Week 2: Random Variables

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September 19/21, 2016

[1] These slides are heavily influenced by Adam Glynn, Justin Grimmer, Jens Hainmueller, Teppei Yamamoto. Many illustrations by Shay O'Brien.

Where We've Been and Where We're Going...

- Last Week
  - welcome and outline of course
  - described uncertain outcomes with probability.
- This Week
  - Monday:
    - summarize one random variable using expectation and variance
    - show how to condition on a variable
  - Wednesday:
    - properties of joint distributions
    - conditional expectations
    - covariance, correlation, independence
- Next Week
  - estimating these features from data
  - estimating uncertainty
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

Example: New Hampshire Primaries

Evidence suggests that candidates gain a small advantage from ballot order.

As a response, in 2008 New Hampshire chose a letter from the alphabet and then listed the candidates in alphabetical order starting with that letter.

We can use probability to assess the “fairness” of this process.

We will do this by introducing a random variable X to be Barack Obama’s position on the 2008 New Hampshire primary ballot.

Example: Assessing Racial Prejudice

- We often want to ask sensitive questions which a survey respondent is unlikely to honestly answer
- A list experiment asks respondents how many items on a list they agree with
  - for example, what proportion of people would be upset by a black family moving in next door to them (Kuklinski et al 1997).
  - randomly split survey into two halves
  - first half ask how many of the following items upset you:
    1. the federal government increasing the tax on gasoline
    2. professional athletes getting million-dollar salaries
    3. large corporations polluting the environment.
  - second half, add a fourth item
    4. a black family moving in next door
  - use the answers to infer the proportion upset by the fourth item.
- To do this we need to understand random variables

What is a Random Variable?

Intuition: functions that map outcomes to numbers.

Formal: X is a function that maps the sample space to the real numbers.

Imagine an experiment of two coin flips

$$\mathbf{S} = \big\{\{heads, heads\}, \{heads, tails\}, \{tails, heads\}, \{tails, tails\}\big\}$$

we could define a random variable $X(s)$ to be the function that returns the number of heads for each element of $\mathbf{S}$.

- $X(\{heads, heads\}) = 2$
- $X(\{heads, tails\}) = 1$
- $X(\{tails, heads\}) = 1$
- $X(\{tails, tails\}) = 0$

A Brief Note on Notation

- We almost always use capital roman letters for the “name” of the random variable such as X
- We refer to a particular value with a lower case letter x
- So we might write $P(X = x)$ to be the probability that the number of heads is equal to x.
- For more complicated random variables we often write out values as follows

$$X = \begin{cases} 1 \text{ if heads} \\ 0 \text{ if tails} \end{cases}$$

- Sometimes the sample space is already numeric so its more obvious (e.g. how long until the train arrives)

Quick FAQ

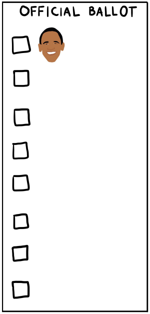
- Why have random variables at all?
  it makes the math easier, even across very different sample spaces.
- Why are they random variables?
  realizations of a stochastic process (i.e. randomness in the outcome, not the mapping)
- Is it really easier this way? It seems hard.
  yep. seriously. let's do an example!

NH Ballot Order Example

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$X = \begin{cases} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{cases}$$



A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

X is a random variable indicating Obama's position on the ballot. Highlighted letters are those leading to a given ballot position. Highlighted individual is first.

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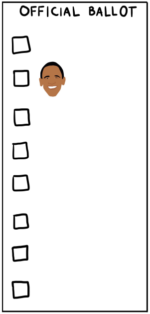

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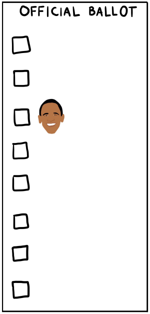

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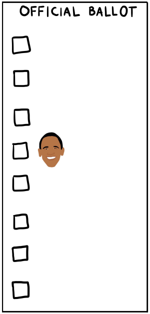

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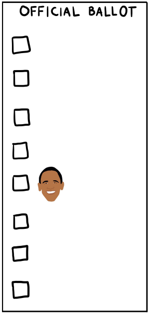

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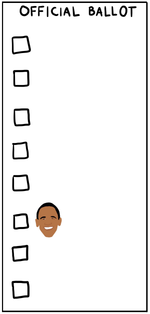

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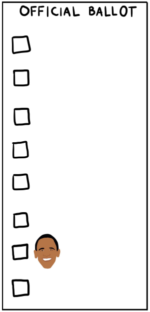

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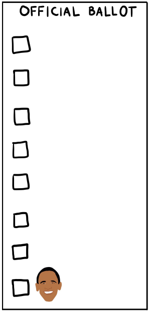

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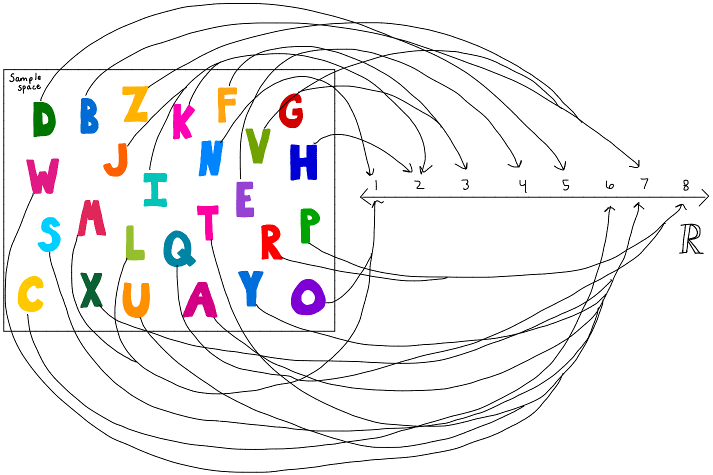
Discrete Distributions

- For discrete distributions, the random variable X takes on a finite, or a countably infinite number of values.
- A common shorthand is to think of discrete RVs taking on distinct values.
- A probability mass function (pmf) and a cumulative distribution function (cdf) are two common ways to define the probability distribution for a discrete RV.
- Probability mass functions provide a compact way to represent information about how likely various outcomes are.

Where do Distributions Come From?

The probabilities associated with each realization of the r.v. come from the underlying experiment and sample space.

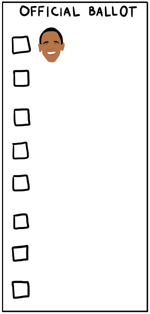


Example: New Hampshire

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$f(x) = \begin{cases} 4/26 & x = 1 \\ 4/26 & x = 2 \\ 2/26 & x = 3 \\ 1/26 & x = 4 \\ 1/26 & x = 5 \\ 1/26 & x = 6 \\ 10/26 & x = 7 \\ 3/26 & x = 8 \end{cases}$$



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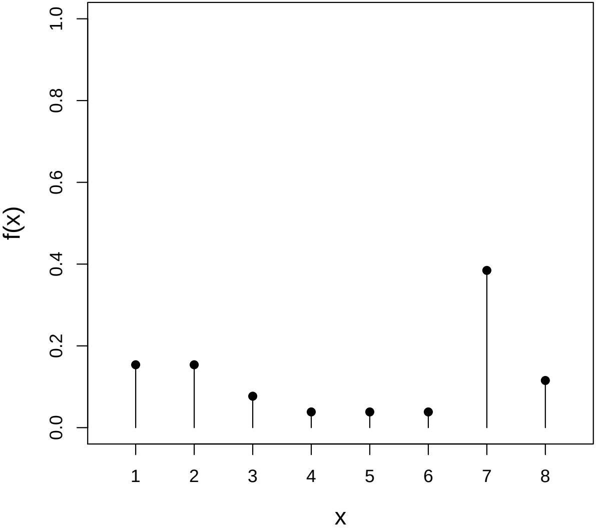
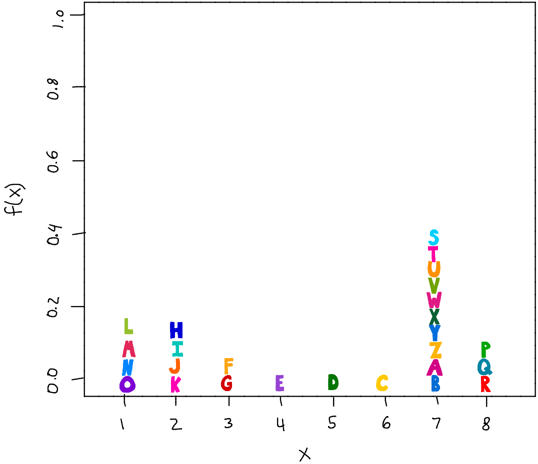
Discrete Probability Mass Functions

A probability mass function $f(x)$ of a random variable X is a non-negative function that gives the probability that $X = x$ and $\sum_x f(x) = 1$.

NH Obama Ballot Position PMF Plot



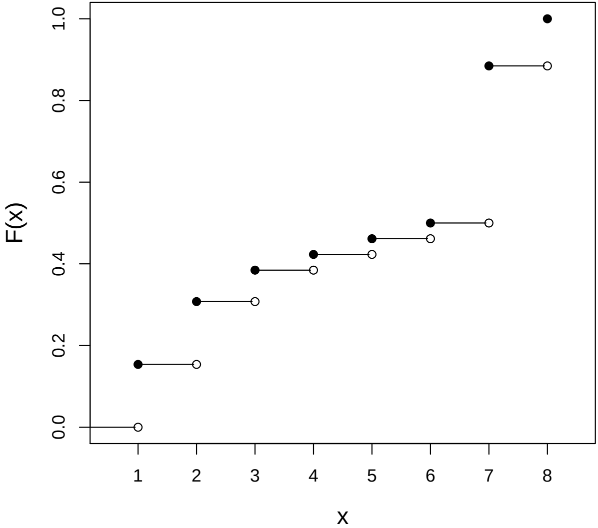
NH Obama Ballot Position PMF Plot



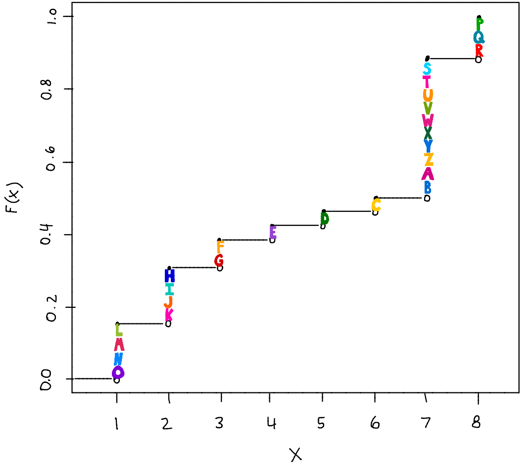
Discrete Cumulative Distribution Function

A cumulative distribution function $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$.

NH Obama Ballot Position CDF Plot



NH Obama Ballot Position CDF Plot



Some Important Discrete Distributions

- Let X be a binary variable with $P(X = 1) = p$ and, thus, $P(X = 0) = 1 - p$, where $p \in [0, 1]$. Then we say that X follows a Bernoulli distribution with the following pmf:

$$f_X(x) = p^x(1-p)^{1-x} \qquad \text{for } x \in \{0, 1\}.$$

- Probably the most famous distribution for a discrete r.v. is the discrete uniform distribution that puts equal probability on each value that X can take:

$$f_X(x) = \begin{cases} 1/k & \text{for } x = 1, \ldots, k \\ 0 & \text{otherwise} \end{cases}$$

- We can summarize these distributions with one number (e.g. the probability of variables being 1)

Empirical Distributions

An empirical mass function $\widehat{f}(x)$ of a variable X is a non-negative function that gives the frequency of the value x from data on X.

An empirical cumulative distribution function $\widehat{F}(x)$ of a variable X is a non-decreasing function that gives the frequency of values of X less than x.

Example: Assessing Racial Prejudice

- We often want to ask sensitive questions which a survey respondent is unlikely to honestly answer
- A list experiment asks respondents how many items on a list they agree with
  - for example, what proportion of people would be upset by a black family moving in next door to them (Kuklinski et al 1997).
  - randomly split survey into two halves
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  - second half, add a fourth item
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  - use the answers to infer the proportion upset by the fourth item.
- To do this we need to understand random variables

Racial Prejudice Example (Kuklinski et al, 1997)

$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3
$f(x)$	?	?	?	?
$\widehat{f}(x)$	0.02	0.27	0.43	0.28
$\widehat{F}(x)$	0.02	0.29	0.72	1.00

$Y = \#$ of angering items on the treatment list for Southerners:

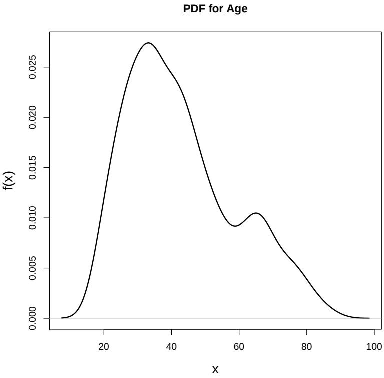
y	0	1	2	3	4
$f(y)$	?	?	?	?	?
$\widehat{f}(y)$	0.02	0.20	0.40	0.28	0.10
$\widehat{F}(y)$	0.02	0.22	0.62	0.90	1.00

Continuous Distributions

- Continuous random variables take on an uncountably infinite number of values.
- This is often a useful approximation when a variable takes on many values.
- A probability density function (pdf) and a cumulative distribution function (cdf) are two common ways to define the distribution for a continuous RV.

Example: Age in the Racial Prejudice Example

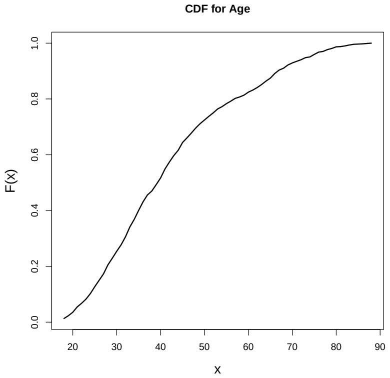
Let X be the age of a randomly selected individual from the Kuklinski et al. (1997) data set.
The probability distribution for this variable is well approximated by a probability density function.



Continuous Cumulative Distribution Functions

A cumulative distribution function $F(x)$ of a random variable X is a non-decreasing function that gives the probability that $X \leq x$. For a continuous RV, the cdf is continuous.

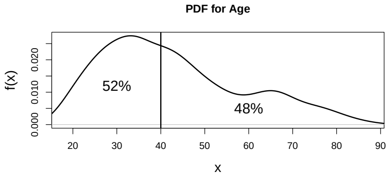
$$F(x) = \int_{-\infty}^{x} f(z)dz$$

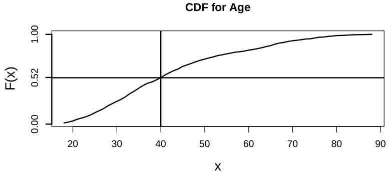


From PDFs to CDFs

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(z)dz$$

$$.52 = P(X \leq 40) = \int_{-\infty}^{40} f(z)dz$$

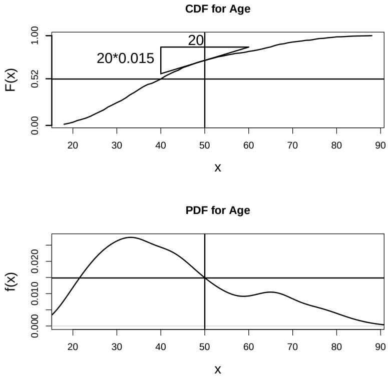




From CDFs to PDFs

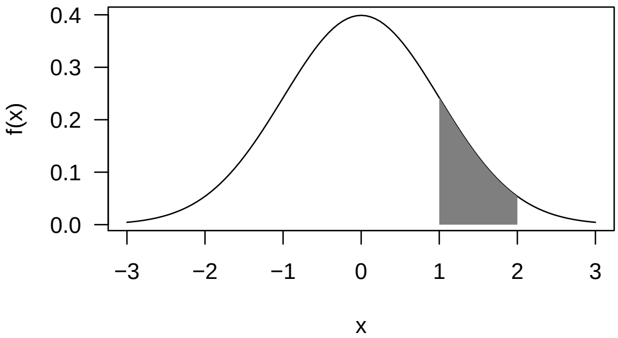
$$f(x) = \frac{dF(x)}{dx}$$

$$.015 = \frac{dF(50)}{dx}$$



Subtleties of Continuous Densities

Remember- the height of the curve is not the probability of x occurring. To get the probability that X will fall in some region, you need the area under the curve.



Expectation

The expected value of a random variable X is denoted by $E[X]$ and is a measure of central tendency of X. Roughly speaking, an expected value is like a weighted average (weighted by probability of occurrence).

The expected value of a discrete random variable X is defined as

$$E[X] = \sum_{\text{all } x} x \cdot f_X(x).$$

The expected value of a continuous random variable X is defined as

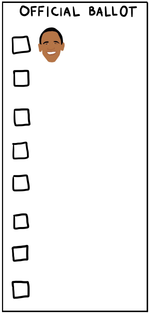
$$E[X] = \int_{-\infty}^{\infty} x \cdot f_X(x) dx.$$

What did we expect for Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

$$\begin{array}{rrr} & 4/26 & \times 1 \\ & 4/26 & \times 2 \\ & 2/26 & \times 3 \\ & 1/26 & \times 4 \\ & 1/26 & \times 5 \\ & 1/26 & \times 6 \\ & 10/26 & \times 7 \\ + & 3/26 & \times 8 \\ \hline & & 4.88 \end{array}$$



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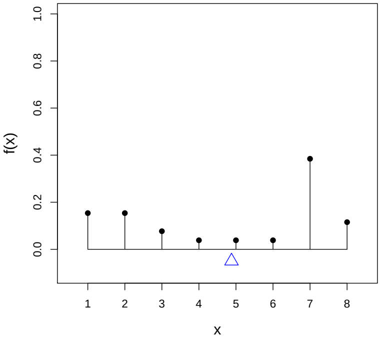
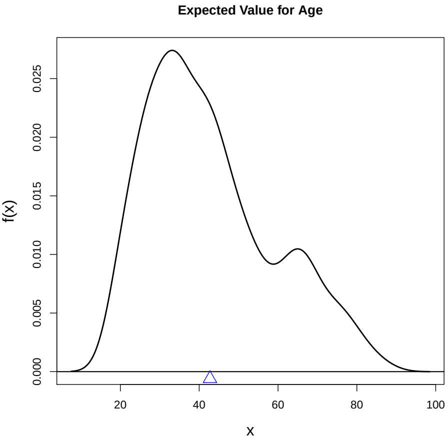
Interpreting Discrete Expected Value

The expected value for a discrete random variable is the balance point of the mass function.



Interpreting Continuous Expected Value

The expected value for a continuous random variable is the balance point of the density function.



Why the Expected Value (Balance Point)?

- It is the probabilistic equivalent of the sample average (mean).
- It is a reasonable measure for the “center” of the data.
- We have some intuition about balance points.
- It has some useful and convenient properties.

Appendix

Population Mean as an Expected Value

Let $x_1, \ldots, x_N$ be our population. Then the population mean is the following

$$\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i$$

This can be re-written in the following form:

$$\bar{x} = \sum_{i=1}^{N} \left\{ x_i \cdot \frac{1}{N} \right\}$$

Note how this resembles the definition of discrete expected value. If all values distinct (i.e. $x_i \neq x_j$ for all $i \neq j$).

$$\bar{x} = \sum_{\text{all } x_i} x_i \cdot f(x_i), \text{ where } f(x_i) = \frac{1}{N}$$

Property 1: Homogeneity

- The expected value of a constant is the constant.
- The expectation of a constant times a RV is the constant times the expectation of the RV.

Suppose a and b are constants and X is a random variable. Then

$$E[b] = b$$

$$E[aX] = aE[X]$$

$$E[aX + b] = aE[X] + b$$

Property 2: Additivity

Expectations of sums are sums of expectations (always).

Suppose we have k random variables $X_1, \ldots, X_k$. If $E[X_i]$ exists for all $i = 1, \ldots, k$, then

$$E\left[\sum_{i=1}^{k} X_i\right] = E[X_1] + \cdots + E[X_k]$$

Property 3: LOTUS

Law of the Unconscious Statistician: If $g(X)$ is a function of a discrete random variable, then

$$E[g(X)] = \sum_x g(x) f_X(x),$$

essentially the expected value of the transformation of the random variable is just the weighted average of the transformed outcomes.

We will come back to this later. But it means that we can can calculate the expected value of $g(X)$ without explicitly knowing the distribution of $g(X)$!

Racial Prejudice Example

$X = \#$ of angering items on the baseline list for Southerners:

x	0	1	2	3	Sum
$\widehat{f}(x)$	0.02	0.27	0.43	0.28	1.00
$x \cdot \widehat{f}(x)$	0.00	0.27	0.86	0.84	1.97

$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4	Sum
$\widehat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$y \cdot \widehat{f}(y)$	0.00	0.20	0.80	0.84	0.40	2.24

Identifying the Percent Angry

Assume that $Y = X + A$, where for a randomly sampled respondent,

- $Y =$ the number of total angering items
- $X =$ the number of angering items on baseline list
- $A = 1$ if angered by a black family moving in next door
- $A = 0$ if not angered by a black family moving in next door.

Exercises for Later:

- Then we know that $E[Y] - E[X] = E[A]$, but can you prove it?
- Noting that A is a Bernoulli RV, how can we interpret $E[A]$?
- What properties and assumptions were necessary?

Variance

The expected value of a function $g()$ of the random variable X, written $g(X)$, is denoted by $E[g(X)]$ and is a measure of central tendency of $g(X)$.

The variance is a special case of this, and the variance of a random variable X (a measure of its dispersion) is given by

$$V[X] = E[(X - E[X])^2]$$

It is the expectation of the squared distances from the mean.

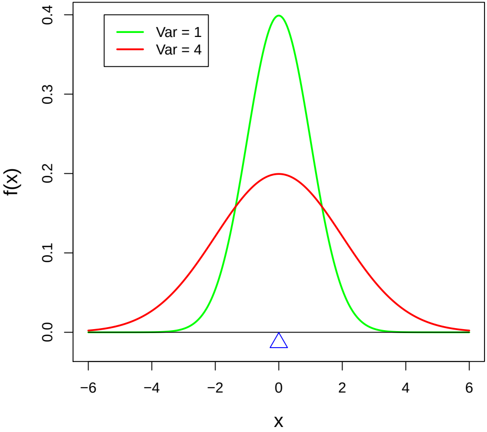
For a discrete random variable X

$$V[X] = \sum_{\text{all } x} (x - E[X])^2 f_X(x)$$

For a continuous random variable X

$$V[X] = \int_{-\infty}^{\infty} (x - E[X])^2 f_X(x) dx$$

Variance Measures the Spread of a Distribution



Why the Variance?

- It is a reasonable measure for the “spread” of a distribution.
- The Normal distribution (bell shaped with thin tails) is completely determined by its expected value (location) and variance (spread).
- The square root of the variance is the standard deviation.
- The variance and standard deviation have some useful properties.

Property 1

- The variance of a constant is zero.
- The variance of a constant times a RV is the constant squared times the variance of the RV.

Suppose a and b are constants and X is a random variable. Then

$$V[b] = 0$$

$$V[aX] = a^2 V[X]$$

$$V[aX + b] = a^2 V[X] + 0$$

Property 2

Variances of sums of independent RVs are sums of variances.

Suppose we have k independent random variables $X_1, \ldots, X_k$. If $V[X_i]$ exists for all $i = 1, \ldots, k$, then

$$V\left[\sum_{i=1}^{k} X_i\right] = V[X_1] + \cdots + V[X_k]$$

NB: Technically independence is sufficient but not necessary.

What was the variance of Obama's NH position?

Candidates:

- Joe Biden
- Hillary Clinton
- Chris Dodd
- John Edwards
- Mike Gravel
- Dennis Kucinich
- Barack Obama
- Bill Richardson

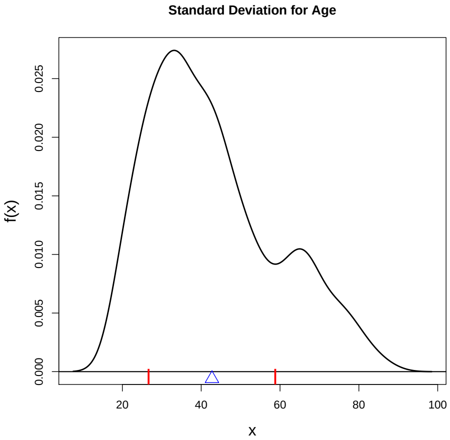
$$
\begin{array}{rrl}
 & 4/26 & \times (1 - 4.88)^2 \\
 & 4/26 & \times (2 - 4.88)^2 \\
 & 2/26 & \times (3 - 4.88)^2 \\
 & 1/26 & \times (4 - 4.88)^2 \\
 & 1/26 & \times (5 - 4.88)^2 \\
 & 1/26 & \times (6 - 4.88)^2 \\
 & 10/26 & \times (7 - 4.88)^2 \\
+ & 3/26 & \times (8 - 4.88)^2 \\
\hline
 & & 2.93
\end{array}
$$

A,B,C,D,E,F,G,H,I,J,K,L,M,N,O,P,Q,R,S,T,U,V,W,X,Y,Z

Does variance matter for fairness?

Interpreting Continuous Standard Deviation

The standard deviation for a continuous random variable is a measure of the spread of the pdf.



Do we lose anything when we use the list experiment?

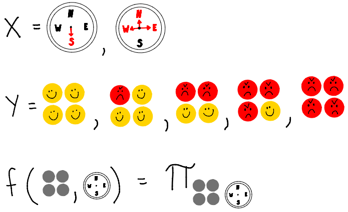
$Y = \#$ of angering items on the treatment list for Southerners:

y	0	1	2	3	4	Sum
$\widehat{f}(y)$	0.03	0.20	0.40	0.28	0.10	1.00
$(y - 2.24)^2 \cdot \widehat{f}(y)$	0.15	0.31	0.02	0.16	0.31	0.95

What is the maximum variance for a Bernoulli random variable?

Joint and Conditional Distributions

- We can describe more than one random variable with joint and conditional distributions.
- For example, suppose we define $X = 0$ (Non-southern), 1 (Southern) and $Y =$ "number of angering items" for a randomly selected respondent receiving the treatment list.
- Furthermore, we define the probability that this respondent will have the values $X = x$ and $Y = y$ to be $f(y, x) = \pi_{yx}$



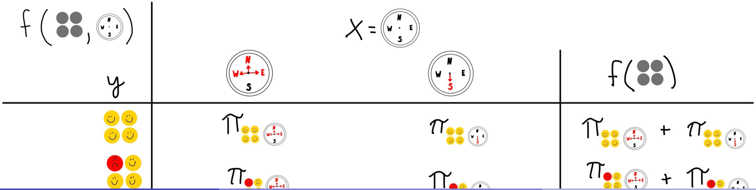
Example Conditional Distribution: Binary X, Discrete Y

Although we cannot observe the responses for the entire population, we can imagine what they might look like as a joint distribution.

Example Conditional Distribution: Binary X, Discrete Y

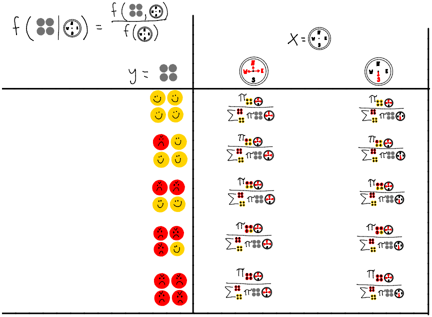
Although we cannot observe the responses for the entire population, we can imagine what they might look like as a joint distribution.

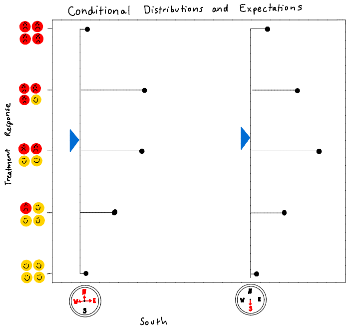
$f(y,x)$	x		
y	0	1	$f(y)$
0	π_{00}	π_{01}	$\pi_{00} + \pi_{01}$
1	π_{10}	π_{11}	$\pi_{00} + \pi_{01}$
2	π_{20}	π_{21}	$\pi_{00} + \pi_{01}$
3	π_{30}	π_{31}	$\pi_{00} + \pi_{01}$
4	π_{40}	π_{41}	$\pi_{00} + \pi_{01}$
$f(x)$	$\sum_{y=0}^{4} \pi_{y0}$	$\sum_{y=0}^{4} \pi_{y1}$	



Discrete Conditional Distribution

Given the joint distribution, we can imagine what the conditional distribution and the conditional expectations would look like.

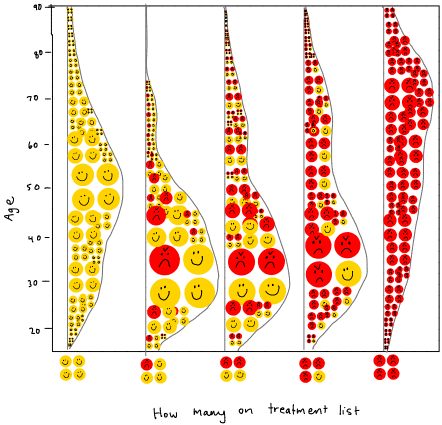




Note that we are just doing what we did before, but now we are doing it twice. In the next example, we will do it many times.

Example: Conditional Distribution with "Continuous" Y

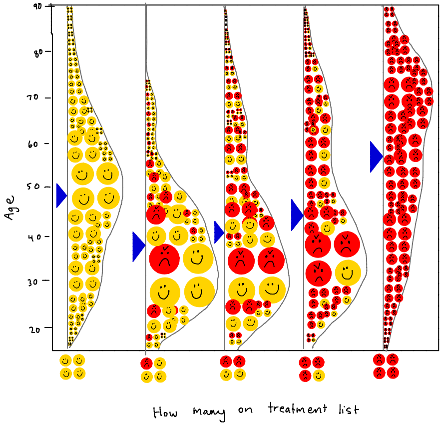
Suppose we define X = "number of angering items" and Y = "age" for a randomly selected respondent receiving the treatment list.



Conditional Expectation Function (CEF)

The conditional expectations form a CEF:

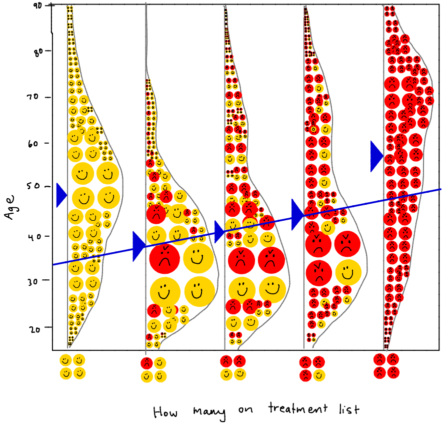
$$E[Y|X = x] = h(x)$$



Linear CEF Assumption

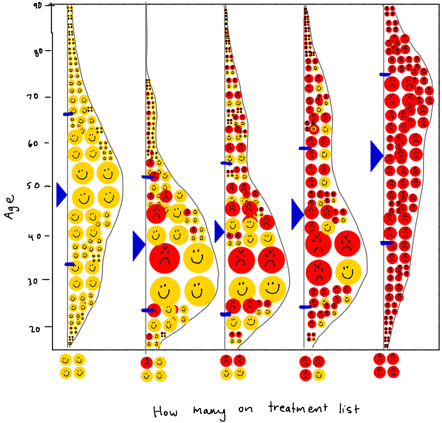
Often we will assume that the CEF is linear:

$$E[Y|X = x] = \beta_0 + \beta_1 x$$



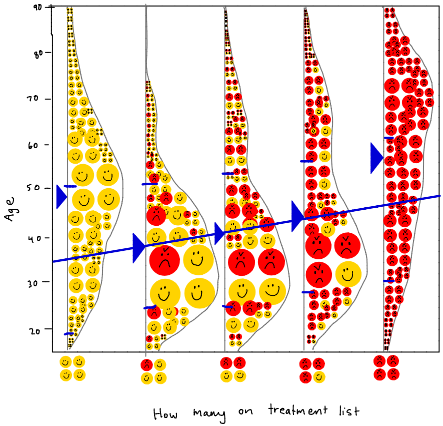
Conditional Variance and Standard Deviation

Similarly, we can assess the conditional standard deviation



Linear CEF and Constant Variance Assumptions

Often, we assume that variance is the same for all values of x.



Interpreting the CEF

Because the CEF is defined merely in terms of the larger population and not in terms of a causal effect (e.g., the causal effect of "number of angering items" on Age), we will utilize a descriptive interpretation of β_0 and β_1.

- For this example, β_0 is the expected age for an individual that is angered by zero items
- β_1 is the expected difference in age between two individuals that have a one unit difference in the number of angering items.

Summary

- Random variables and probability distribution provide useful infrastructure for everything we will do this year.
- Expected value and variance are two useful characteristics of the probability distributions associated with random variables.
- These concepts can be extended by conditioning on other variables.

Fun with Sensitive Questions

Graeme Blair
(slides that follow from Graeme)

Fun with Sensitive Questions

Cannot ask direct questions when there are **incentives to conceal sensitive responses**

1. Social pressure
2. Physical retaliation
3. Legal jeopardy

How to Address Incentives to Conceal

Develop trust with respondents, ask directly

Survey experimental methods

1. **Endorsement experiment** Evaluation bias
2. **List experiment** Aggregation
3. **Randomized response** Random noise

Bias in Direct Questions on Vote Buying

Estimated rate of vote buying from direct survey item

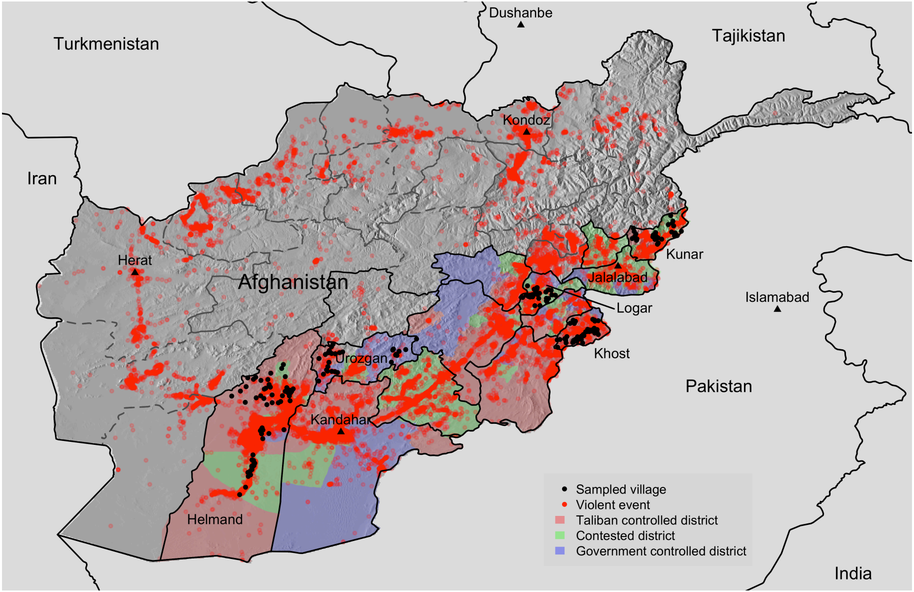
2.4%

Estimate using list experiment

24.3%

Gonzalez-Ocantos et al. 2011, *AJPS*

Question text: "they gave you a gift or did you a favor"


Dushanbe
Turkmenistan
Tajikistan
Iran
Kondoz
Herat
Afghanistan
Kunar
Jalalabad
Islamabad
Logar
Urozgan
Khost
Pakistan
Kandahar
Helmand
Sampled village
Violent event
Taliban controlled district
Contested district
Government controlled district
India

Survey of Civilians in Afghanistan

- 2,754 respondents
- 5 provinces, randomly sampled from 8 Pashtun-dominated provinces (Helmand, Khost, Kunar, Logar, and Urozgan)
- 21 districts, randomly sampled within province
- 204 villages, randomly sampled within district

Outcomes

"Do you support the goals and policies of the **foreign forces**?"

Endorsement experiment design

Control group

It has recently been proposed to allow Afghans to vote in direct elections when selecting leaders for district councils.

Treatment group

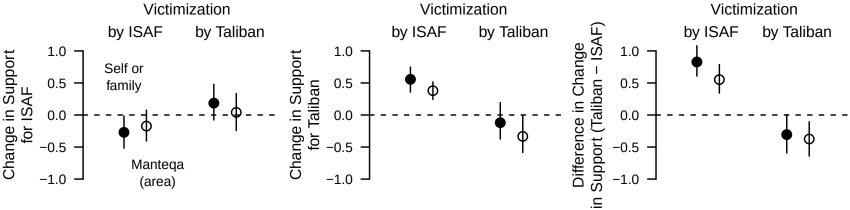
It has recently been proposed **by foreign forces** to allow Afghans to vote in direct elections when selecting leaders for district councils.

Provided for under Electoral Law, these direct elections would increase the transparency of local government as well as its responsiveness to the needs and priorities of the Afghan people. It would also permit local people to actively participate in local administration through voting and by advancing their own candidacy for office in these district councils. How strongly would you support this policy?

5 I strongly agree with this policy
4 I somewhat agree with this policy
3 I am indifferent to this policy
2 I disagree with this policy
1 I strongly disagree with this policy

Refused
Don't know

Conditional results



Controlling for frequency of contact with combatants; education; age; income; Madrassa schooling; tribe; violence levels in village; district territorial control; ...

Lyall, Blair, and Imai 2014

List experiment design

I'm going to read you a list with the names of different groups and individuals on it. After I read the entire list, I'd like you to tell me how many of these groups and individuals you broadly support, meaning that you generally agree with the goals and policies of the group or individual. Please don't tell me which ones you generally agree with; only tell me how many groups or individuals you broadly support.

Control group

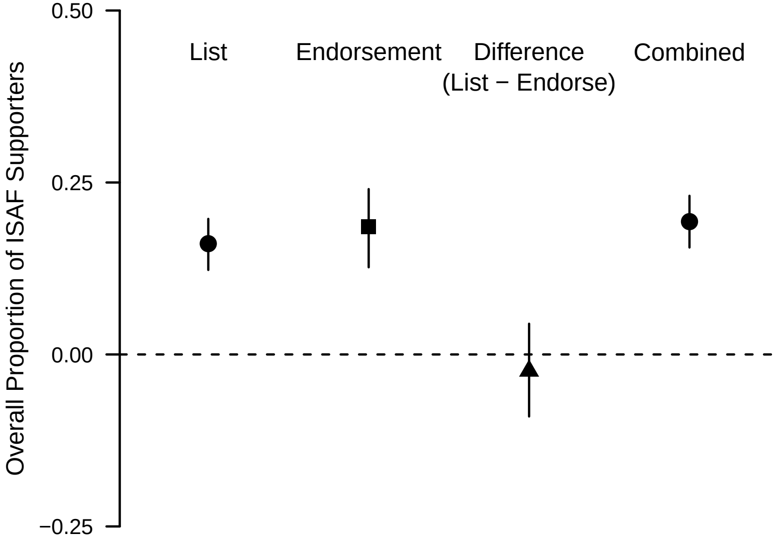
Karzai Government
National Solidarity Program
Local Farmers

Treatment group

Karzai Government
National Solidarity Program
Local Farmers
Foreign forces

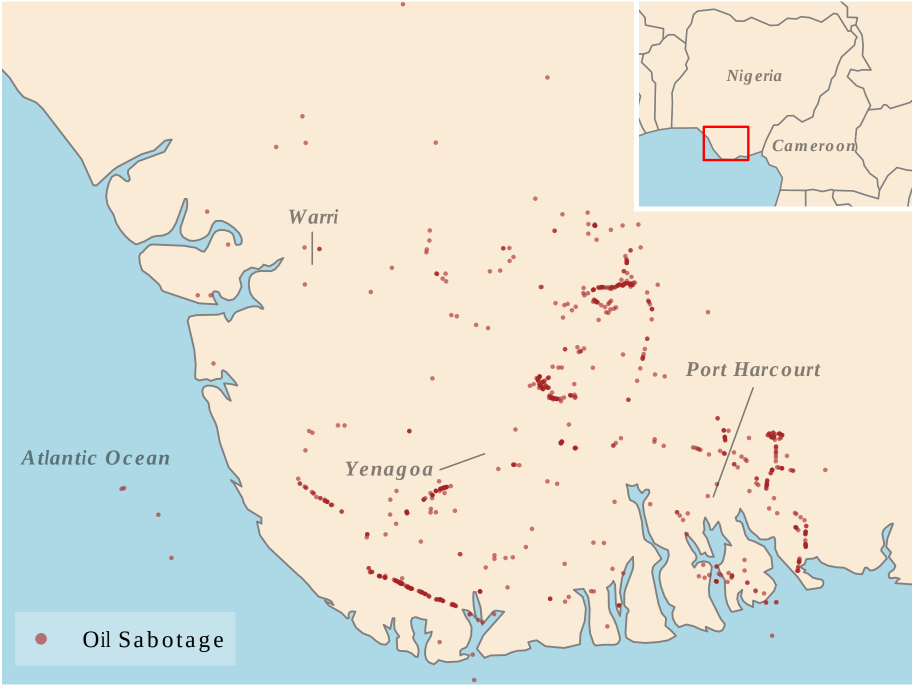
How many, if any, of these individuals and groups do you support?

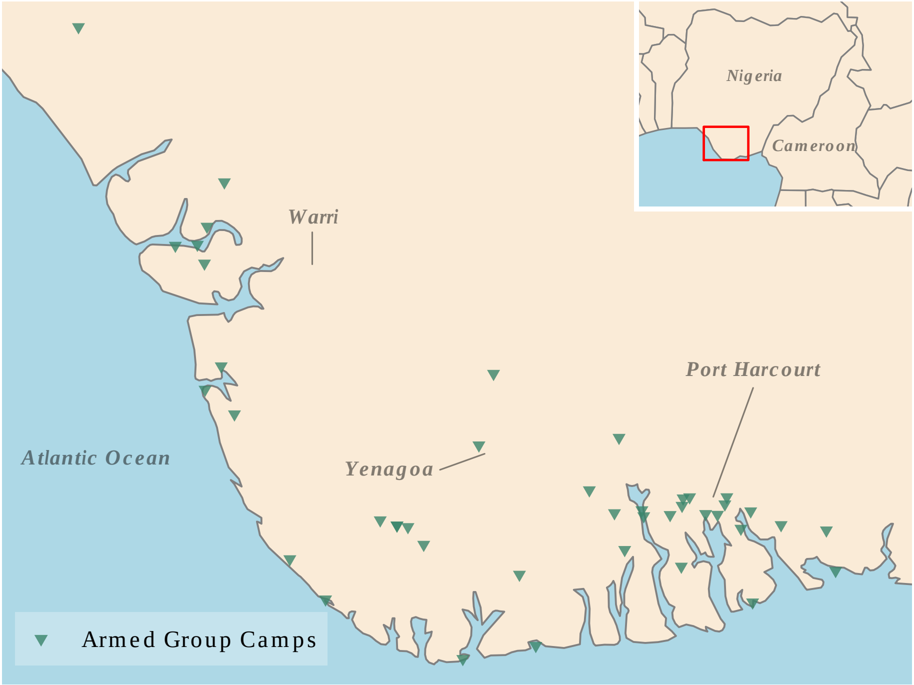
Proportion of ISAF Supporters



Survey

- Survey of 2,448 civilians in the Niger Delta
- Randomly sampled 204 communities near oil interruption sites and camps of armed groups


Nigeria
Cameroon
Warri
Port Harcourt
Atlantic Ocean
Yenagoa
Oil Sabotage


Nigeria
Cameroon
Warri
Port Harcourt
Yenagoa
Atlantic Ocean
Armed Group Camps

Survey

- Survey of 2,448 civilians in the Niger Delta
- Random sample of 204 communities near and far from oil interruption sites and armed group camps
- Interviewed 12 people per community
  Random walk pattern to select households; Kish grid within household

Funded by the International Growth Centre

Outcome

"Did you share information with **militants** about their enemies in the community, state counterinsurgency forces, or oil facility activities?"

Problems with using list or endorsement experiments

Too sensitive for list experiment

Often difficult to define "control" condition in endorsement experiment for behaviors

Alternative: **Randomized response technique**

Randomized response technique

How? Introducing random noise

- Roll the dice in private
- If you roll a 1, tell me "no"
- If you roll a 6, tell me "yes"
- Otherwise, answer: "Did you share information with armed groups"

Analysis of the randomized response technique

1. Used fair dice, and actually rolled it.
2. **Compliance.** Complied with "forced" response.
3. **No Liars.** When not forced, answered truthfully.

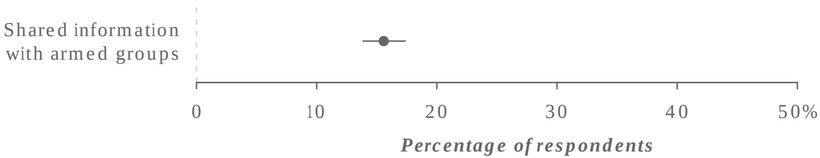
Proportion answered yes

$= 2/3 \cdot$ Proportion yes to sensitive item $+ 1/6$

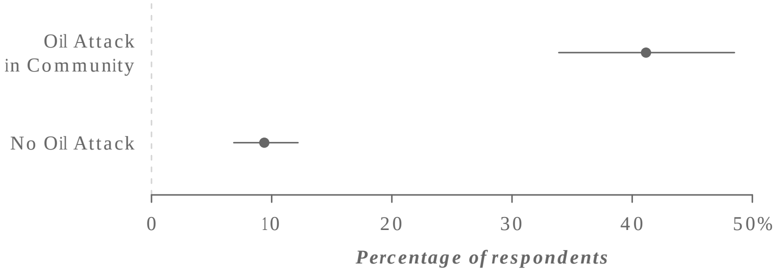
Proportion yes to sensitive item

$= 3/2 \cdot ($ Proportion answered yes $- 1/6)$

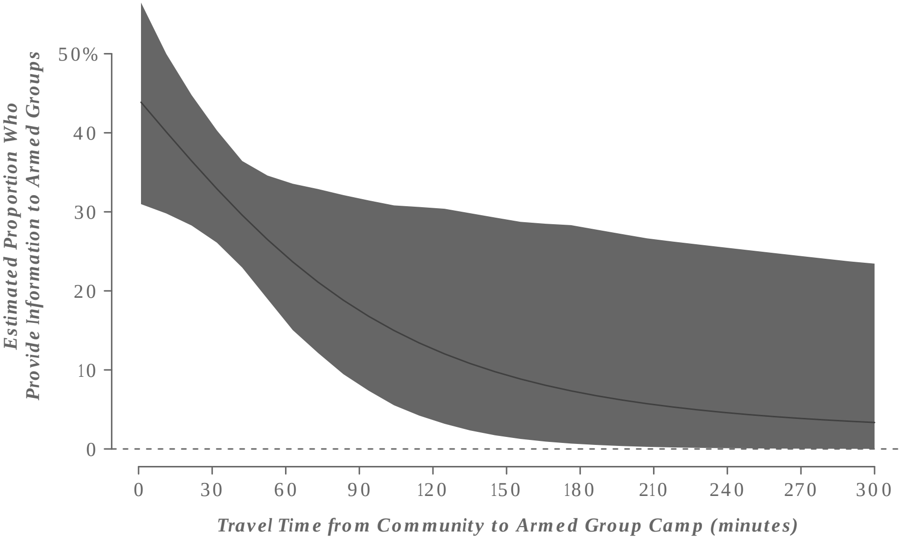
1. Civilians share information regularly with armed groups



2. Civilians near oil interruptions dominate collaboration



3. Civilians near armed group camps dominate collaboration



Three techniques for sensitive survey items

- **Endorsement experiment** Baseline attitudes
- **List experiment** Aggregation
- **Randomized response** Random noise

Alternative methods

- Physical separation from respondents
  Scacco 2012
- Self-administered questionnaires (e.g. MP3)
  Chauchard 2013
- Incentives for honest responses
  Bursztyn et al. 2014

Design Advice and Software for Analysis

- `rr` package in R for randomized response

  Blair with Yang-Yang Zhou and Kosuke Imai

- `list` package in R for list experiments

  Blair with Kosuke Imai

- `endorse` package in R for endorsement experiments

  Yuki Shiraito and Kosuke Imai

Why Do We Focus on Means?

Back

- Population means ($\bar{y} = \frac{1}{N}\sum_{i=1}^{N} y_i$) often provide a “good” summary of the center of the data (and it is relatively easy to tell when they provide bad summaries).
- The accuracy of means is relatively easy to describe.
- Randomized experiments identify average causal effects (more on this later)

The Mean as a Least Squares Summary

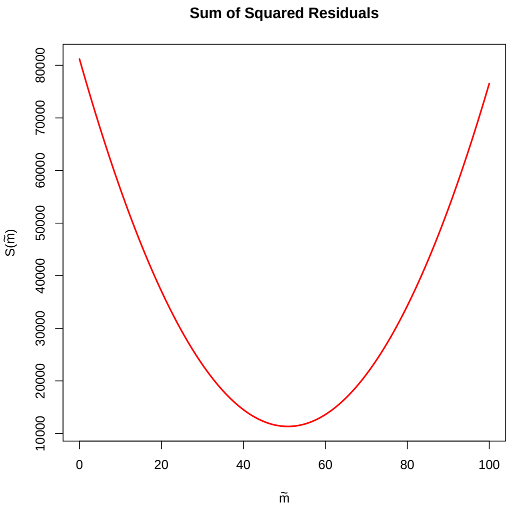
Suppose we want to pick a single number (m) for the middle of the data that summarizes all the values for y, by minimizing the sum of squared residuals (i.e., least squares).

$$SSR(\tilde{m}) = \sum_{i=1}^{N} (y_i - \tilde{m})^2.$$

One way to calculate the least squares estimator

1. Calculate the derivative of SSR with respect to $\tilde{m}$
2. Set the derivative equal to 0
3. Solve for m

The Objective Function for SSR CLlib



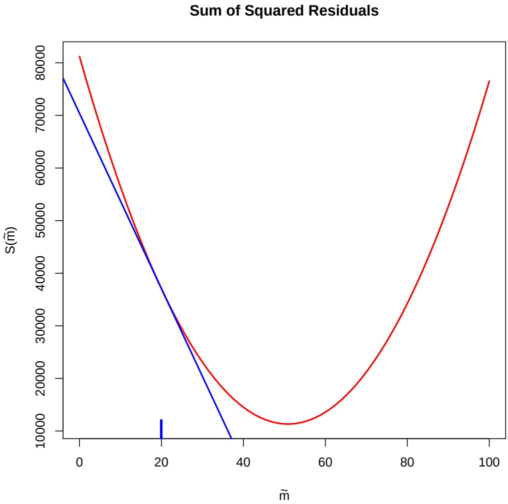
1.

$$SSR(\tilde{m}) = \sum_{i=1}^{N} (y_i - \tilde{m})^2$$

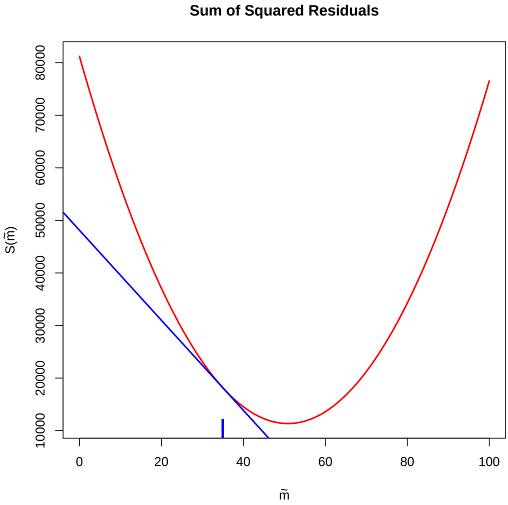
$$= \sum_{i=1}^{N} (y_i^2 - 2y_i\tilde{m} + \tilde{m}^2)$$

$$\frac{\partial SSR(\tilde{m})}{\partial \tilde{m}} = \sum_{i=1}^{N} (-2y_i + 2\tilde{m})$$

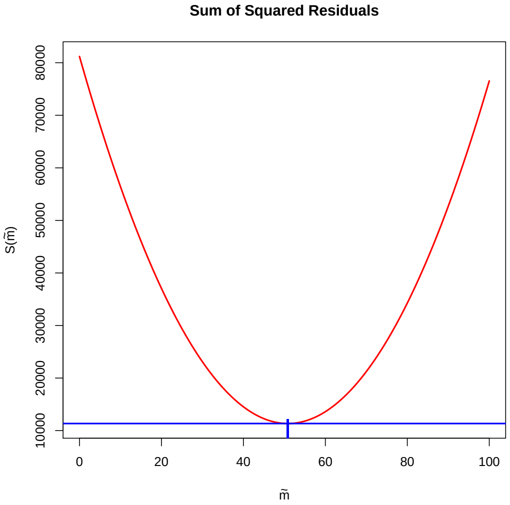
The Slope of the Tangent Line for SSR CLlib



The Slope of the Tangent Line for SSR CLlib



The Slope of the Tangent Line for SSR CLlib



1.

$$SSR(\tilde{m}) = \sum_{i=1}^{N}(y_i - \tilde{m})^2$$
$$= \sum_{i=1}^{N}(y_i^2 - 2y_i\tilde{m} + \tilde{m}^2)$$

$$\frac{\partial S(m)}{\partial \tilde{m}} = \sum_{i=1}^{N}(-2y_i + 2\tilde{m})$$

2.

$$0 = \sum_{i=1}^{N}(-2y_i + 2m)$$

3.

$$m = \frac{1}{N}\sum_{i=1}^{N} y_i$$

Variance and Standard Deviation

- Mean is the least squares predictor in terms of SSR.
- How small is "least" (i.e., how much spread around the mean)?
- Variance and Standard Deviation are useful transformations of SSR for the mean.

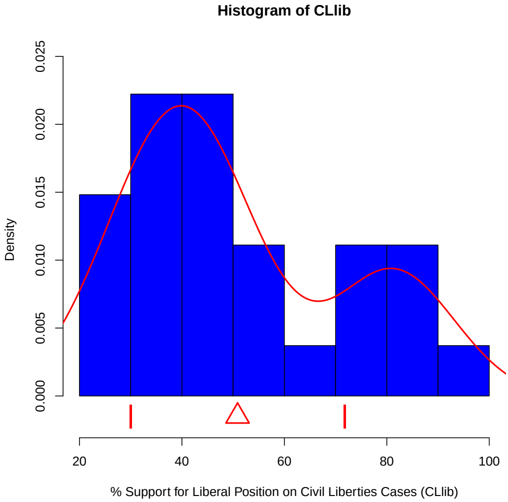
Population Variance

$$S^2 = \frac{\sum_{i=1}^{N}(y_i - \bar{y})^2}{N-1} = \frac{SSR}{N-1}$$

Population Standard Deviation

$$S = \sqrt{S^2}$$

Mean and Standard Deviation



References

- Kuklinski et al. 1997 "Racial prejudice and attitudes toward affirmative action" *American Journal of Political Science*
- Glynn 2013 "What can we learn with statistical truth serum? Design and analysis of the list experiment"
- All the Blair papers above.

Where We've Been and Where We're Going...

- Last Week
  - welcome and outline of course
  - described uncertain outcomes with probability.
- This Week
  - Monday:
    - summarize one random variable using expectation and variance
    - show how to condition on a variable
  - Wednesday:
    - properties of joint distributions
    - conditional expectations
    - covariance, correlation, independence
- Next Week
  - estimating these features from data
  - estimating uncertainty
- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

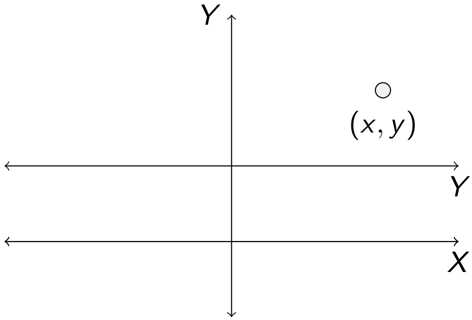
Questions?

Joint Distributions

- We've talked about joint probabilities of events—what was the probability of A and B occurring: $P(A \cap B)$
- We also talked about the conditional probability of A given that B occurred.
- We also need to think about more than one r.v. at the same time.
  - in regression we think about how the distribution of one variable changes under different values of another variable
  - e.g. does running more negative ads decrease election turnout?
- The joint distribution of two (or more) variables describes the pairs of observations that we are more or less likely to see.

Understanding Joint Distributions

- Consider two r.v.s now, X and Y, each on the real line, $\mathbb{R}$.
- The pair form a two-dimensional space, or $\mathbb{R} \times \mathbb{R}$
- One realization of the r.v. is a point in that space



Understanding Joint Distributions

- Imagine we are throwing darts on a two-dimensional board: the joint distribution tells us where the darts are more likely to land.
- Distributions can be limited to a subset of the real line
  - e.g. two uniform random variables might be between 0 and 1
  - e.g. discrete random variables typically only include integers
- With two r.vs. there are now two dimensions to deal with.
- Often, we are interested in two random variables that are qualitatively different:
  - Y (response, outcome, dependent variable, etc.)
    $=$ the random variable we want to explain, or predict.
  - X (predictor, explanatory/independent variable, covariate, etc.)
    $=$ the random variable with which we want to explain Y.

Joint Probability Mass Function

Definition

For two discrete random variables X and Y the joint PMF $P_{X,Y}(x,y)$ gives the probability that $X = x$ and $Y = y$ for all x and y:

$$P_{X,Y}(x,y) \; = \; \Pr(X = x \text{ and } Y = y)$$

Restrictions:

- $P_{X,Y}(x,y) \geq 0$ and $\sum_x \sum_y P_{X,Y}(x,y) = 1$.

Joint Probability Mass Function

Definition

For two discrete random variables X and Y the joint PMF $P_{X,Y}(x,y)$ gives the probability that $X = x$ and $Y = y$ for all x and y:

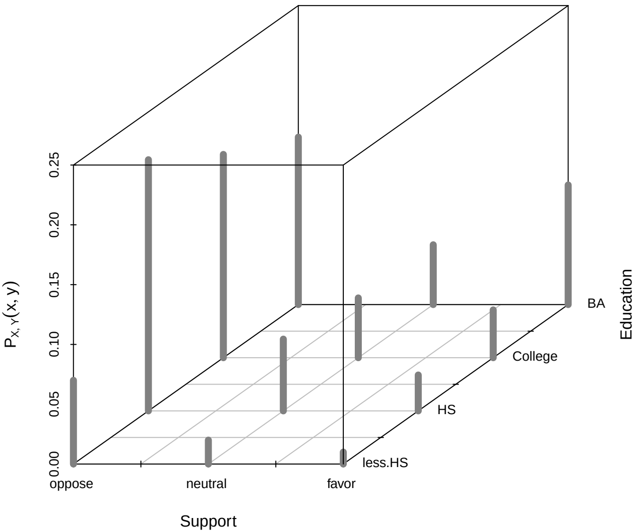
$$P_{X,Y}(x,y) \; = \; \Pr(X = x \text{ and } Y = y)$$

Should the U.S. allow more immigrants to come and live here?

		X: Education			
		less HS	HS	College	BA
Y: Support	oppose	0.07	0.22	0.18	0.15
	neutral	0.02	0.06	0.05	0.05
	favor	0.01	0.03	0.04	0.11

With discrete r.v.s this is very similar to thinking about a cross-tab, with frequencies/ probabilities in the cells instead of raw numbers.

Joint Probability Mass Function



From Joint to Marginal PMF

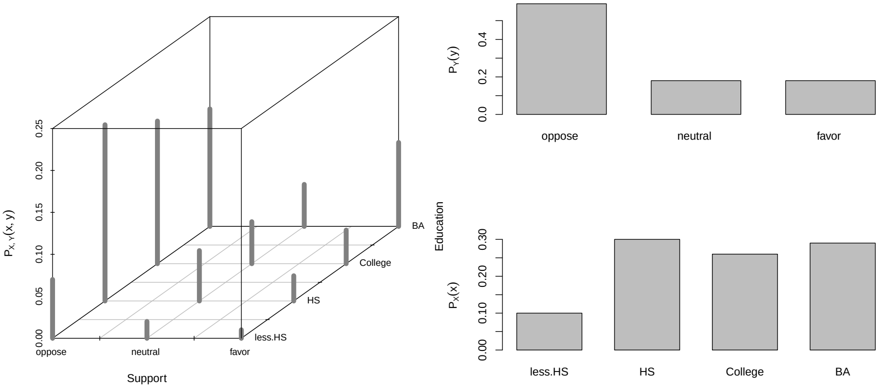
Given the joint PMF $P_{X,Y}(x,y)$ can we recover the marginal PMF $P_Y(y)$ (distribution over a single variable)?

		X: Education				
		less HS	HS	College	BA	$P_Y(y)$
Y: Support	oppose	0.07	0.21	0.17	0.14	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.10	0.19

To obtain $P_Y(y)$ we marginalize the joint probability function $P_{X,Y}(x,y)$ over X:

$$P_Y(y) = \sum_x P_{X,Y}(x,y) = \sum_x \Pr(X = x, Y = y)$$

Joint and Marginal Probability Mass Functions



Why Does Marginalization Work?

Begin with discrete case. Consider jointly distributed discrete random variables, X and Y. We'll suppose they have joint pmf,

$$P(X = x, Y = y) = p(x, y)$$

Suppose that the distribution allocates its mass at $x_1, x_2, \ldots, x_M$ and $y_1, y_2, \ldots, y_N$.

Define the conditional mass function $P(X = x|Y = y)$ as,

$$\begin{aligned} P(X = x|Y = y) \equiv &= p(x|y) \\ &= p(x, y)/p(y) \end{aligned}$$

Then it follows that:

$$p(x, y) = p(x|y)p(y)$$

Marginalizing over y to get $p(x)$ is then,

$$p(x_j) = \sum_{i=1}^{N} p(x_j|y_i)p(y_i)$$

A Table

	Y = 0	Y = 1	
X = 0	p(0,0)	p(0, 1)	$p_X(0)$
X = 1	p(1,0)	p(1,1)	$p_X(1)$
	p_Y (0)	p_Y (1)	

	Y = 0	Y = 1	
X = 0	0.01	0.05	?
X = 1	0.25	0.69	?
	0.26	0.74	

$$\begin{aligned} p_X(0) &= p(0|y=0)p(y=0) + p(0|y=1)p(y=1) \\ &= \frac{0.01}{0.26} \times 0.26 + \frac{0.05}{0.74} \times 0.74 \\ &= 0.06 \end{aligned}$$

$$\begin{aligned} p_X(1) &= p(1|y=0)p(y=0) + p(1|y=1)p(y=1) \\ &= \frac{0.25}{0.26} \times 0.26 + \frac{0.69}{0.74} \times 0.74 \\ &= 0.94 \end{aligned}$$

Conditional PMF

Definition

The conditional PMF of Y given X, $P_{Y|X}(y|x)$, is the PMF of Y when X is known to be at a particular value $X = x$:

$$P_{Y|X}(y|x) = \frac{\Pr(X = x \text{ and } Y = y)}{\Pr(X = x)} = \frac{P_{X,Y}(x,y)}{P_X(x)}$$

Key relationships:

- $P_{X,Y}(x,y) = P_{Y|X}(y|x)P_X(x)$ (multiplicative rule)
- $P_{Y|X}(y|x) = P_{X|Y}(x|y)P_Y(y)/P_X(x)$ (Bayes' rule)

Conditional PMFs are just like ordinary PMFs, but refer to a universe where the "conditioning event" $(X = x)$ is known to have occurred.

Conditional distributions are key in statistical modeling because they inform us how the distribution of Y varies across different levels of X.

From Joint to Conditional: $P_{Y|X}(y|x) = \frac{P_{X,Y}(x,y)}{P_X(x)}$

Table: Joint PMF $P_{X,Y}(x, y)$ and Marginal PMFs $P_X(x), P_Y(y)$

		Education				
	$P_{X,Y}(x, y)$	less HS	HS	College	BA	$P_Y(y)$
Support	oppose	0.07	0.22	0.18	0.15	0.62
	neutral	0.02	0.06	0.05	0.05	0.19
	favor	0.01	0.03	0.04	0.11	0.19
	$P_X(x)$	0.11	0.32	0.27	0.31	1.00

Table: Conditional PMF $P_{Y|X}(y|x)$

		Education				
	$P_{Y|X}(y|x)$	less HS	HS	College	BA	
Support	oppose	0.70	0.70	0.65	0.48	0.62
	neutral	0.20	0.20	0.19	0.17	0.19
	favor	0.10	0.10	0.15	0.34	0.19
		1.00	1.00	1.00	1.00	1.00

Joint and Conditional Probability Mass Functions

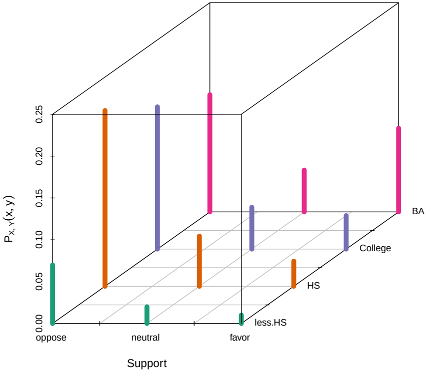


Figure: Joint

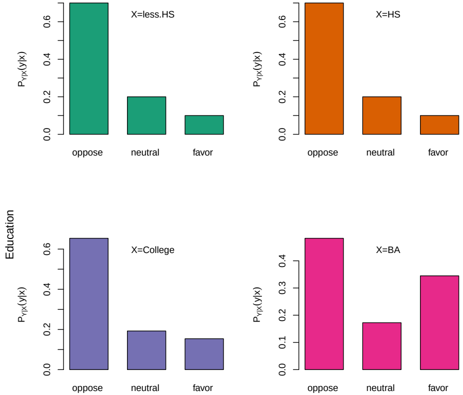


Figure: Conditional

Joint Probability Density Function

Definition

For two continuous random variables X and Y the joint PDF $f_{X,Y}(x,y)$ gives the density height where $X = x$ and $Y = y$ for all x and y.

The multiplicative rule:

$$f_{X,Y}(x,y) = f_{Y|X}(y|x) f_X(x)$$

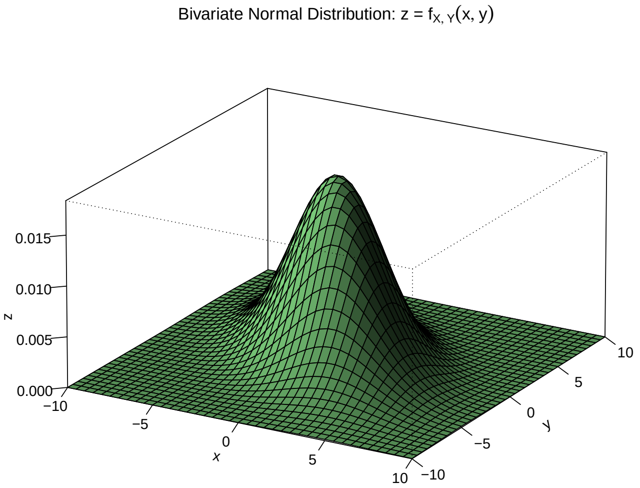
where

- $f_{Y|X}(y|x)$: Conditional PDF of Y given $X = x$
- $f_X(x)$: Marginal PDF of X

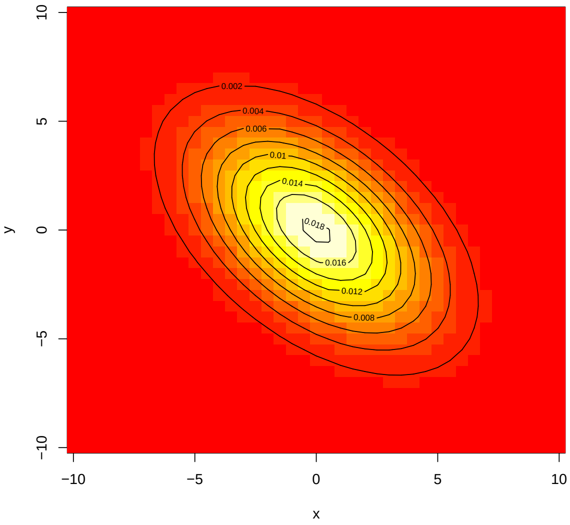
Restrictions:

- $\int_x \int_y f_{X,Y}(x,y)\, dy\, dx = 1$

3D Plot of a Joint Probability Density Function



Contour Plot of a Joint Probability Density Function

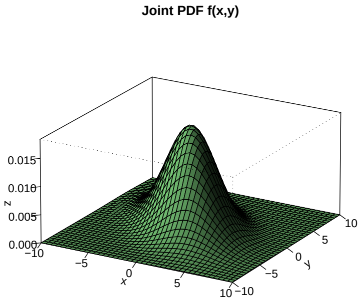


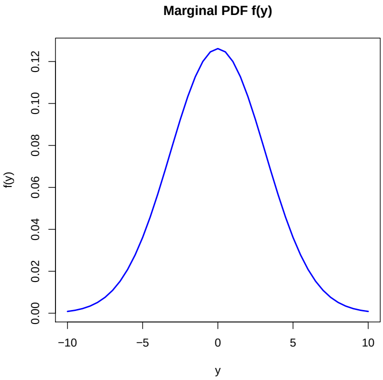
From Joint to Marginal PDF

How can we obtain $f_Y(y)$ from $f_{X,Y}(x,y)$?

We marginalize the joint probability function $f_{X,Y}(x,y)$ over X:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$





Conditioning on X

- Goal in statistical modeling is often to characterize the conditional distribution of the outcome variable $f_{Y|X}(y|x)$ across different levels of $X = x$.

- Typically, we summarize the conditional distributions with a few parameters such as the conditional mean of $E[Y|X = x]$ and the conditional variance $V[Y|X = x]$

- Moreover, we are often interested in estimating $E[Y|X]$, i.e. the conditional expectation function that describes how the conditional mean of Y varies across all possible values of X (we sometimes call this the population regression function)

Conditional Expectation

Definition (Conditional Expectation (Discrete))

Let Y and X be discrete random variables. The conditional expectation of Y given $X = x$ is defined as:

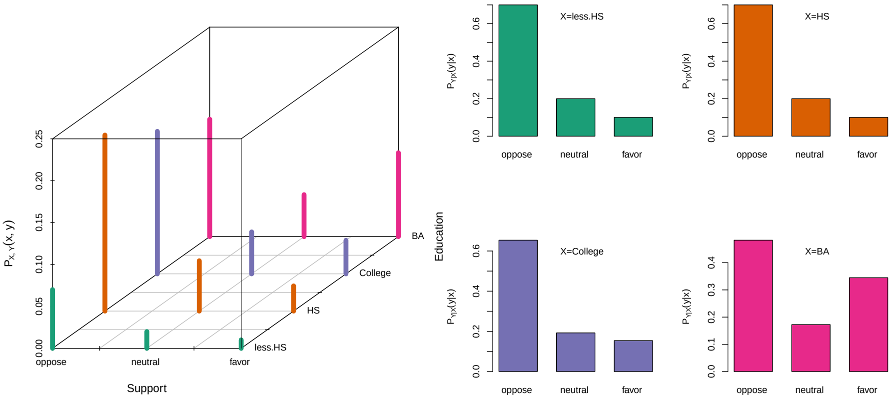
$$E[Y|X = x] = \sum_{y} y \Pr(Y = y|X = x) = \sum_{y} y \, P_{Y|X}(y|x)$$

Definition (Conditional Expectation (Continuous))

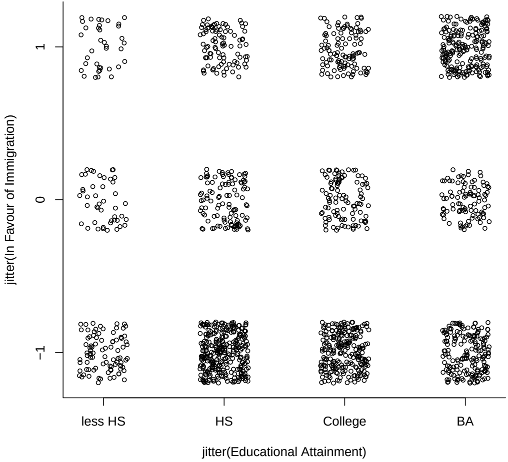
Let Y and X be continuous random variables. The conditional expectation of Y given $X = x$ is given by:

$$E[Y|X = x] = \int_{-\infty}^{\infty} y \, f_{Y|X}(y|x) dy$$

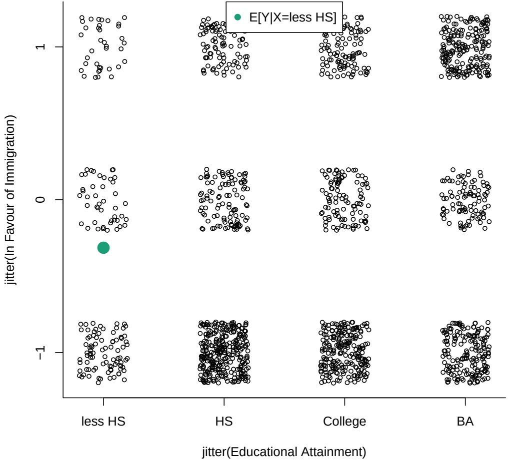
Joint and Conditional Probability Mass Functions



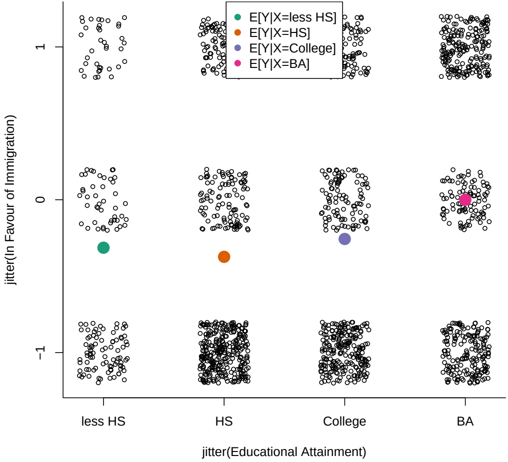
Conditional PMF $P_{Y|X}(y|x)$



Conditional Expectation $E[Y|X=1]$



Conditional Expectation Function $E[Y|X]$



Law of Iterated Expectations

Theorem (Law of Iterated Expectations)

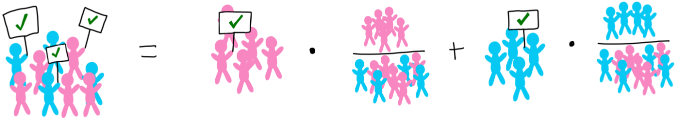
For two random variables X and Y,

$$E[Y] = E[E[Y|X]] = \begin{cases} \sum_{\text{all } x} E[Y|X=x] \cdot P_X(x) & (\textit{discrete } X) \\ \int_{-\infty}^{\infty} E[Y|X=x] \cdot f_X(x)dx & (\textit{continuous } X) \end{cases}$$

Note that the outer expectation is taken with respect to the distribution of X.

Example: Y (support) and $X \in \{1, 0\}$ (gender). Then, the LIE tells us:

$$\begin{aligned} \underbrace{E[Y]}_{\text{Average Support}} &= E[E[Y|X]] \\ &= \underbrace{E[Y|X=1]}_{\text{Average Support}|\text{Woman}} \cdot \underbrace{P_X(1)}_{\Pr(\text{Woman})} + \underbrace{E[Y|X=0]}_{\text{Average Support}|\text{Man}} \cdot \underbrace{P_X(0)}_{\Pr(\text{Man})} \end{aligned}$$



Properties of Conditional Expectation

Conditional expectations have some convenient properties

1. $E[c(X)|X] = c(X)$ for any function $c(X)$.
   - Basically, any function of X is a constant with regard to the conditional expectation. If we know X, then we also know X^2, for instance.
2. If $E[Y^2] < \infty$ and $E[g(X)^2] < \infty$ for some function g, then $E[(Y - E[Y|X])^2|X] \leq E[(Y - g(X))^2|X]$ and $E[(Y - E[Y|X])^2] \leq E[(Y - g(X))^2]$

The second property is quite important. It says that the conditional expectation is the function of X that minimizes the squared prediction error for Y across any possible function of X.

Conditional Variance

Conditional expectation gives us information about the central tendency of a random variable given another random variable.

We also want to know the conditional variance to understand our uncertainty about the conditional distribution.

Remember, the conditional distribution of $Y|X$ is basically like any other probability distribution, so we are going to want to summarize the center *and* spread.

Conditional Variance

Definition

The conditional variance of Y given $X = x$ is defined as:

$$V[Y|X=x] \; = \; \begin{cases} \displaystyle\sum_{\text{all } y}(y - E[Y|X=x])^2 P_{Y|X}(y|x) & \text{(discrete } Y\text{)} \\ \displaystyle\int_{-\infty}^{\infty}(y - E[Y|X=x])^2 f_{Y|X}(y|x)dy & \text{(continuous } Y\text{)} \end{cases}$$

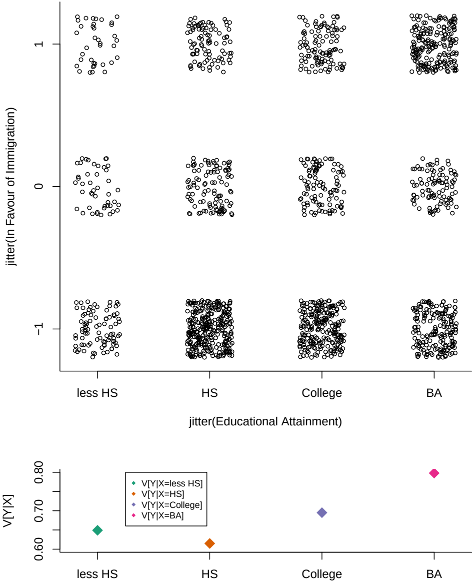
A useful rule related to conditional variance is the law of total variance:

$$\underbrace{V[Y]}_{\text{Total variance}} \quad = \quad \underbrace{E[V[Y|X]]}_{\text{Average of Group Variances}} \quad + \quad \underbrace{V[E[Y|X]]}_{\text{Variance in Group Averages}}$$

Example: Y (support) and $X \in \{1, 0\}$ (gender). The LTV says that the total variance in support can be decomposed into two parts:

1. On average, how much support varies within gender groups (within variance)
2. How much average support varies between gender groups (between variance)

Conditional Variance Function $V[Y|X]$



Independence

Definition (Independence of Random Variables)

Two random variables Y and X are independent if

$$f_{X,Y}(x, y) = f_X(x)f_Y(y)$$

for all x and y. We write this as $Y \perp\!\!\!\perp X$.

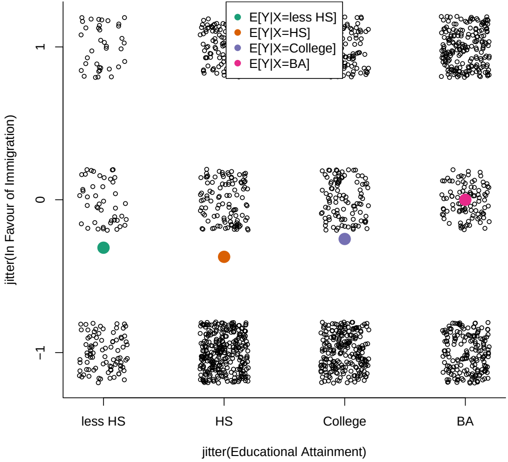
Independence implies

$$f_{Y|X}(y|x) = f_Y(y)$$

and thus

$$E[Y|X = x] = E[Y]$$

Is $Y \perp\!\!\!\perp X$?



Expected Values with Independent Random Variables

If random variables X and Y are independent, then

$$E[XY] = E[X]E[Y]$$

Proof: For discrete X and Y,

$$\begin{aligned}
E[XY] &= \sum_{\text{all } x} \sum_{\text{all } y} x\, y\, P_{X,Y}(x, y) \\
&= \sum_{\text{all } x} \sum_{\text{all } y} x\, y\, P_X(x) P_Y(y) \\
&= \sum_{\text{all } x} x\, P_X(x) \sum_{\text{all } y} y\, P_Y(y) \\
&= E[X]E[Y]
\end{aligned}$$

We can prove the continuous case by following the same steps, with $\sum$ replaced by $\int$.

Covariance

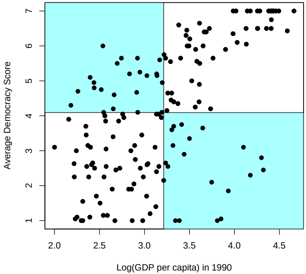
Definition

The covariance of X and Y is defined as:

$$Cov[X, Y] = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

- Covariance measures the linear association between two random variables .
- If $Cov[X, Y] > 0$, observing an X value greater than $E[X]$ makes it more likely to also observe a Y value greater than $E[Y]$, and vice versa.

  - Points in upper right and lower left quadrants (relative to the means) add to the covariance.
  - Points in the upper left and lower right quadrants subtract from the covariance.



Covariance and Independence

Does $X \perp\!\!\!\perp Y$ imply $Cov[X, Y] = 0$? Yes!

Proof:

$$\begin{aligned} Cov[X, Y] &= E[XY] - E[X]E[Y] \\ &= E[X]E[Y] - E[X]E[Y] \quad \text{(independence)} \\ &= 0. \end{aligned}$$

Does $Cov[X, Y] = 0$ imply $X \perp\!\!\!\perp Y$? No!

Counterexample: Suppose $X \in \{-1, 0, 1\}$ with $P_X(x) = 1/3$ and $Y = X^2$.

Is $X \perp\!\!\!\perp Y$? No, because $P_{Y|X}(y \mid x) \neq P_Y(y)$
(Learning about X gives meaningful information about Y.)

What is $Cov[X, Y]$?

$$\begin{aligned} Cov[X, Y] &= E[XX^2] - E[X]E[X^2] \;=\; E[X^3] - E[X]E[X^2] \\ &= E[X] - E[X]E[X^2] \;=\; 0 - 0 \cdot E[X^2] \;=\; 0. \end{aligned}$$

Therefore, $X \perp\!\!\!\perp Y \implies Cov[X, Y] = 0$, but not vice versa.

Important Identities for Variances and Covariances

1. For random variables X and Y and constants a, b and c,

$$V[aX + bY + c] = a^2 V[X] + b^2 V[Y] + 2ab\, Cov[X, Y]$$

2. Important special cases:

$$V[X + Y] = V[X] + V[Y] + 2Cov[X, Y]$$
$$V[X - Y] = V[X] + V[Y] - 2Cov[X, Y]$$

3. Furthermore, if X and Y are independent,

$$V[X \pm Y] = V[X] + V[Y]$$

Proof: Plug in to the definition of variance and expand (try it yourself!)

Correlation

- $Cov[X, Y]$ depends not only on the strength of (linear) association between X and Y, but also the scale of X and Y.
- Can we have a pure measure of association that is scale-independent?

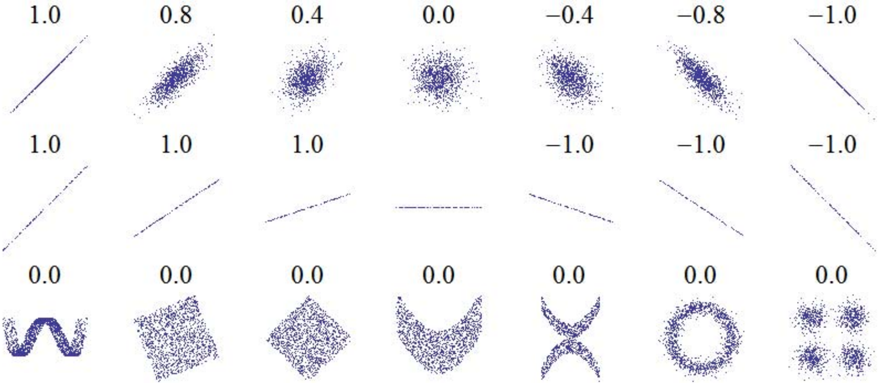
Definition (Correlation)

The correlation between two random variables X and Y is defined as

$$Cor[X, Y] = \frac{Cov[X, Y]}{\sqrt{V[X]V[Y]}} = \frac{Cov[X, Y]}{SD[X]SD[Y]}.$$

- $Cor[X, Y]$ is a standardized measure of linear association between X and Y.
- Always satisfies: $-1 \leq Cor[X, Y] \leq 1$.

Correlation is *Linear*



- $Cor[X, Y] = \pm 1$ iff $Y = aX + b$ where $a \neq 0$.
- Like covariance, correlation measures the linear association between X and Y.

Conditional Independence

Definition (Conditional Independence of Random Variables)

Random variables Y and X are conditionally independent given Z iff

$$f_{X,Y|Z}(x, y|z) \; = \; f_{Y|Z}(y|z) \cdot f_{X|Z}(x|z)$$

for all x, y, and z. This is often written as $Y \perp\!\!\!\perp X \mid Z$.

- Can also be written as

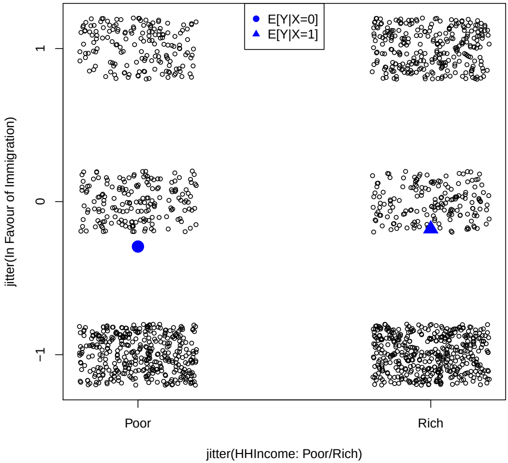
$$f_{Y|X,Z}(y \mid x, z) \; = \; f_{Y|Z}(y \mid z)$$

- Interpretation: Once we know Z, X contains no meaningful information about likely values of Y.
  (Z has all the information about Y contained in X, if any.)
- $Y \perp\!\!\!\perp X \mid Z$ implies

$$E[Y|X = x, Z = z] = E[Y|Z = z].$$

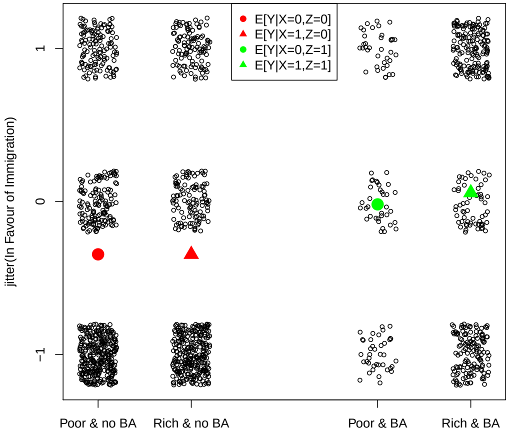
Is $Y \perp\!\!\!\perp X$?

Example: X = wealth, Y = support for immigration, Z = education.



Is $Y \perp\!\!\!\perp X | Z$?

Example: $X =$ wealth, $Y =$ support for immigration, $Z =$ education.



Distributions

- We like random variables because they take complex real world phenomena and represent them with a common mathematical infrastructure
- We can work with arbitrary pmf/pdfs but we will often work with particular families of distributions
  - members of the same family have similar forms determined by parameters
  - the parameters determine the shape of the distribution
- When we can work with an existing set of distributions, it makes calculations simpler
- Examples: Bernoulli, Binomial, Gamma, Normal, Poisson, t-distribution

Bernoulli Random Variable

Definition

Suppose X is a random variable, with $X \in \{0, 1\}$ and $P(X = 1) = \pi$.
Then we will say that X is Bernoulli random variable,

$$p(X = x) \quad = \quad \pi^x (1 - \pi)^{1-x}$$

for $x \in \{0, 1\}$ and $p(X = x) = 0$ otherwise.
We will (equivalently) say that

$$X \quad \sim \quad \text{Bernoulli}(\pi)$$

Bernoulli Random Variable Mean and Variance

Suppose $X \sim \text{Bernoulli}(\pi)$

$$
\begin{aligned}
E[X] &= 1 \times P(X = 1) + 0 \times P(X = 0) \\
&= \pi + 0(1 - \pi) = \pi \\
\text{var}(X) &= E[X^2] - E[X]^2 \\
E[X^2] &= 1^2 P(X = 1) + 0^2 P(X = 0) \\
&= \pi \\
\text{var}(X) &= \pi - \pi^2 \\
&= \pi(1 - \pi)
\end{aligned}
$$

$E[X] = \pi$
$\text{var}(X) = \pi(1 - \pi)$
Importantly, we can also just look this up!

Normal/Gaussian Random Variables

Definition

Suppose X is a random variable with $X \in \Re$ and density

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

Then X is a normally distributed random variable with parameters μ and σ^2.
Equivalently, we'll write

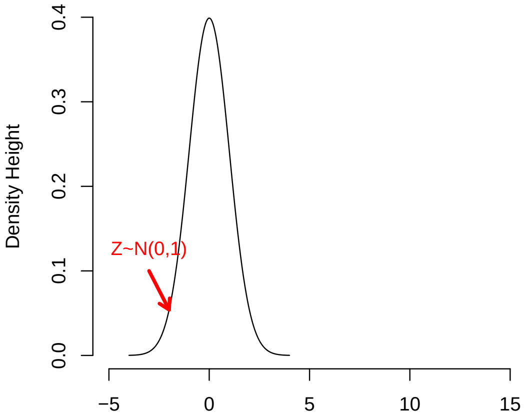
$$X \sim \text{Normal}(\mu, \sigma^2)$$

Expected Value/Variance of Normal Distribution

Z is a standard normal distribution if

Intuition

Suppose $Z \sim \text{Normal}(0, 1)$.



Proof: $Z \sim N(0, 1)$ and $Y = aZ + b$, then $Y \sim N(b, a^2)$

To prove

Proof: $Z \sim N(0,1)$ and $Y = aZ + b$, then $Y \sim N(b, a^2)$

So, we can work with $F_Z(\frac{x-b}{a})$.

Expectation and Variance

Assume we know:

$$
\begin{aligned}
E[Z] &= 0 \\
Var(Z) &= 1
\end{aligned}
$$

Multivariate Normal

Definition

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$ is a vector of random variables. If $\boldsymbol{X}$ has pdf

$$f(\boldsymbol{x}) = (2\pi)^{-N/2} \det(\boldsymbol{\Sigma})^{-1/2} \exp\left(-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}(\boldsymbol{x}-\boldsymbol{\mu})\right)$$

Then we will say $\boldsymbol{X}$ has a Multivariate Normal Distribution,

$$\boldsymbol{X} \sim \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

Multivariate Normal Distribution

Consider the (bivariate) special case where $\boldsymbol{\mu} = (0, 0)$ and

$$\boldsymbol{\Sigma} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Properties of the Multivariate Normal Distribution

Suppose $\boldsymbol{X} = (X_1, X_2, \ldots, X_N)$

$$
\begin{aligned}
E[\boldsymbol{X}] &= \boldsymbol{\mu} \\
\text{cov}(\boldsymbol{X}) &= \boldsymbol{\Sigma}
\end{aligned}
$$

So that,

$$
\boldsymbol{\Sigma} = \begin{pmatrix}
\text{var}(X_1) & \text{cov}(X_1, X_2) & \ldots & \text{cov}(X_1, X_N) \\
\text{cov}(X_2, X_1) & \text{var}(X_2) & \ldots & \text{cov}(X_2, X_N) \\
\vdots & \vdots & \ddots & \vdots \\
\text{cov}(X_N, X_1) & \text{cov}(X_N, X_2) & \ldots & \text{var}(X_N)
\end{pmatrix}
$$

One Step Deeper: Exponential Family

Nearly every distribution we will discuss is in the exponential family. An exponential family distribution has the density of the following form:

$$f_Y(y;\theta,\phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y,\phi)\right\}$$

Example: Poisson(μ):

$$\Pr(Y_i = y \mid \mu) \;=\; \exp\{y \log \mu - \exp(\log \mu) - \log y!\}$$

$\Longrightarrow \theta = \log \mu$, $\phi = 1$, $a(\phi) = \phi, b(\theta) = \exp(\theta)$, and $c = -\log y!$

Many other examples, including: Normal, Bernoulli/binomial, Gamma, multinomial, exponential, negative binomial, beta, uniform, chi-squared, etc.

One Step Deeper: Properties of the Exponential Family

- Mean is a function of θ and given by

$$\mathbb{E}(Y) \equiv \mu = b'(\theta)$$

- Variance is a function of θ and ϕ and given by

$$\mathbb{V}(Y) \equiv V = b''(\theta)a(\phi)$$

- Common forms of $a(\phi)$: 1 (Poisson, Bernoulli), ϕ (normal, Gamma), and ϕ/ω_i (binomial)
- $b''(\theta)$ is called the variance function
- In the Poisson model, $\theta_i = \log \mu_i$, $a(\phi) = 1$ and $b(\theta_i) = \exp(\theta_i)$
  $\Rightarrow \mathbb{E}(Y_i) = \frac{db(\theta_i)}{d\theta_i} = \exp(\theta_i) = \mu_i$ and $\mathbb{V}(Y_i) = \frac{d^2 b(\theta_i)}{d\theta_i^2} = \exp(\theta_i) = \mu_i$

Summary

- Random variables and probability distributions provide useful models of the world
- We can characterize distributions in terms of their expectation (location) and variance (spread).
- Joint and conditional distributions capture the relationship between random variables.
- There is a common set of famous distributions such as the Normal distribution.

Next Week

- Learning From Random Samples
- Point estimation
- Interval estimation
- Reading
  - Aronow and Miller 3.1-3.1.5 (estimation)
  - Aronow and Miller 3.2.1 (intervals)
  - Fox Chapter 3: Examining Data
  - Optional: Imai 7.1 (estimation/inference)

Fun With Spam

Fun With: Building a Spam Filter

Suppose we have an email i, ($i = 1, \ldots, N$) which we represent as a count of J words

Example: Building a Spam Filter

Goal: For each document $\boldsymbol{x}_i$, we want to infer most likely category

Example: Building a Spam Filter

Estimating the Naïve Bayes Classifier

Two components to estimate:

- $p(C_k) = \frac{\text{No. Documents in } k}{\text{No. Documents}}$
- $p(\boldsymbol{x}_i|C_k) = \prod_{j=1}^{J} p(x_{ij}|C_k)$

$$p(x_{im} = z|C_k) = \frac{\text{No(Docs}_{ij} = \text{z and C} = \text{C}_k \text{)}}{\text{No(C= C}_k\text{)}}$$

Algorithm steps:

1) Learn $\hat{p}(C)$ and $\hat{p}(\boldsymbol{x}_i|C_k)$ on labeled data
2) Use this to identify most likely C_k for each document i in unlabeled data

Simple intuition about Naïve Bayes:

- Learn what documents in class j look like
- Find class k that document i is most similar to

Example: Building a Spam Filter

Scoring the algorithm is easy.

$$p(C_k|\boldsymbol{x}_i) \propto p(C_k) \prod_{j=1}^{J} p(x_{i,j}|C_k)^{x_{ij}}$$

which is simply the probability of the class multiplied by the product of the probabilities for the words that are observed in the test document.

Example: Building a Spam Filter

- Learn the most probable class using Bayes Rule and a powerful but "naïve" independence assumption
- Despite that the model is "wrong" it classifies spam quite well
- Shares the basic structure of many models, is a building block for more complex models
- This was a complicated example, it is okay if you didn't follow all of it.

Questions?