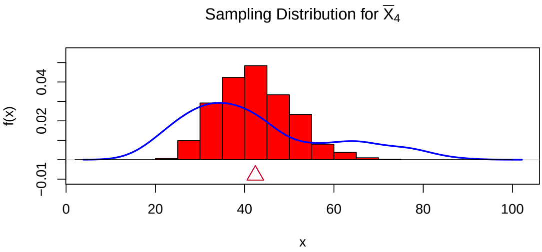
- Estimators 1,2, and 4 are unbiased because they get the right answer on average.
- Estimator 3 is biased.

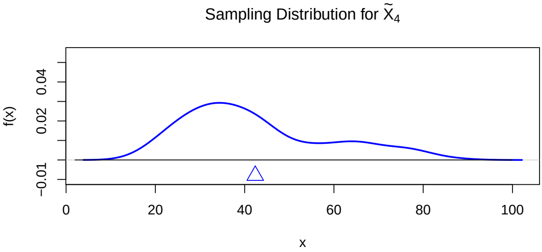
Age population distribution in blue





Age population distribution in blue, sampling distributions in red





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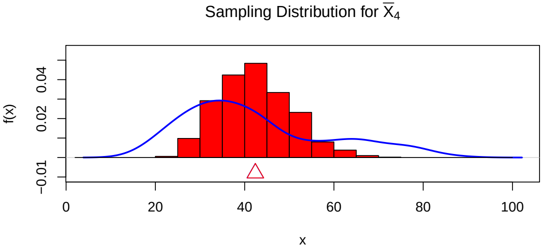

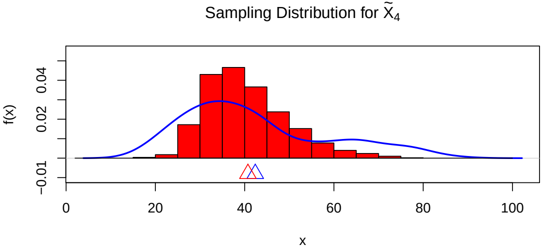

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- Note that this does **not** imply that a particular estimate is always close to the true parameter value
- The standard deviation of the sampling distribution of an estimator, $\sqrt{V[\hat{\theta}]}$, is often called the standard error of the estimator

Variance of Example Estimators

What is the variance of our estimators?

1. $V[Y_1] =$
2. $V[\frac{1}{2}(Y_1 + Y_n)] =$
3. $V[42] =$
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Among the unbiased estimators, the sample average has the smallest variance. This means that Estimator 4 (the sample average) is likely to be closer to the true value μ, than Estimators 1 and 2.

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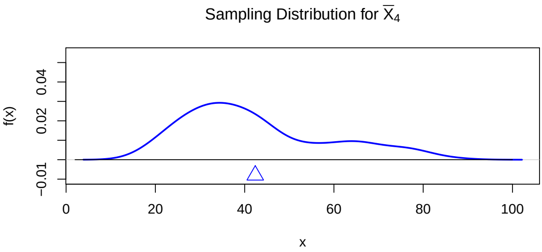

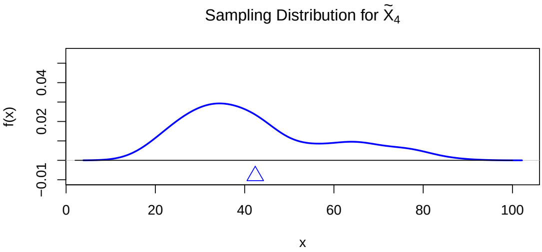

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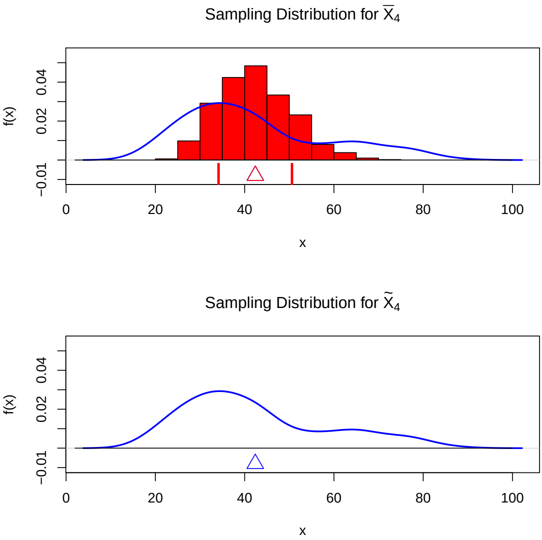

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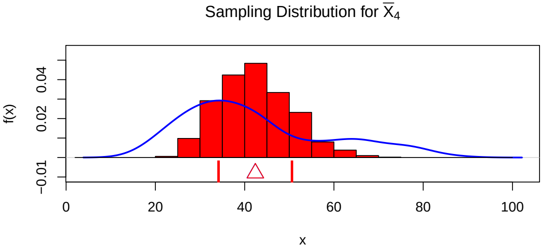

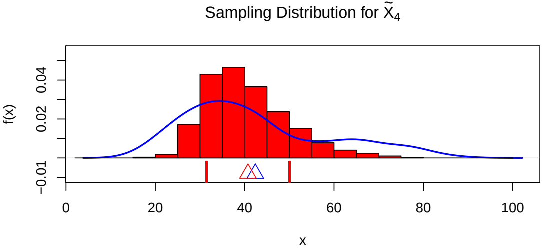

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- Asymptotic properties of an estimator are defined by the behavior of $\hat{\theta}_1, ...\hat{\theta}_n$ when n goes to infinity.

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- Two types of stochastic convergence are of particular importance:
  1. Convergence in probability: values in the sequence eventually take a constant value
     (i.e. the limiting distribution is a point mass)
  2. Convergence in distribution: values in the sequence continue to vary, but the variation eventually comes to follow an unchanging distribution
     (i.e. the limiting distribution is a well characterized distribution)

Convergence in Probability

Definition (Convergence in Probability)

A sequence $X_1, ..., X_n$ of random variables converges in probability towards a real number a if, for all accuracy levels $\varepsilon > 0$,

$$\lim_{n \to \infty} \Pr\left(|X_n - a| \geq \varepsilon\right) = 0$$

We write this as

$$X_n \xrightarrow{p} a \quad \text{or} \quad \operatorname*{plim}_{n \to \infty} X_n = a.$$

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- For example, the sample mean $\bar{X}_n$ converges to the population mean μ in probability because

$$E[\bar{X}_n] = \mu \quad \text{and} \quad V[\bar{X}_n] = \sigma^2/n \to 0 \quad \text{as} \quad n \to \infty$$

3: Consistency (does it get closer to the right answer as sample size increases)

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An estimator θ_n is consistent if the sequence $\theta_1, ..., \theta_n$ converges in probability to the true parameter value θ as sample size n grows to infinity:

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Deriving Consistency of Estimators

Our candidate estimators:

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2. $\widehat{\mu}_2 = 4$
3. $\widehat{\mu}_3 = \overline{Y}_n \equiv \frac{1}{n}(Y_1 + \cdots + Y_n)$
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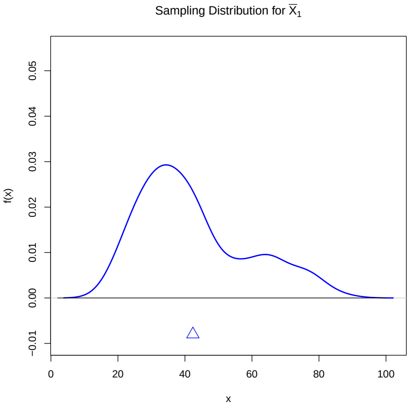

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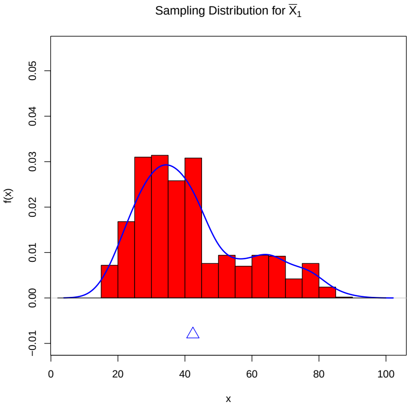

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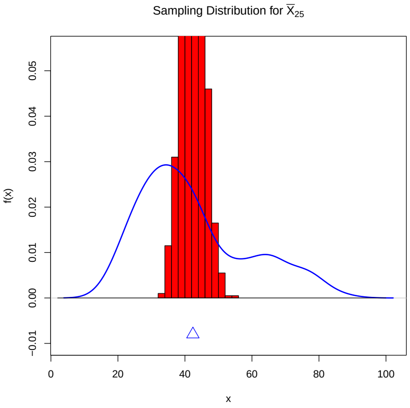

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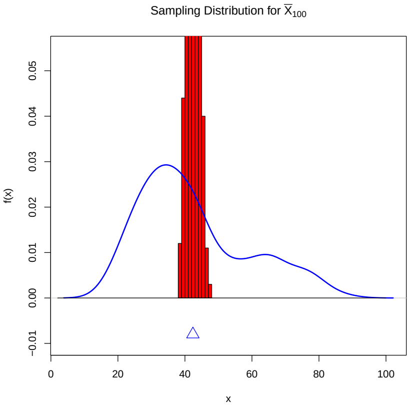

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Consider the median estimator: $\tilde{X}_n =$ median$(Y_1, ..., Y_n)$

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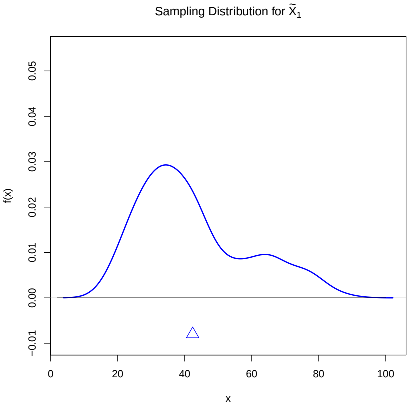

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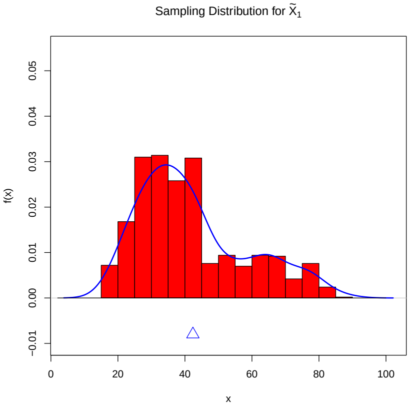

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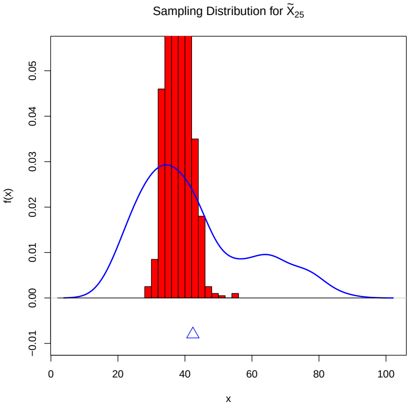

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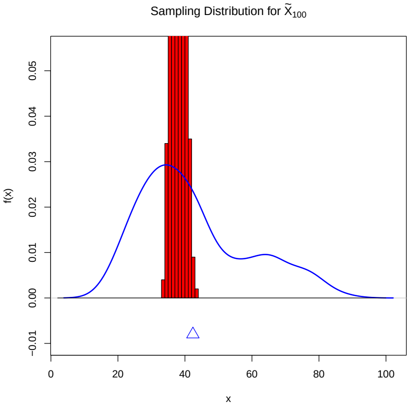

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The sampling distributions of many estimators converge towards a normal distribution.

For example, we've seen that the sampling distribution of the sample mean converges to the normal distribution.

Mean Squared Error

How can we choose between an unbiased estimator and a biased, but more efficient estimator?

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Definition (Mean Squared Error)

To compare estimators in terms of both efficiency and unbiasedness we can use the Mean Squared Error (MSE), the expected squared difference between $\hat{\theta}$ and θ:

$$MSE(\hat{\theta}) = E[(\hat{\theta} - \theta)^2] = Bias(\hat{\theta})^2 + V(\hat{\theta}) = \left[E[\hat{\theta}] - \theta\right]^2 + V(\hat{\theta})$$

Review and Example

Gerber, Green, and Larimer (*American Political Science Review*, 2008)

Dear Registered Voter:

WHAT IF YOUR NEIGHBORS KNEW WHETHER YOU VOTED?

Why do so many people fail to vote? We've been talking about the problem for years, but it only seems to get worse. This year, we're taking a new approach. We're sending this mailing to you and your neighbors to publicize who does and does not vote.

The chart shows the names of some of your neighbors, showing which have voted in the past. After the August 8 election, we intend to mail an updated chart. You and your neighbors will all know who voted and who did not.

DO YOUR CIVIC DUTY—VOTE!

MAPLE DR	Aug 04	Nov 04	Aug 06
9995 JOSEPH JAMES SMITH	Voted	Voted	______
9995 JENNIFER KAY SMITH		Voted	______
9997 RICHARD B JACKSON		Voted	______
9999 KATHY MARIE JACKSON		Voted	______

Basic Analysis

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```
load("gerber_green_larimer.RData")
## turn turnout variable into a numeric
social$voted <- 1 * (social$voted == "Yes")
neigh.mean <- mean(social$voted[social$treatment == "Neighbors"])
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Is this a "real" effect? Is it big?

Population vs. Sampling Distribution

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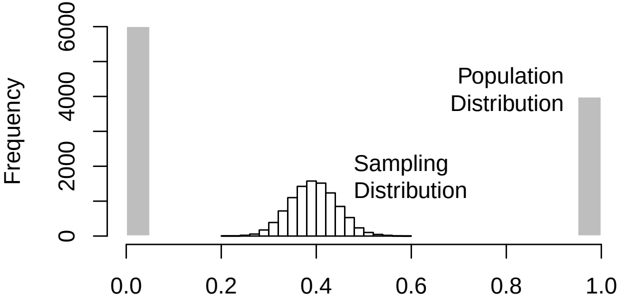

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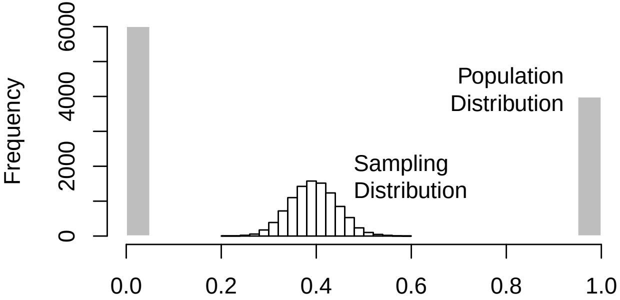

But remember that we only get to see one draw from the sampling distribution. Thus ideally we want an estimator with good properties.

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- By asymptotic Normality $(\hat{\theta} - 0)/\text{SE}(\hat{\theta}) \sim N(0, 1)$
- By the properties of Normals, we know that this implies that $\hat{\theta} \sim \mathcal{N}(0, \text{SE}(\hat{\theta}))$

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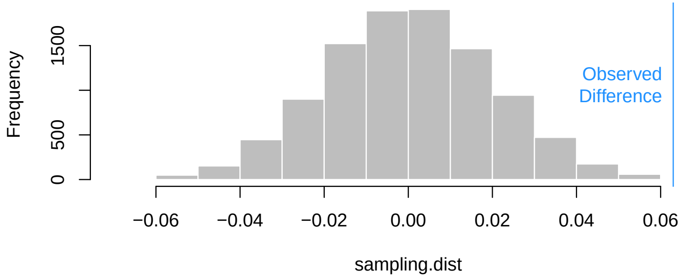

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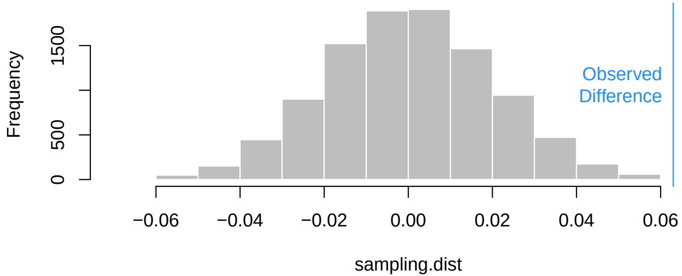

Does the observed difference in means seem plausible if there really were no difference between the two groups in the population?

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Next Class: interval estimation

Summary of Properties

Concept	Criteria	Intuition
Unbiasedness	$E[\hat{\mu}] = \mu$	Right on average
Efficiency	$V[\hat{\mu}_1] < V[\hat{\mu}_2]$	Low variance
Consistency	$\hat{\mu}_n \xrightarrow{p} \mu$	Converge to estimand as $n \to \infty$
Asymptotic Normality	$\hat{\mu}_n \overset{\text{approx.}}{\sim} N(\mu, \frac{\sigma^2}{n})$	Approximately normal in large n

Fun with Hidden Populations

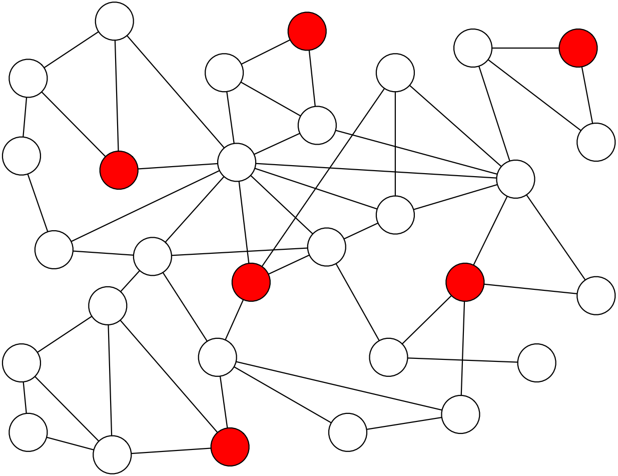
Dennis M. Feehan and Matthew J. Salganik "Generalizing the Network Scale-Up Method: A New Estimator for the Size of Hidden Populations"

Slides graciously provided by Matt Salganik.

Scale-up Estimator

Basic insight from Bernard et al. (1989)

Network scale-up method



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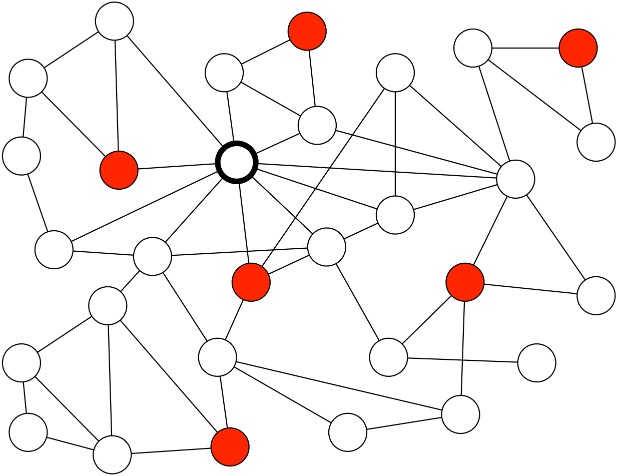

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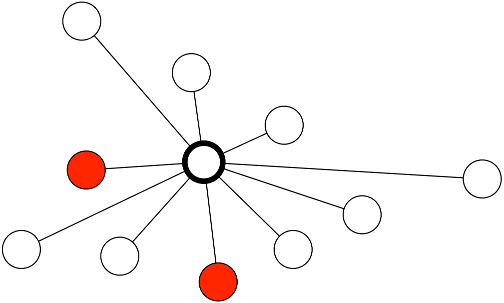

$$\hat{N}_T = \frac{2}{10} \times 30 = 6$$

If $\underbrace{y_{i,k} \sim Bin(d_i, N_k/N)}_{\text{basic scale-up model}}$, then maximum likelihood estimator is

$$\hat{N}_T = \frac{\sum_i y_{i,T}}{\sum_i \hat{d}_i} \times N$$

- $\hat{N}_T$: number of people in the target population
- $y_{i,T}$: number of people in target population known by person i
- $\hat{d}_i$: estimated number of people known by person i
- N: number of people in the population

See Killworth et al., (1998)

Target population	Location	Citation
Mortality in earthquake	Mexico City, Mexico	Bernard et al. (1989)
Rape victims	Mexico City, Mexico	Bernard et al. (1991)
HIV prevalence, rape, & homelessness	U.S.	Killworth et al. (1998)
Heroin use	14 U.S. cities	Kadushin et al. (2006)
Choking incidents in children	Italy	Snidero et al. (2007, 2009)
Groups most at-risk for HIV/AIDS	Ukraine	Paniotto et al. (2009)
Heavy drug users	Curitiba, Brazil	Salganik et al. (2011)
Men who have sex with men	Japan	Ezoe et al. (2012)
Groups most at risk for HIV/AIDS	Almaty, Kazakhstan	Scutelniciuc (2012a)
Groups most at risk for HIV/AIDS	Moldova	Scutelniciuc (2012b)
Groups most at risk for HIV/AIDS	Thailand	Aramrattan (2012)
Groups most at risk for HIV/AIDS	Chongqing, China	Guo (2012)
Groups most at risk for HIV/AIDS	Rwanda	Rwanda Biomedical Center (2012)

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- They show that for the estimator to be unbiased and consistent requires a particular assumption that average personal network size is the same in the hidden population as the remainder.
- This was unknown up to this point!
- Analyzing the estimator let them see that the problem can be addressed by collecting a new kind of data on the visibility of hidden population (which can easily be collected with respondent driven sampling)

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- Studying estimators can not only expose problems but suggest solutions
- Another example of creative and interesting ideas coming from the applied people
- Formalizing methods is important because it is what allows them to be studied- it was a long time before anyone discovered the bias/consistency concerns!

References

- Kuklinski et al. 1997 “Racial prejudice and attitudes toward affirmative action” *American Journal of Political Science*
- Gerber, Alan S., Donald P. Green, and Christopher W. Larimer. "Social pressure and voter turnout: Evidence from a large-scale field experiment." *American Political Science Review* 102.01 (2008): 33-48.

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Questions?

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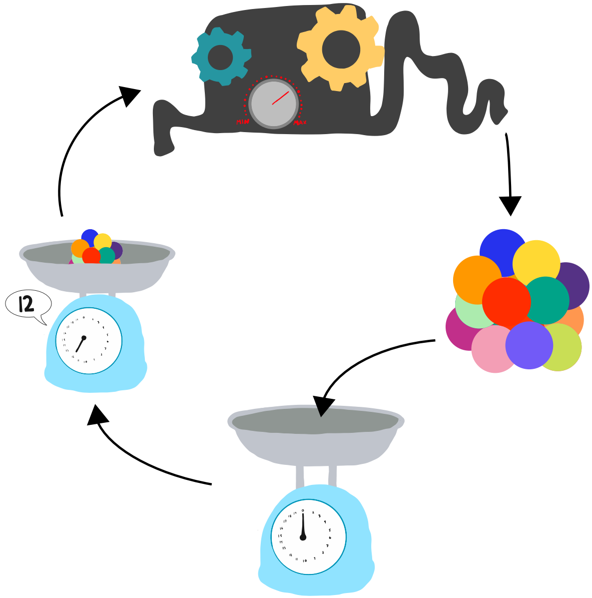

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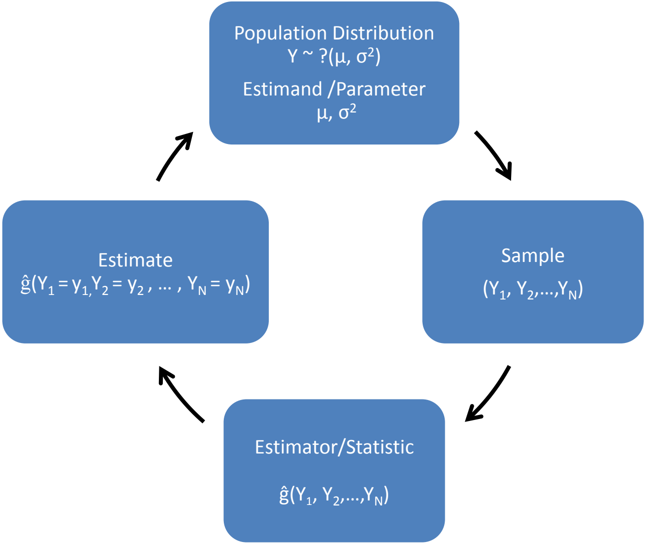

Population Distribution
Y ~ ?(μ, σ²)
Estimand /Parameter
μ, σ²
Sample
(Y₁, Y₂,…,Y_N)
Estimator/Statistic
ĝ(Y₁, Y₂,…,Y_N)
Estimate
ĝ(Y₁ = y₁, Y₂ = y₂ , … , Y_N = y_N)

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- An interval estimate is a realized value from an interval estimator. The estimated interval typically forms what we call a confidence interval, which we will define shortly.

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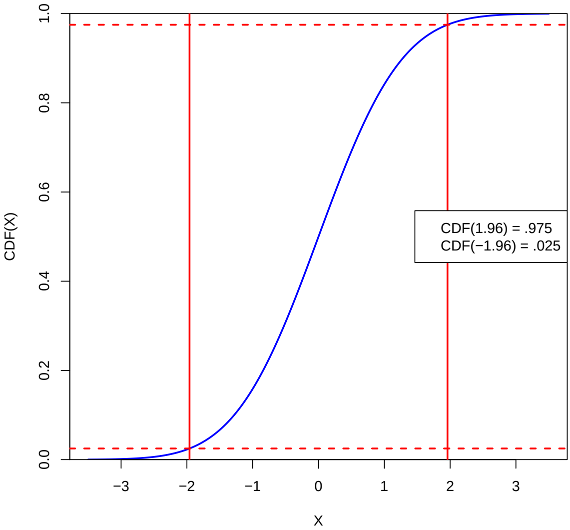

Constructing a Confidence Interval with Known σ^2

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We call this estimator a 95% confidence interval for μ.

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$$\overline{Y} \sim_{approx} ?(?, ?)$$

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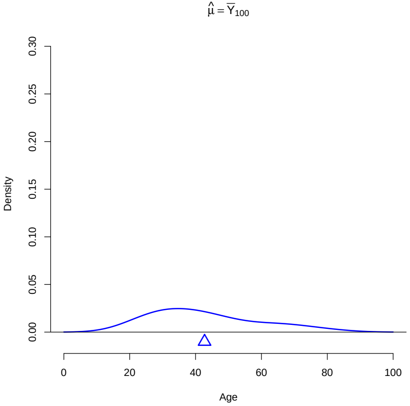

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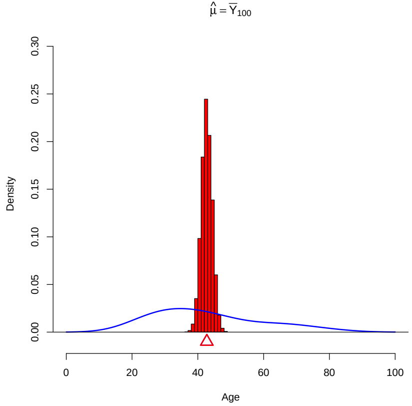

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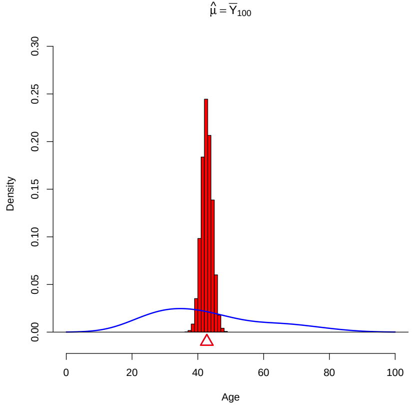

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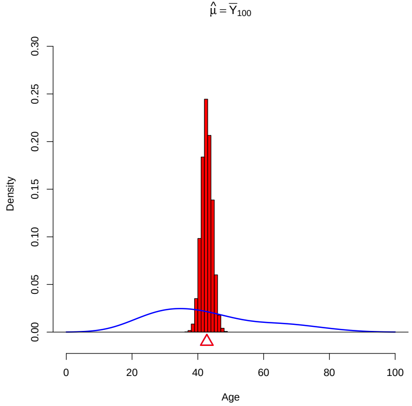


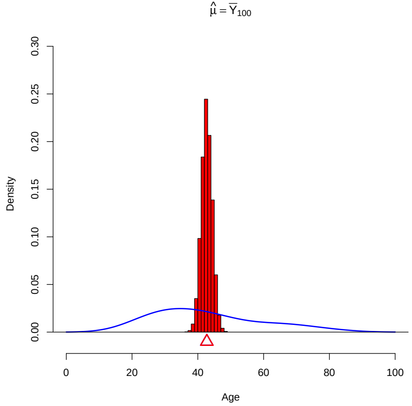
Kuklinski Example

$$\overline{Y} \sim_{approx} N(\mu, \sigma^2/n)$$

Suppose the 1,161 respondents in the Kuklinski data set were the population, with $\mu = 42.7$ and $\sigma^2 = 257.9$.

If we sampled 100 respondents, the sampling distribution of $\overline{Y}_{100}$ is:

$$\overline{Y}_{100} \sim_{approx} N(42.7, 2.579)$$



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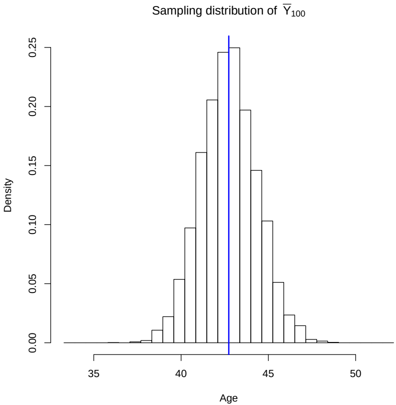

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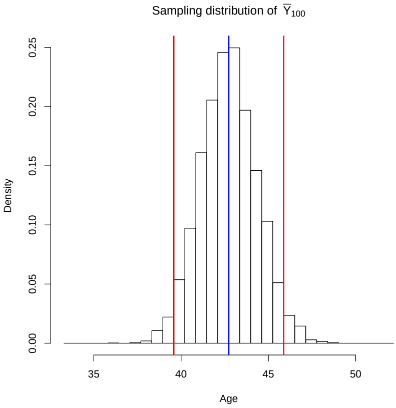

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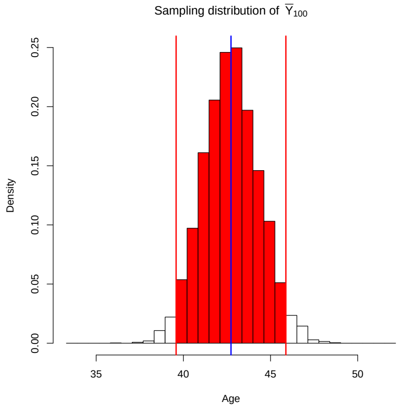

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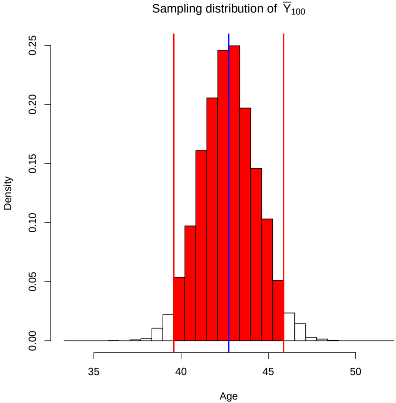
But the 1,161 is actually the sample (not the population).



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But we can **not** directly use this now because σ^2 is unknown.

Instead, we need an estimator of σ^2, $\hat{\sigma}^2$.

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Two possible estimators of population variance:

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S_{1n}^2 (unbiased and consistent) is commonly called the sample variance.

Estimating σ and the SE

Returning to Kulinski et. al...

We will use the sample variance:

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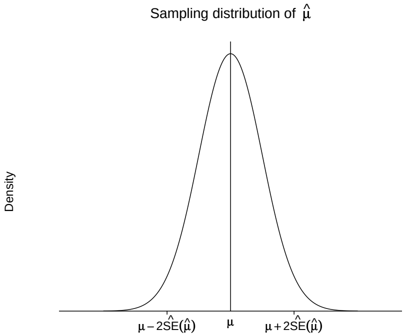
We will plug in S for σ and our estimated standard error will be

$$\widehat{SE}[\hat{\mu}] = \frac{S}{\sqrt{n}}$$

95% Confidence Intervals

If $X_1, ..., X_n$ are i.i.d. and n is large, then

$$\widehat{\mu} \quad \sim \quad N(\mu, (\widehat{SE}[\hat{\mu}])^2)$$



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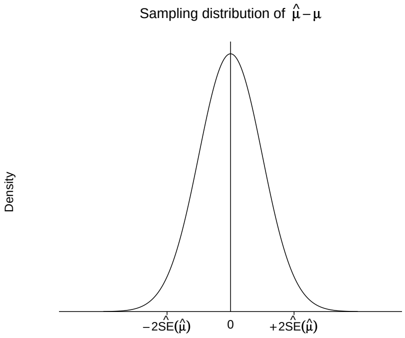

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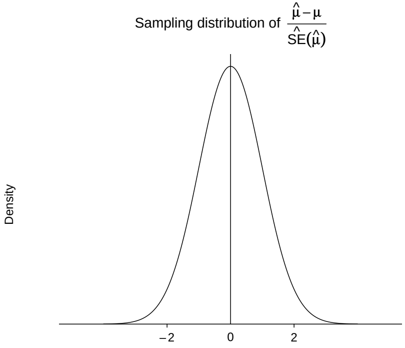

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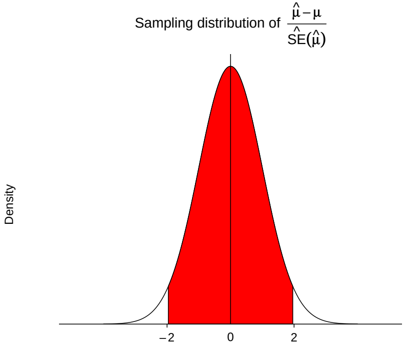
We know that

$$P\left(-1.96 \leq \frac{\widehat{\mu} - \mu}{\widehat{SE}[\hat{\mu}]} \leq 1.96\right) = 95\%$$



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We can work backwards from this:

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The random quantities in this statement are $\widehat{\mu}$ and $\widehat{SE}[\hat{\mu}]$.
Once the data are observed, nothing is random!

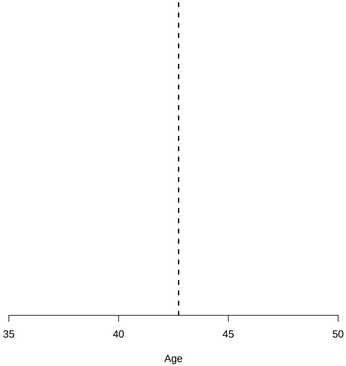
What does this mean?

We can simulate this process using the Kuklinski data:

1) Draw a sample of size 100:

2) Calculate $\hat{\mu}$ and $\widehat{SE}[\hat{\mu}]$:

3) Construct the 95% CI:



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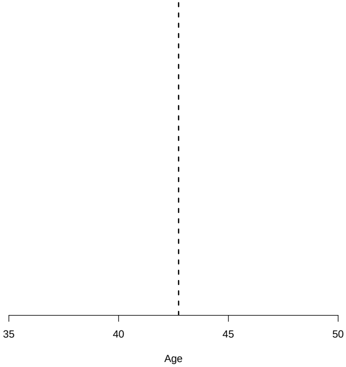

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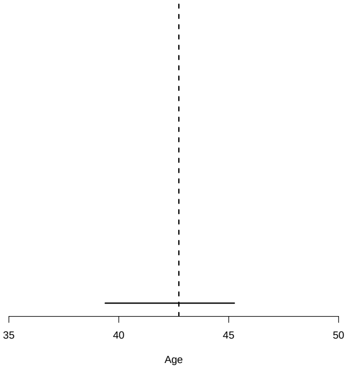

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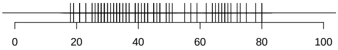

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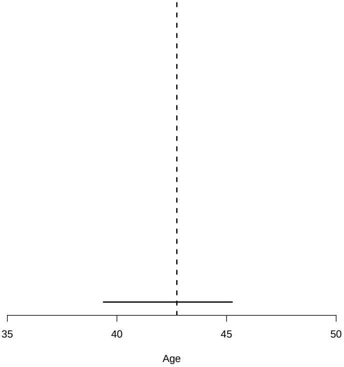

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$$\hat{\mu} = 41.93 \quad \widehat{SE}[\hat{\mu}] = 1.604$$

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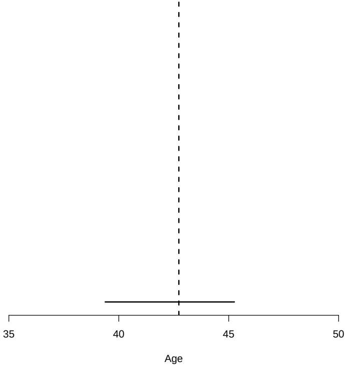

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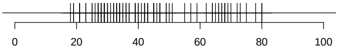

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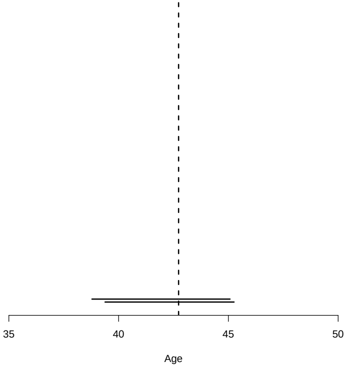

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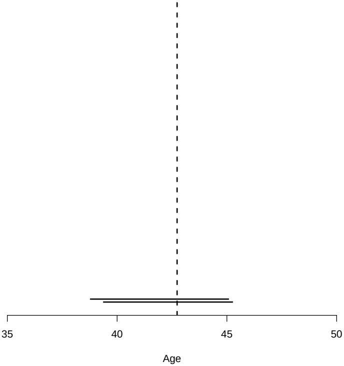

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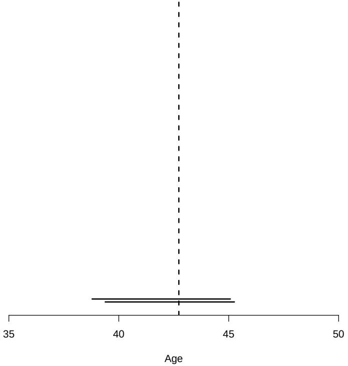

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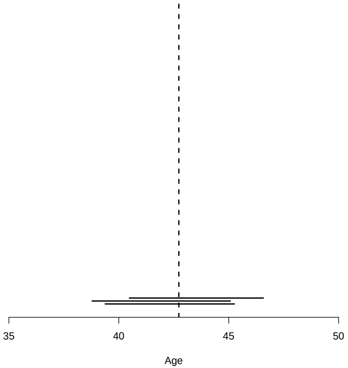

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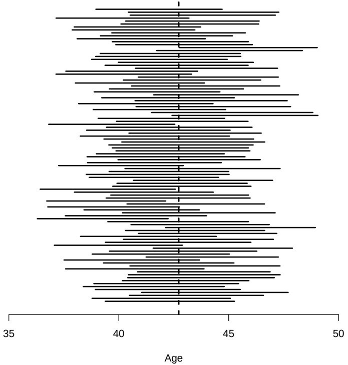
By repeating this process, we generate the sampling distribution of the 95% CIs.



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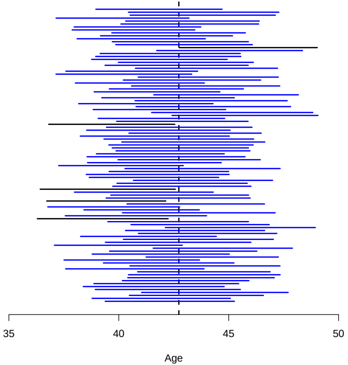
Most of the CIs cover the true μ;



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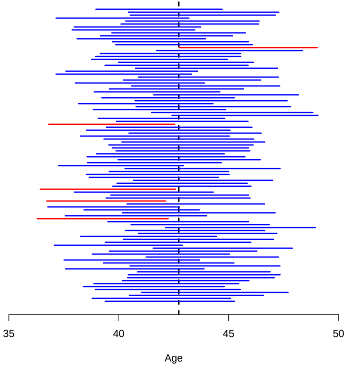

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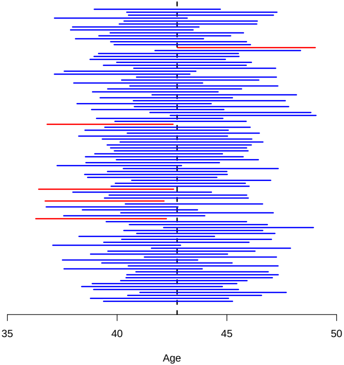
In the long run, we expect 95% of the CIs generated to contain the true value.



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Interpreting a Confidence Interval

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- Best: for a fixed confidence level/coverage probability, find the smallest interval

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$$P\left(-z \leq \frac{\widehat{\mu} - \mu}{\widehat{SE}[\hat{\mu}]} \leq z\right) = (1 - \alpha)\%$$

How can we find z?

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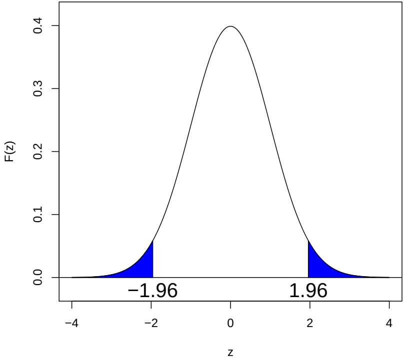

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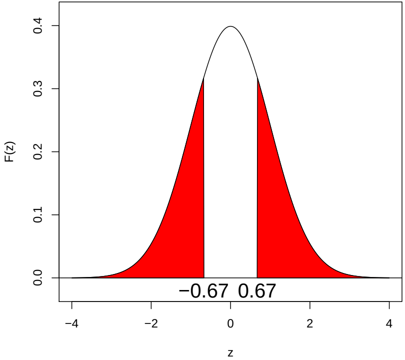

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When the sample size is small, we need to know something about the distribution in order to construct confidence intervals with the correct coverage (because we can't appeal to the CLT or assume that S is a good approximation of σ).

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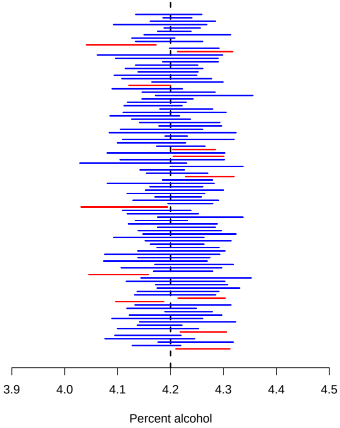

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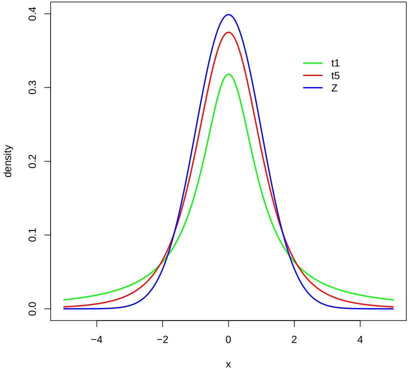

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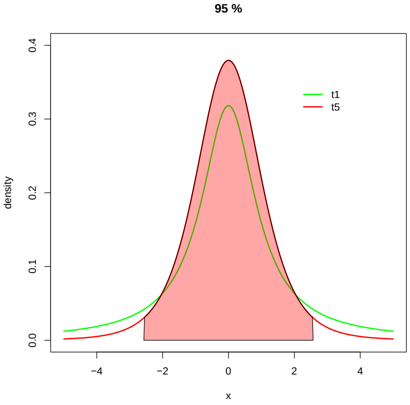

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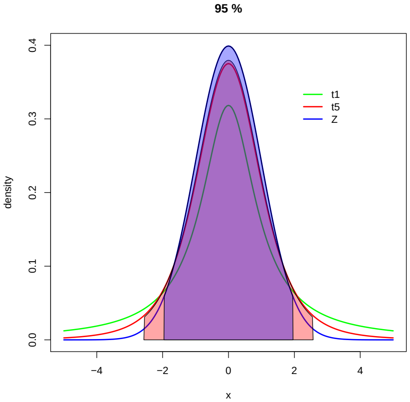

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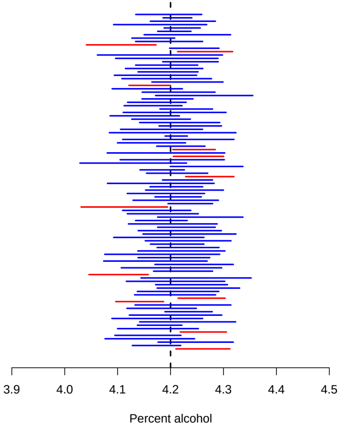

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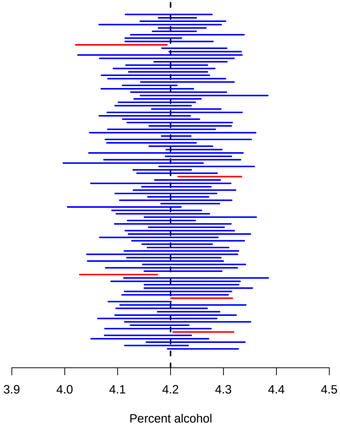

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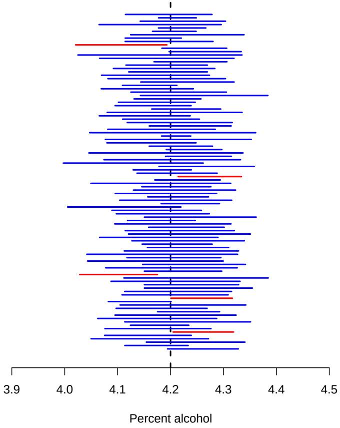

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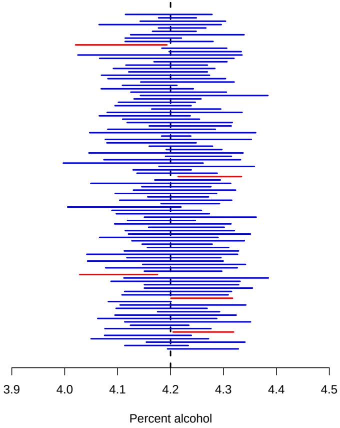

Another Rationale for the t-Distribution

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Thus, we need to derive the sampling distribution of the new random variable. It turns out that T_n follows Student's t-distribution with $n-1$ degrees of freedom.

Theorem (Distribution of t-Value from a Normal Population)

Suppose we have an i.i.d. random sample of size n from $N(\mu, \sigma^2)$. Then, the sample mean $\overline{X}_n$ standardized with the estimated standard error $S_n/\sqrt{n}$ satisfies,

$$T_n \equiv \frac{\overline{X}_n - \mu}{S_n/\sqrt{n}} \sim \tau_{n-1}$$

Appendix

Kuklinski Example Returns

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We will usually be interested in comparing μ_1 to μ_2, although we will sometimes need to compare σ_1^2 to σ_2^2 in order to make the first comparison.

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CIs for $\mu_1 - \mu_2$

Using the same type of argument that we used for the univariate case, we write a $(1 - \alpha)$% CI as the following:

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Interval estimation of the population proportion

- Let's say that we have a sample of iid Bernoulli random variables, $Y_1, \ldots, Y_n$, where each takes $Y_i = 1$ with probability π. Note that this is also the population proportion of ones. We have shown in previous weeks that the expectation of one of these variable is just the probability of seeing a 1: $E[Y_i] = \pi$.

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- **Problem** Show that the sample proportion, $\hat{\pi} = \frac{1}{n}\sum_{i=1}^{n} Y_i$, of the above iid Bernoulli sample, is unbiased for the true population proportion, π, and that the sampling variance is equal to $\frac{\pi(1-\pi)}{n}$.

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- Note that if we have an estimate of the population proportion, $\hat{\pi}$, then we also have an estimate of the sampling variance: $\frac{\hat{\pi}(1-\hat{\pi})}{n}$.

Interval estimation of the population proportion

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- Note that if we have an estimate of the population proportion, $\hat{\pi}$, then we also have an estimate of the sampling variance: $\frac{\hat{\pi}(1-\hat{\pi})}{n}$.
- Given the facts from the previous problem, we just apply the same logic from the population mean to show the following confidence interval:

$$P\left(\hat{\pi} - z_{\alpha/2} \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}} \leq \pi \leq \hat{\pi} + z_{\alpha/2} \times \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}\right) = (1-\alpha)$$

Gerber, Green, and Larimer experiment

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N of Individuals	191,243	38,218	38,204	38,218	38,201

- Let's use what we have learned up until now and the information in the table to calculate a 95% confidence interval for the difference in proportions voting between the Neighbors group and the Civic Duty group.

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- You may assume that the samples with in each group are iid and the two samples are independent.

Calculating the CI for social pressure effect

- We know distribution of sample proportion turned among Civic Duty group $\hat{\pi}_C \sim N(\pi_C, (\pi_C(1 - \pi_C))/n_C)$

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- Remember that we can calculate the sample variance for a sample proportion like so: $(\hat{\pi}_C(1-\hat{\pi}_C))/n_C$

Gerber, Green, and Larimer experiment

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```
n.n <- 38201
samp.var.n <- (0.378 * (1 - 0.378))/n.n
n.c <- 38218
samp.var.c <- (0.315 * (1 - 0.315))/n.c
se.diff <- sqrt(samp.var.n + samp.var.c)
## lower bound
(0.378 - 0.315) - 1.96 * se.diff
## [1] 0.05626701
## upper bound
(0.378 - 0.315) + 1.96 * se.diff
## [1] 0.06973299
```

Thus, the confidence interval for the effect is $[0.056267, 0.069733]$.

Summary of Interval Estimation

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- Interval estimates should be interpreted in terms of repeated sampling.

Next Week

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  - Aronow and Miller 3.2.2
  - Fox Chapter 2: What is Regression Analysis?
  - Fox Chapter 5.1 Simple Regression
  - Aronow and Miller 4.1.1 (bivariate regression)
  - "Momentous Sprint at the 2156 Olympics" by Andrew J Tatem et al. *Nature* 2004

Fun with Anscombe's Quartet

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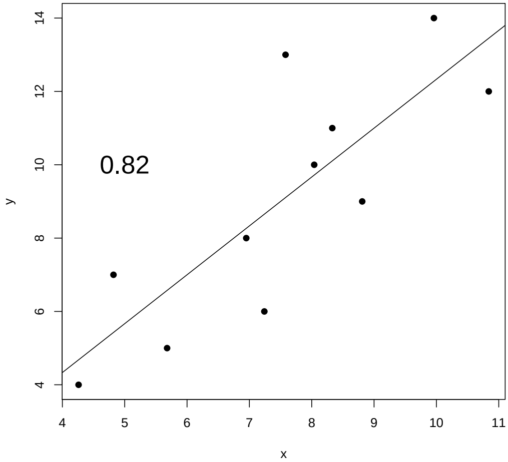

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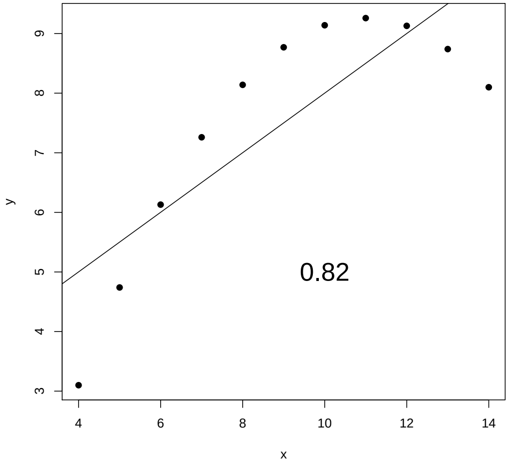

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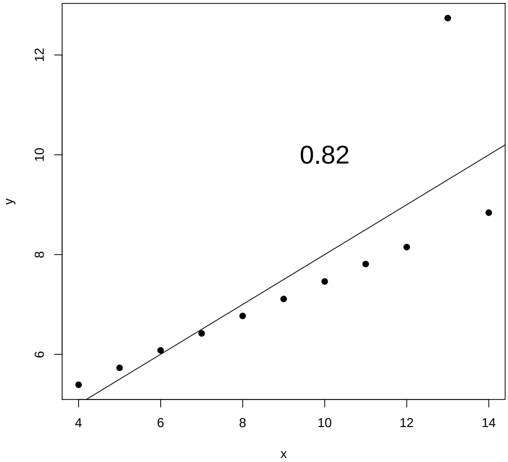

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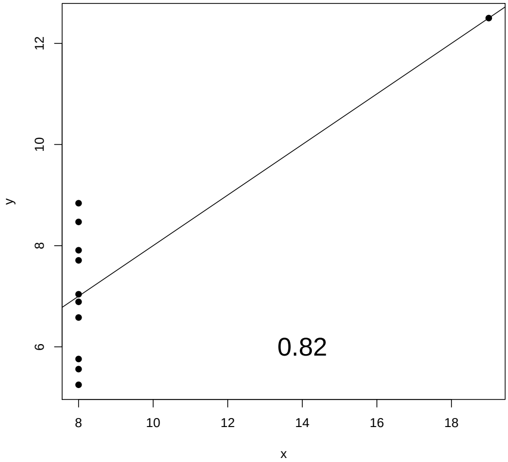

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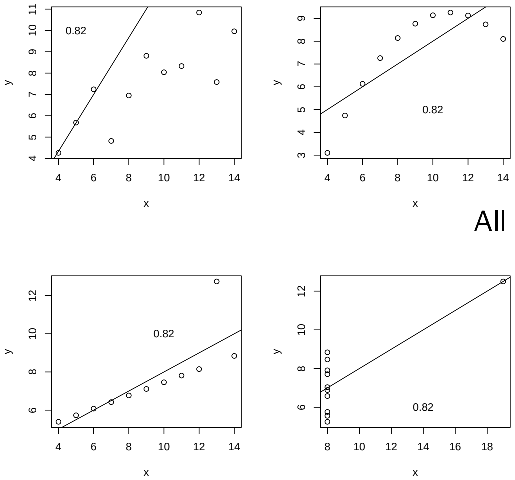

All yield same regression model!

Fun Beyond Correlation

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Reshef, D. N.; Reshef, Y. A.; Finucane, H. K.; Grossman, S. R.; McVean, G.; Turnbaugh, P. J.; Lander, E. S.; Mitzenmacher, M.; Sabeti, P. C. (2011). "Detecting Novel Associations in Large Data Sets". *Science* 334 (6062): 1518-1524.

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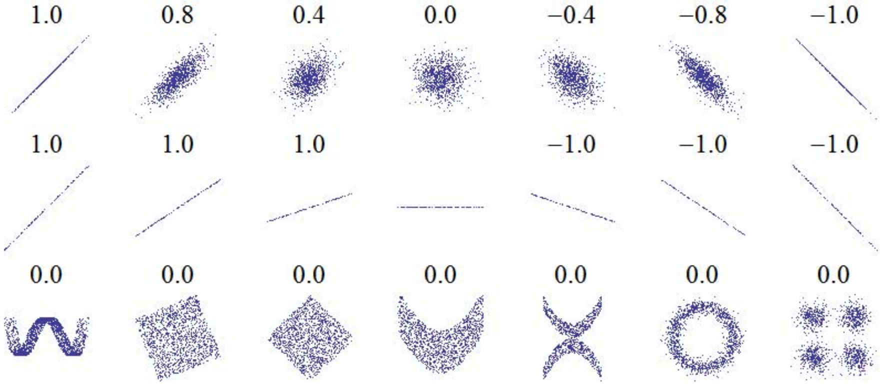

Fun with Correlation Part 2

Enter the Maximal Information Coefficient

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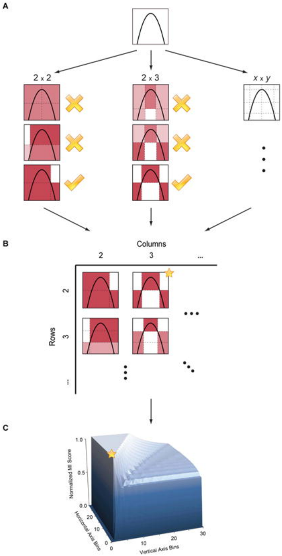

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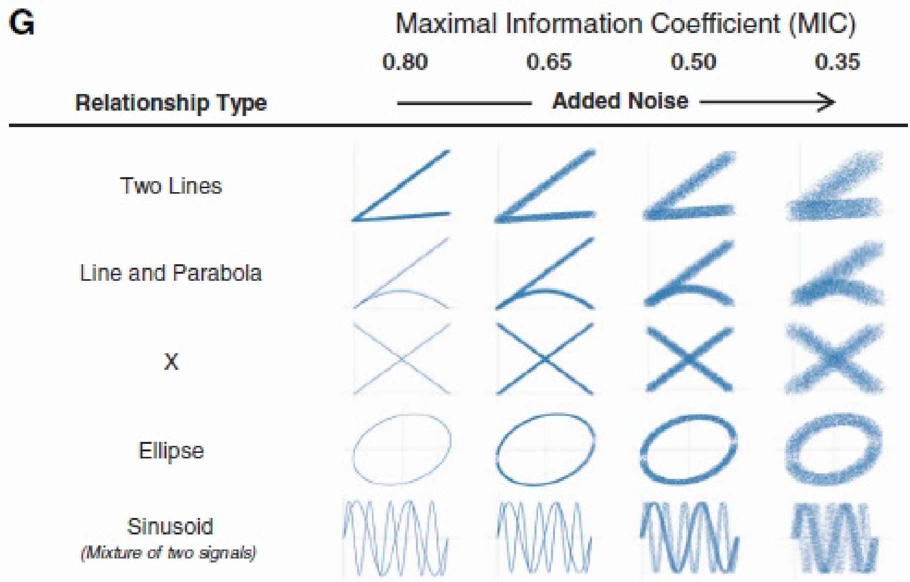

Fun with Correlation Part 2

Enter the Maximal Information Coefficient

MATHEMATICS

A Correlation for the 21st Century

Terry Speed

Bioinformatics Division, Walter and Eliza Hall Institute of M
Department of Statistics, University of California, Berkeley
E-mail: terry{at}stat.berkeley.edu

Fun with Correlation Part 2

Concerns with MIC

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- low power

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This is still an open issue!

A Sketch of why the Student t-distribution

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- Some math will show our distribution is going to be of the form $\sum Z^2$ where $Z \sim N(0, 1)$.
- Let's figure out what distribution that will be

χ^2 Distribution

Suppose $Z \sim \text{Normal}(0, 1)$.

χ^2 Distribution

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The pdf then is

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The pdf then is

$$\frac{\partial F_X(x)}{\partial x} = f_Z(\sqrt{x}) \frac{1}{2\sqrt{x}} + f_Z(-\sqrt{x}) \frac{1}{2\sqrt{x}}$$

Definition

Suppose X is a continuous random variable with $X \geq 0$, with pdf

$$f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{n/2-1} e^{-x/2}$$

Then we will say X is a χ^2 distribution with n degrees of freedom. Equivalently,

$$X \sim \chi^2(n)$$

χ^2 Properties

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1 &= E[Z_i^2] - 0 \\
E[X] &= N
\end{aligned}
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$$\text{var}(X) = \sum_{i=1}^{N} \text{var}(Z_i^2)$$

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Student's t-Distribution

Definition

Suppose $Z \sim \text{Normal}(0, 1)$ and $U \sim \chi^2(n)$. Define the random variable Y as,

$$Y = \frac{Z}{\sqrt{\frac{U}{n}}}$$

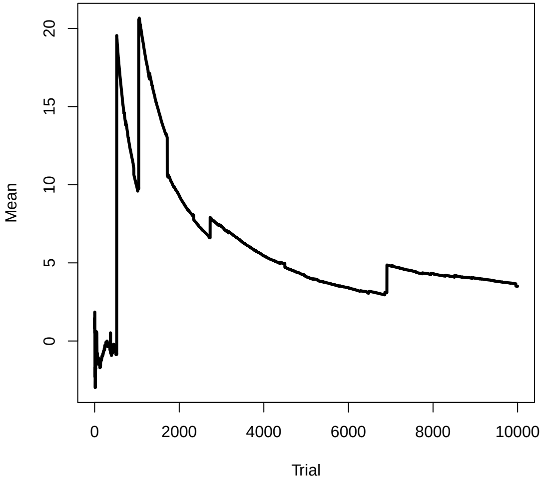
If Z and U are independent then $Y \sim t(n)$, with pdf

$$f(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi n}\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-\frac{n+1}{2}}$$

We will use the t-distribution extensively for test-statistics

Student's t-Distribution, Properties

Suppose $n = 1$, Cauchy distribution



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If $X \sim t(2)$
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Suppose $n = 1$, Cauchy distribution
If $X \sim$ Cauchy(1), then:
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var$(X) =$ undefined
If $X \sim t(2)$
E[X] $= 0$
var$(X) =$ undefined

Student's t-Distribution, Properties

Suppose $n > 2$, then
$var(X) = \frac{n}{n-2}$
As $n \to \infty$ var$(X) \to 1$.