Week 4: Testing/Regression

Brandon Stewart[1]

Princeton

October 3/5, 2016

[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller.

Where We've Been and Where We're Going...

- Last Week
  - inference and estimator properties
  - point estimates, confidence intervals
- This Week
  - Monday:
    - hypothesis testing
    - what is regression?
  - Wednesday:
    - nonparametric regression
    - linear approximations
- Next Week
  - inference for simple regression
  - properties of OLS
- Long Run
  - probability → inference → regression

Questions?

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

A Running Example for Testing

Statistics play an important role in determining which drugs are approved for sale by the FDA.

There are typically three phases of clinical trials before a drug is approved:

- Phase I: Toxicity (Will it kill you?)
- Phase II: Efficacy (Is there any evidence that it helps?)
- Phase III: Effectiveness (Is it better than existing treatments?)

Phase I trials are conducted on a small number of healthy volunteers, Phase II trial are either randomized experiments or within-patient comparisons, and Phase III trials are almost always randomized experiments with control groups.

Example

Consider a Phase II efficacy trial reported in Sowers et al. (2006), for a drug combination designed to treat high blood pressure in patients with metabolic syndrome.

- The trial included 345 patients with initial systolic blood pressure between 140-159.
- Each subject was assigned to take the drug combination for 16 weeks.
- Systolic blood pressure was measured on each subject before and after the treatment period.

Example

Subject	SBP_{before}	SBP_{after}	Decrease
1	147	135	12
2	153	122	31
3	142	119	23
4	141	134	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$
345	155	115	40

Example

- The drug was administered to 345 patients.
- On average, blood pressure was 21 points lower after treatment.
- The standard deviation of changes in blood pressure was 14.3.

Question: Should the FDA allow the drug to proceed to the next stage of testing?

The FDA's Decision

We can think of the FDA's problem in terms of two dimensions:

- The true state of the world
- The decision made by the FDA

	Drug works	Drug doesn't work
FDA approves	Good!	Bad!
FDA doesn't approve	Bad!	Good!

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Elements of a Hypothesis Test

Hypothesis testing gives us a systematic framework for making decisions based on observed data.

Terms to know:

- Null Hypothesis
- Alternative Hypothesis
- Test Statistic
- Rejection Region

Null and Alternative Hypotheses

- Null Hypothesis: The conservatively assumed state of the world (often "no effect")

  Example: The drug does not reduce blood pressure on average ($\mu_{decrease} \leq 0$)

- Alternative Hypothesis: Claim to be tested (research hypothesis)

  Example: The drug does reduce blood pressure on average ($\mu_{decrease} > 0$)

More Examples

Null Hypothesis Examples (H_0):

- The drug does not change blood pressure on average ($\mu_{decrease} = 0$)

Alternative Hypothesis Examples (H_a):

- The drug does change blood pressure on average ($\mu_{decrease} \neq 0$)

The FDA's Decision

Back to the two dimensions of the FDA's problem:

- The true state of the world
- The decision made by the FDA

	Drug works (H_0 False)	Drug doesn't work (H_0 True)
FDA approves (reject H_0)	Correct	Type I error
FDA doesn't approve (don't reject H_0)	Type II error	Correct

Test Statistics, Null Distributions, and Rejection Regions

Test Statistic: A function of the sample and the null hypothesis value of the parameter. For example:

$$\frac{\overline{X} - \mu_0}{\frac{S}{\sqrt{n}}}$$

Null Distribution: the sampling distribution of the statistic/test statistic <u>assuming that the null is true</u>.

Null Distributions

The CLT tells us that in large samples,

$$\overline{X} \sim_{approx} N(\mu, \sigma^2/n).$$

We know from our previous discussion that in large samples,

$$S/\sqrt{n} \approx \sigma/\sqrt{n}$$

If we assume that the null hypothesis is true such that $\mu = \mu_0$, then

$$\begin{aligned} \overline{X} &\sim_{approx} N(\mu_0, S^2/n) \\ \frac{\overline{X} - \mu_0}{\frac{S}{\sqrt{n}}} &\sim_{approx} N(0, 1) \end{aligned}$$

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

α

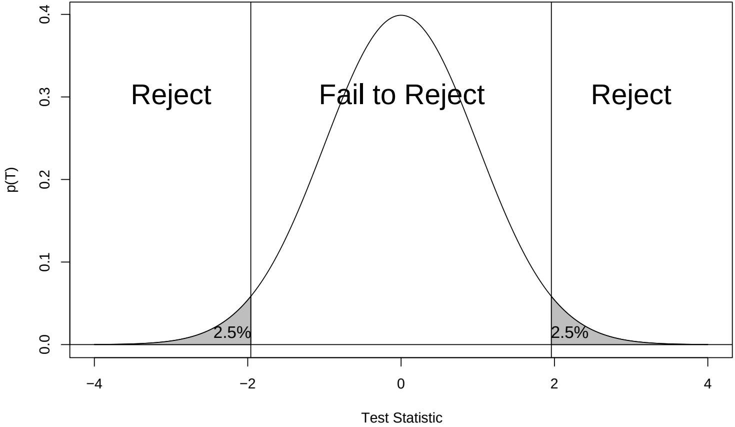
α is the probability of Type I error.

We usually pick an α that we are comfortable with in advance, and using the null distribution for the test statistic and the alternative hypothesis, we define a rejection region.

Example: Suppose $\alpha =$5%, the test statistic is $\frac{\overline{X}-\mu_0}{\frac{S}{\sqrt{n}}}$, the null hypothesis is $H_0 : \mu = \mu_0$, and the alternative hypothesis is $H_a : \mu \neq \mu_0$.

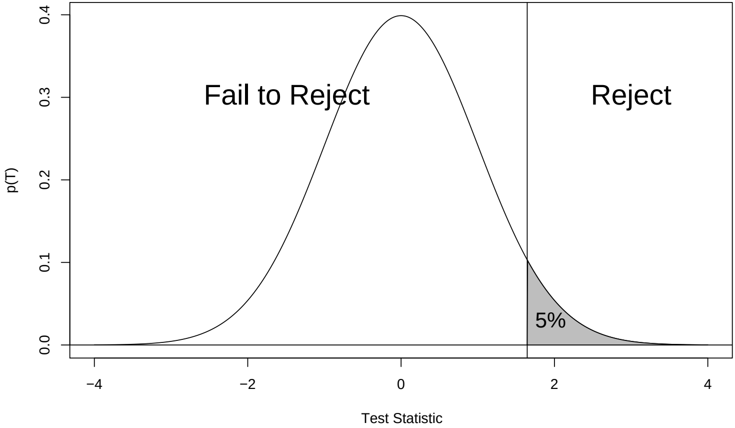
Two-sided rejection region

Rejection region with $\alpha = .05$, $H_0 : \mu = 0$, $H_A : \mu \neq 0$:



One-sided Rejection Region

Rejection region with $\alpha = .05$, $H_0 : \mu \leq 0$, $H_A : \mu > 0$:



Example

So, should the FDA approve further trials?

Recall the null and alternative hypotheses:

$$H_0 : \mu_{decrease} \leq 0$$

$$H_a : \mu_{decrease} > 0$$

Example

We can calculate the test statistic:

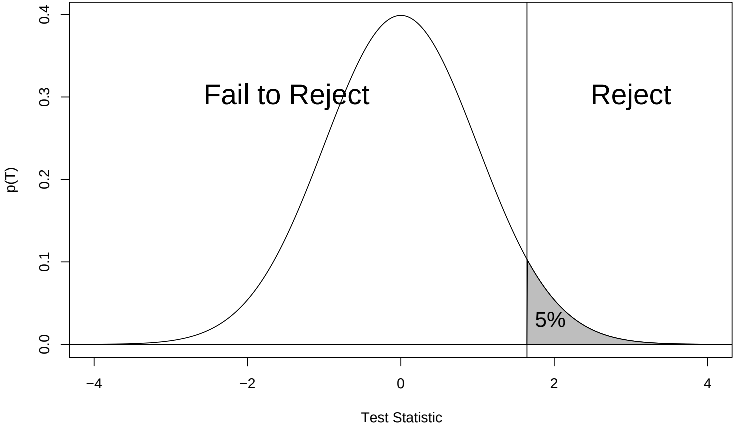
- $\overline{x} = 21.0$
- $s = 14.3$
- $n = 345$

Therefore,

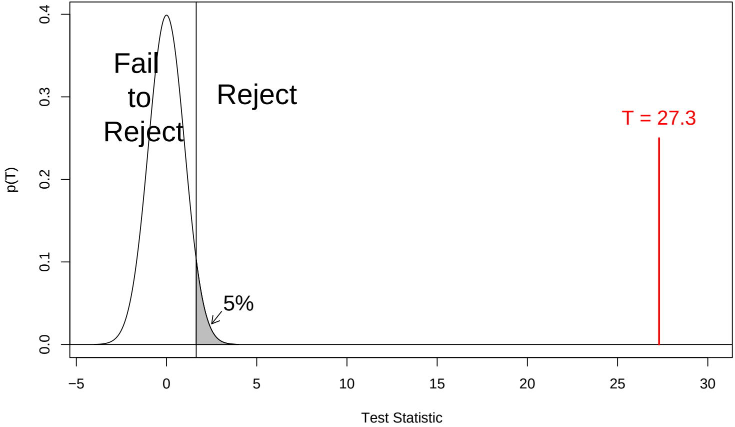
$$T = \frac{21.0 - 0}{\frac{14.3}{\sqrt{345}}} = 27.3$$

What is the decision?

Rejection Region with $\alpha = .05$



Rejection Region with $\alpha = .05$



1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

P-value

The appropriate level (α) for a hypothesis test depends on the relative costs of Type I and Type II errors.

What if there is disagreement about these costs?

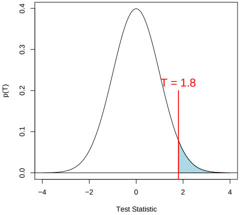
We might like a quantity that summarizes the strength of evidence against the null hypothesis without making a yes or no decision.

P-value: Assuming that the null hypothesis is true, the probability of getting something at least as extreme as our observed test statistic, where extreme is defined in terms of the alternative hypothesis.

P-value

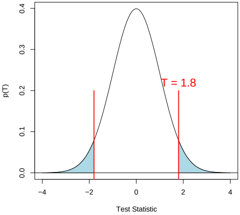
The p-value depends on both the realized value of the test statistic and the alternative hypothesis.

$$H_a : \mu > 0$$



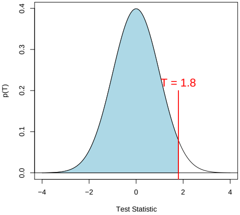
$$p = 0.036$$

$$H_a : \mu \neq 0$$



$$p = .072$$

$$H_a : \mu < 0$$

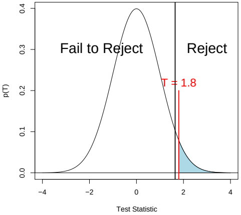


$$p = 0.964$$

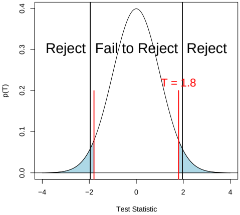
Rejection Regions and P-values

What is the relationship between p-values and the rejection region of a test? Assume that $\alpha = .05$:

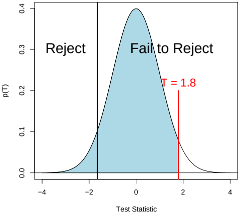
$H_a : \mu > 0$



$H_a : \mu \neq 0$



$H_a : \mu < 0$



If $p < \alpha$, then the test statistic falls in the rejection region for the α-level test.

Example 1

Recall the drug testing example, where $H_0 : \mu_0 \leq 0$ and $H_a : \mu_0 > 0$:

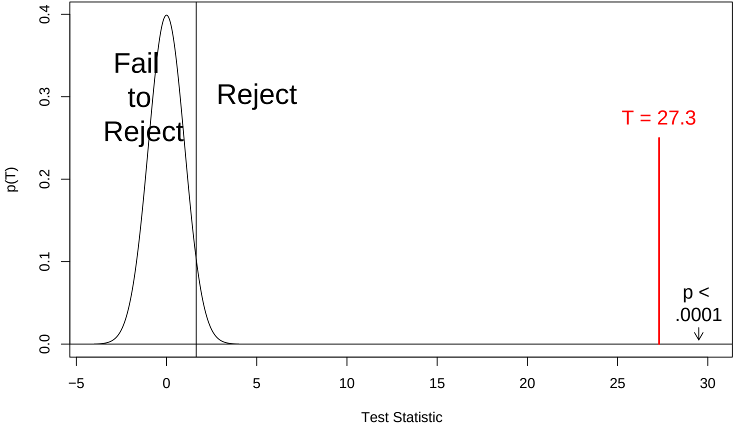
- $\overline{x} = 21.0$
- $s = 14.3$
- $n = 345$

Therefore,

$$T = \frac{21.0 - 0}{\frac{14.3}{\sqrt{345}}} = 27.3$$

What is the probability of observing a test statistic greater than 27.3 if the null is true?

Example 1


Fail
to
Reject
Reject
T = 27.3
p <
.0001
p(T)
0.0
0.1
0.2
0.3
0.4
−5
0
5
10
15
20
25
30
Test Statistic

α Rejection Regions and $1-\alpha$ CIs

Up to this point, we have defined rejection regions in terms of the test statistic.

In some cases, we can define an equivalent rejection region in terms of the parameter of interest.

For a two-sided, large-sample test, we reject if:

$$\frac{\overline{X}-\mu_0}{\frac{s}{\sqrt{n}}} > z_{\alpha/2} \text{ or } \frac{\overline{X}-\mu_0}{\frac{s}{\sqrt{n}}} < -z_{\alpha/2}$$

$$\overline{X}-\mu_0 > z_{\alpha/2}\times\frac{s}{\sqrt{n}} \text{ or } \overline{X}-\mu_0 < -z_{\alpha/2}\times\frac{s}{\sqrt{n}}$$

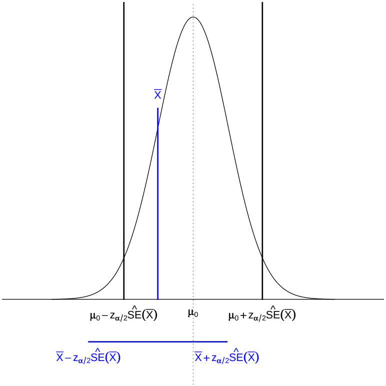
$$\overline{X} > \mu_0 + z_{\alpha/2}\times\frac{s}{\sqrt{n}} \text{ or } \overline{X} < \mu_0 - z_{\alpha/2}\times\frac{s}{\sqrt{n}}$$

α Rejection Regions and $1-\alpha$ CIs

The rescaled rejection region is related to $1-\alpha$ CI:

- If the observed $\overline{X}$ is in the α rejection region, the $1-\alpha$ CI does not contain μ_0.
- If the observed $\overline{X}$ is not in the α rejection region, the $1-\alpha$ CI contains μ_0.

Therefore, we can use the $1-\alpha$ CI to test the null hypothesis at the α level.


$\overline{X}$
$\mu_0 - z_{\alpha/2}\widehat{SE}(\overline{X})$
μ_0
$\mu_0 + z_{\alpha/2}\widehat{SE}(\overline{X})$
$\overline{X} - z_{\alpha/2}\widehat{SE}(\overline{X})$
$\overline{X} + z_{\alpha/2}\widehat{SE}(\overline{X})$

Another interpretation of CIs

The form of the "fail to reject" region of an α-level hypothesis test is:

$$\left(\mu_0 - z_{\alpha/2} \times \frac{s}{\sqrt{n}}, \mu_0 + z_{\alpha/2} \times \frac{s}{\sqrt{n}}\right)$$

The form of a region of a $1-\alpha$ CI is:

$$\left(\overline{X} - z_{\alpha/2} \times \frac{s}{\sqrt{n}}, \overline{X} + z_{\alpha/2} \times \frac{s}{\sqrt{n}}\right)$$

So the $1-\alpha$ CI is the set of null hypotheses μ_0 that would not be rejected at the α level.

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous *X*
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Hypothesis Testing: Setup

Goal: test a hypothesis about the value of a parameter.

Statistical decision theory underlies such hypothesis testing.

Trial Example:

Suppose we must decide whether to convict or acquit a defendant based on evidence presented at a trial. There are four possible outcomes:

		Defendant	
		Guilty	Innocent
Decision	Convict	Correct	Type I Error
	Acquit	Type II Error	Correct

We could make two types of errors:

- Convict an innocent defendant (type-I error)
- Acquit a guilty defendant (type-II error)

Our goal is to limit the probability of making these types of errors.

However, creating a decision rule which minimizes both types of errors at the same time is impossible. We therefore need to balance them.

Hypothesis Testing: Error Types

		Defendant	
		Guilty	Innocent
Decision	Convict	Correct	Type-I error
	Acquit	Type-II error	Correct

Now, suppose that we have a statistical model for the probability of convicting and acquitting, conditional on whether the defendant is actually guilty or innocent.

Then, our decision-making rule can be characterized by two probabilities:

- $\alpha \;=\; \text{Pr(type-I error)} \;=\; \Pr(\text{convict} \mid \text{innocent})$
- $\beta \;=\; \text{Pr(type-II error)} \;=\; \Pr(\text{acquit} \mid \text{guilty})$

The probability of making a correct decision is therefore $1 - \alpha$ (if innocent) and $1 - \beta$ (if guilty).

Hypothesis testing follows an analogous logic, where we want to decide whether to reject (= convict) or fail to reject (= acquit) a null hypothesis (= defendant) using sample data.

Hypothesis Testing: Steps

		Null Hypothesis (H_0)	
		False	True
Decision	Reject	$1-\beta$	α
	Fail to Reject	β	$1-\alpha$

1. Specify a null hypothesis H_0 (e.g. the defendant = innocent)

2. Pick a value of $\alpha = \Pr(\text{reject } H_0 \mid H_0)$ (e.g. 0.05). This is the maximum probability of making a type-I error we decide to tolerate, and called the significance level of the test.

3. Choose a test statistic T, which is a function of sample data and related to H_0 (e.g. the count of testimonies against the defendant)

4. Assuming H_0 is true, derive the null distribution of T (e.g. standard normal)

Hypothesis Testing: Steps

		Null Hypothesis (H_0)	
		False	True
Decision	Reject	$1-\beta$	α
	Fail to Reject	β	$1-\alpha$

5. Using the critical values from a statistical table, evaluate how unusual the observed value of T is under the null hypothesis:
    - If the probability of drawing a T at least as extreme as the observed T is less than α, we reject H_0.
      (e.g. there are too many testimonies against the defendant for her to be innocent, so reject the hypothesis that she was innocent.)
    - Otherwise, we fail to reject H_0.
      (e.g. there is not enough evidence against the defendant, so give her the benefit of the doubt.)

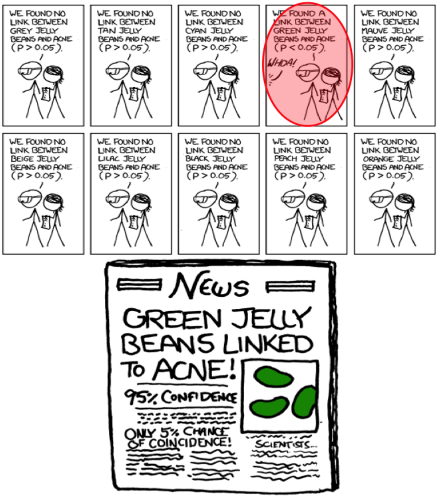
1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. **The Significance of Significance**
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Practical versus Statistical Significance

$$\frac{\overline{X} - \mu_0}{S/\sqrt{n}} \sim t_{n-1}$$

- What are the possible reasons for rejecting the null?
  1. $\overline{X} - \mu_0$ is large (big difference between sample mean and mean assumed by H_0)
  2. n is large (you have a lot of data so you have a lot of precision)
  3. S is small (the outcome has low variability)
- We need to be careful to distinguish:
  - practical significance (e.g. a big effect)
  - statistical significance (i.e. we reject the null)
- In large samples even tiny effects will be significant, but the results may not be very important substantively. Always discuss both!

Star Chasing (aka there is an XKCD for everything)


WE FOUND NO LINK BETWEEN GREY JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN TAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN CYAN JELLY BEANS AND ACNE (P > 0.05).
WE FOUND A LINK BETWEEN GREEN JELLY BEANS AND ACNE (P < 0.05).
WHOA!
WE FOUND NO LINK BETWEEN MAUVE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BEIGE JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN LILAC JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN BLACK JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN PEACH JELLY BEANS AND ACNE (P > 0.05).
WE FOUND NO LINK BETWEEN ORANGE JELLY BEANS AND ACNE (P > 0.05).
NEWS
GREEN JELLY BEANS LINKED TO ACNE!
95% CONFIDENCE
ONLY 5% CHANCE OF COINCIDENCE!
SCIENTISTS...

Multiple Testing

- If we test all of the coefficients separately with a t-test, then we should expect that 5% of them will be significant just due to random chance.
- Illustration: randomly draw 21 variables, and run a regression of the first variable on the rest.
- By design, no effect of any variable on any other, but when we run the regression:

Multiple Test Example

```
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0280393  0.1138198  -0.246  0.80605
## X2          -0.1503904  0.1121808  -1.341  0.18389
## X3           0.0791578  0.0950278   0.833  0.40736
## X4          -0.0717419  0.1045788  -0.686  0.49472
## X5           0.1720783  0.1140017   1.509  0.13518
## X6           0.0808522  0.1083414   0.746  0.45772
## X7           0.1029129  0.1141562   0.902  0.37006
## X8          -0.3210531  0.1206727  -2.661  0.00945 **
## X9          -0.0531223  0.1079834  -0.492  0.62412
## X10          0.1801045  0.1264427   1.424  0.15827
## X11          0.1663864  0.1109471   1.500  0.13768
## X12          0.0080111  0.1037663   0.077  0.93866
## X13          0.0002117  0.1037845   0.002  0.99838
## X14         -0.0659690  0.1122145  -0.588  0.55829
## X15         -0.1296539  0.1115753  -1.162  0.24872
## X16         -0.0544456  0.1251395  -0.435  0.66469
## X17          0.0043351  0.1120122   0.039  0.96923
## X18         -0.0807963  0.1098525  -0.735  0.46421
## X19         -0.0858057  0.1185529  -0.724  0.47134
## X20         -0.1860057  0.1045602  -1.779  0.07910 .
## X21          0.0021111  0.1081179   0.020  0.98447
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9992 on 79 degrees of freedom
## Multiple R-squared:  0.2009, Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.4797
```

Multiple Testing Gives False Positives

- Notice that out of 20 variables, one of the variables is significant at the 0.05 level (in fact, at the 0.01 level).
- But this is exactly what we expect: $1/20 = 0.05$ of the tests are false positives at the 0.05 level
- Also note that $2/20 = 0.1$ are significant at the 0.1 level. Totally expected!

Problem of Multiple Testing

- The multiple testing (or "multiple comparison") problem occurs when one considers a set of statistical tests simultaneously.
- Consider $k = 1, ..., m$ independent hypothesis tests (e.g. control versus various treatment groups). Even if each test is carried out at a low significance level (e.g., $\alpha = 0.05$) the overall type I error rate grows very fast:
  $\alpha_{overall} = 1 - (1 - \alpha_k)^m$.
- That's right - it grows exponentially. E.g., given test 7 tests at $\alpha = .1$ level the overall type I error is .52.
- Even if all null hypotheses are true we will reject at least one of them with probability .52.
- Same for confidence intervals: probability that all 7 CI cover the true values simultaneously over repeated samples is .52.So for each coefficient you have a .90 confidence interval, but overall a .52 percent confidence interval.

Problem of Multiple Testing

- Several statistical techniques have been developed to "adjust" for this inflation of overall type I errors for multiple testing.
- To compensate for the number of tests, these corrections generally require a stronger level of evidence to be observed in order for an individual comparison to be deemed "significant"
- The most prominent adjustments include:
  - Bonferroni: for each individual test use significance level of $\alpha_{k,BFer} = \alpha_k/m$
  - Sidak: for each individual test use significance level of $\alpha_{k,Sid} = 1 - (1 - \alpha_k)1/m$
  - Scheffe (for confidence intervals)
- It remains a heated debate.

Summary of Testing

- Hypothesis testing provides a principled framework for making decisions between alternatives.
- The level of a test determines how often the researcher is willing to reject a correct null hypothesis.
- Reporting p-values allows the researcher to separate the analysis from the decision.
- There is a close relationship between the results of an α level hypothesis test and the coverage of a $(1-\alpha)\%$ confidence interval.

Taking Stock

- What we've been up to: estimating parameters of population distributions. Generally we've been learning about a single variable.
- From here on out, we'll be interested in the relationships between variables. How does one variable change as we change the values of another variable? This question will be the bread and butter of the class moving forward.

What is a relationship and why do we care?

- Most of what we want to do in the social science is learn about how two variables are related
- Examples:
  - Does turnout vary by types of mailers received?
  - Is the quality of political institutions related to average incomes?
  - Does parental incarceration affect intergenerational mobility for child?

Notation and conventions

- Y - the dependent variable or outcome or regressand or left-hand-side variable or response
  - Voter turnout
  - Log GDP per capita
  - Income relative to parent
- X - the independent variable or explanatory variable or regressor or right-hand-side variable or treatment or predictor
  - Social pressure mailer versus Civic Duty Mailer
  - Average Expropriation Risk
  - Incarcerated parent
- Generally our goal is to understand how Y varies as a function of X:

$$Y = f(X) + \text{error}$$

Three uses of regression

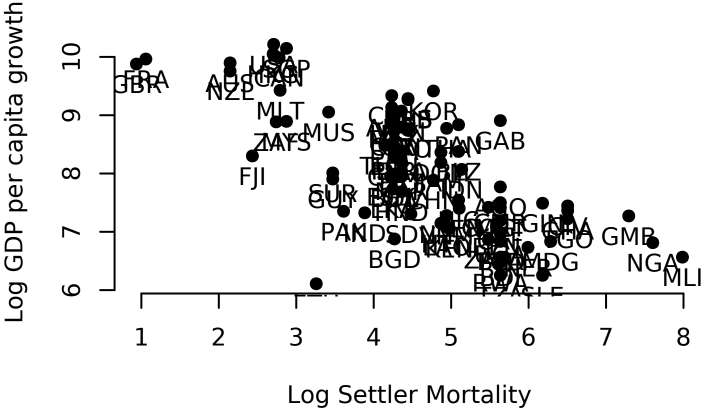
1. Description - parsimonious summary of the data
2. Prediction/Estimation/Inference - learn about parameters of the joint distribution of the data
3. Causal Inference - evaluate counterfactuals

Describing relationships

- Remember that we had ways to summarize the relationship between variables in the population.
- Joint densities, covariance, and correlation were all ways to summarize the relationship between two variables.
- But these were population quantities and we only have samples, so we may want to estimate these quantities using their sample analogs (plug-in principle or analogy principle)

Scatterplots

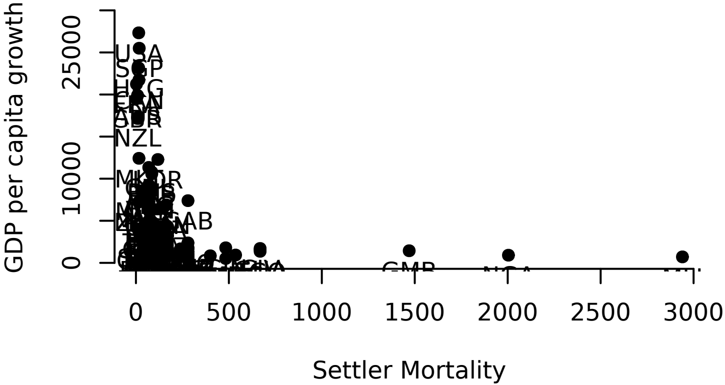
- Sample version of joint probability density.
- Shows graphically how two variables are related



Data from Acemoglu, Johnson and Robinson

Non-linear relationship

- Example of a non-linear relationship, where we use the unlogged version of GDP and settler mortality:



Sample Covariance

The sample version of population covariance, $\sigma_{XY} = E[(X - E[X])(Y - E[Y])]$.

Definition (Sample Covariance)

The **sample covariance** between Y_i and X_i is

$$S_{XY} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X}_n)(Y_i - \overline{Y}_n)$$

Sample Correlation

The sample version of population correlation, $\rho = \sigma_{XY}/\sigma_X\sigma_Y$.

Definition (Sample Correlation)

The **sample correlation** between Y_i and X_i is

$$\hat{\rho} = r = \frac{S_{XY}}{S_X S_Y} = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{\sqrt{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2}}$$

Regression is About Conditioning on X

- Regression quantifies how an outcome variable Y varies as a function of one or more predictor variables X
- Many methods, but the common idea: conditioning on X
- Goal is to characterize $f(Y|X)$, the conditional probability distribution of Y for different levels of X
- Instead of modeling the whole conditional density of Y given X, in regression we usually only model the conditional mean of Y given X: $E[Y|X=x]$
- Our key goal is to approximate the conditional expectation function $E[Y|X]$, which summarizes how the average of Y varies across all possible levels of X (also called the population regression function)
- Once we have estimated $E[Y|X]$, we can use it for prediction and/or causal inference, depending on what assumptions we are willing to make

Review: Conditional expectation

It will be helpful to review a core concept:

Definition (Conditional Expectation Function)

The **conditional expectation function** (CEF) or the **regression function** of Y given X, denoted

$$r(x) = E[Y|X = x]$$

is the function that gives the mean of Y at various values of x.

- Note that this is a function of the population distributions. We will want to produce estimates $\widehat{r}(x)$.

CEF for binary covariates

- We've been writing μ_1 and μ_0 for the means in different groups.
- For example, on the homework, you are looking at the expected value of the loan amount conditional on gender. There we had μ_m and μ_w.
- Note that these are just conditional expectations. Define Y to be the loan amount, $X = 1$ to indicate a man, and $X = 0$ to indicate a woman and then we have:

$$\mu_m = r(1) = E[Y|X = 1]$$
$$\mu_w = r(0) = E[Y|X = 0]$$

- Notice here that since X can only take on two values, 0 and 1, then these two conditional means completely summarize the CEF.

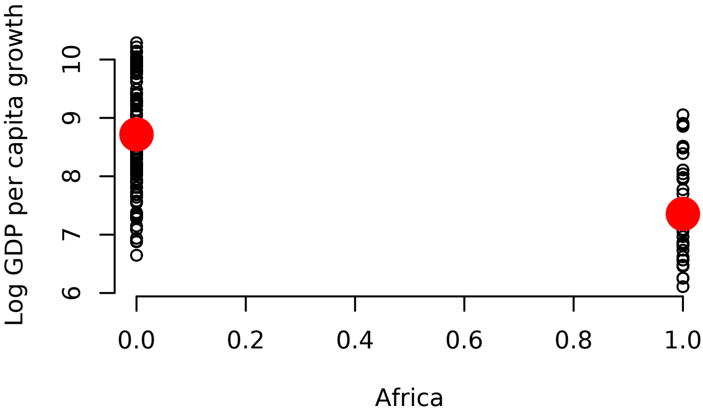
Estimating the CEF for binary covariates

- How do we estimate $\widehat{r}(x)$?
- We've already done this: it's just the usual sample mean among the men and then the usual sample mean among the women:

$$\widehat{r}(1) = \frac{1}{n_1} \sum_{i:X_i=1} Y_i$$
$$\widehat{r}(0) = \frac{1}{n_0} \sum_{i:X_i=0} Y_i$$

- Here we have $n_1 = \sum_{i=1}^{n} X_i$ is the number of men in the sample and $n_0 = n - n_1$ is the number of women.
- The sum here $\sum_{i:X_i=1}$ is just summing only over the observations i such that have $X_i = 1$, meaning that i is a man.
- This is very straightforward: estimate the mean of Y conditional on X by just estimating the means within each group of X.

Binary covariate example CEF plot



CEF: Estimands, Estimators, and Estimates

- The conditional expectation function $E[Y|X]$ is the estimand (or parameter) we are interested in

- $\widehat{E}[Y|X]$ is the estimator of this parameter of interest, which is a function of X

- For a given sample dataset, we obtain an estimate of $E[Y|X]$.

- We want to extend the regression idea to the case of multiple X variables, but we will start this week with the simple bivariate case where we have a single X

1. Testing: Making Decisions
    - Hypothesis testing
    - Forming rejection regions
    - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. **Fun With Salmon**
6. Nonparametric Regression
    - Discrete X
    - Continuous X
    - Bias-Variance Tradeoff
7. Linear Regression
    - Combining Linear Regression with Nonparametric Regression
    - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Fun With Salmon

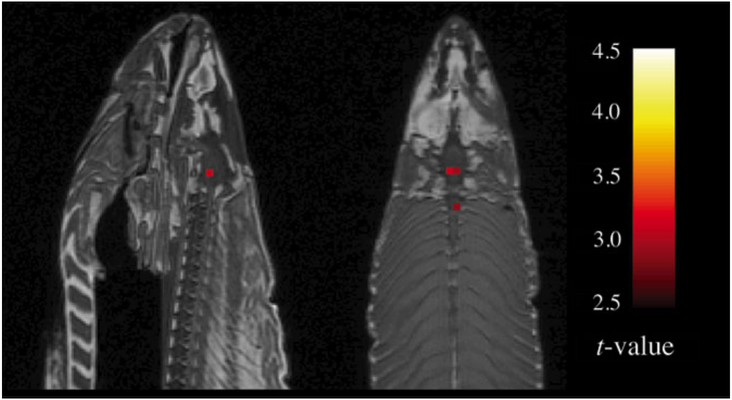
Bennett, Baird, Miller and Wolford. (2009). “Neural correlates of interspecies perspective taking in the post-mortem Atlantic Salmon: an argument for multiple comparisons correction.”

Methods

(a.k.a. the greatest methods section of all time)

- Subject
  "One mature Atlantic Salmon (Salmo salar) participated in the fMRI study. The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning."
- Task
  "The task administered to the salmon involved completing an open-ended mentalizing task. The salmon was shown a series of photographs depicting human individuals in social situations with a specified emotional valence. The salmon was asked to determine what emotion the individual in the photo must have been experiencing."
- Design
  "Stimuli were presented in a block design with each photo presented for 10 seconds followed by 12 seconds of rest.A total of 15 photos were displayed. Total scan time was 5.5 minutes."

Results



"Several active voxels were discovered in a cluster located within the salmon's brain cavity. The size of this cluster was 81 mm^3 with a cluster-level significance of $p = .001$."

Where We've Been and Where We're Going...

- Last Week
  - inference and estimator properties
  - point estimates, confidence intervals
- This Week
  - Monday:
    - hypothesis testing
    - what is regression?
  - Wednesday:
    - nonparametric regression
    - linear approximations
- Next Week
  - inference for simple regression
  - properties of OLS
- Long Run
  - probability → inference → regression

Questions?

1. Testing: Making Decisions
    - Hypothesis testing
    - Forming rejection regions
    - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
    - Discrete X
    - Continuous X
    - Bias-Variance Tradeoff
7. Linear Regression
    - Combining Linear Regression with Nonparametric Regression
    - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Nonparametric Regression with Discrete X

- Let's take a look at some data on education and income from the American National Election Study
- We use two variables:
  - Y: income
  - X: educational attainment
- Goal is to characterize the conditional expectation $E[Y|X]$, i.e. how average income varies with education level

Nonparametric Regression with Discrete X

`educ`: Respondent's education:

- 1. 8 grades or less and no diploma or
- 2. 9-11 grades
- 3. High school diploma or equivalency test
- 4. More than 12 years of schooling, no higher degree
- 5. Junior or community college level degree (AA degrees)
- 6. BA level degrees; 17+ years, no postgraduate degree
- 7. Advanced degree

Nonparametric Regression with Discrete X

`income`: Respondent's family income:

- 1. None or less than $2,999
- 2. $3,000-$4,999
- 3. $5,000-$6,999
- 4. $7,000-$8,999
- 5. $9,000-$9,999
- 6. $10,000-$10,999

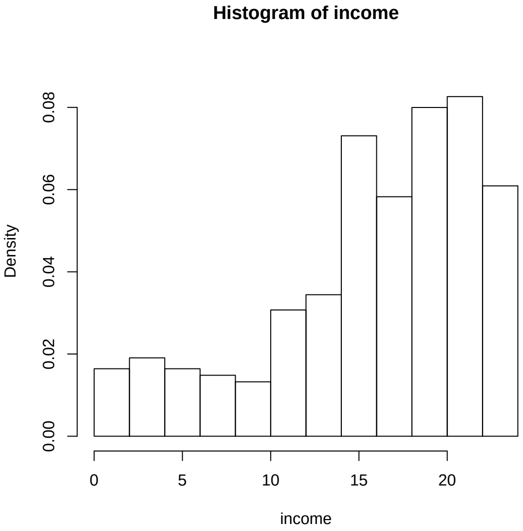
⋮

- 17. $35,000-$39,999
- 18. $40,000-$44,999

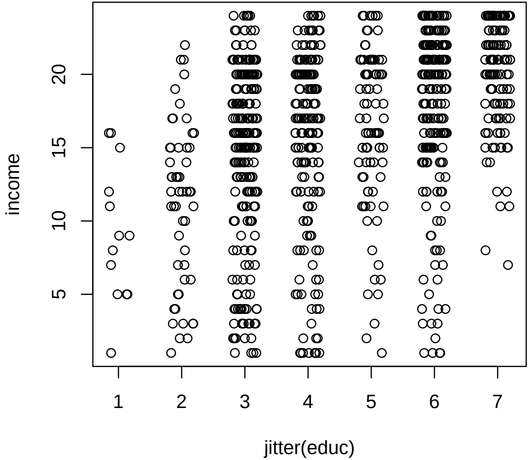
⋮

- 23. $90,000-$104,999
- 24. $105,000 and over

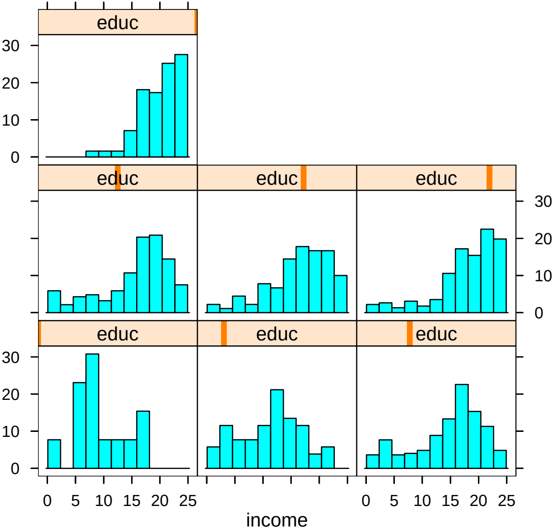
Marginal Distribution of Y



Income and Education



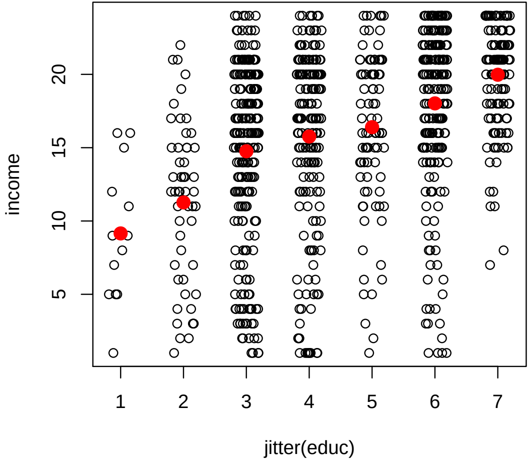
Distribution of income given education $p(y|x)$



Nonparametric Regression with Discrete X

- Hard to decode what is going on in the histograms
- Let's try to find a more parsimonious summary measure: $E[Y|X]$
- Here our X variable education has a small number of levels (7) and there are a reasonable number of observations in each level
- In situations like this we can estimate $E[Y|X=x]$ as the sample mean of Y at each level of $x \in X$ (just like the binary case)

Nonparametric Regression with Discrete X



Nonparametric Regression

- This approach makes minimal assumptions
- It works well as long as
    - X is discrete
    - there are a small number values of X
    - a small number of X variables
    - a lot of observations at each X value
- This method does not impose any specific functional form on the relationship between Y and X (i.e. the shape of $E[Y|X]$)
- → It is called a nonparametric regression
- But what do we do when X is continuous and has many values?

Nonparametric Regression with Continuous X

Consider the Chirot data:

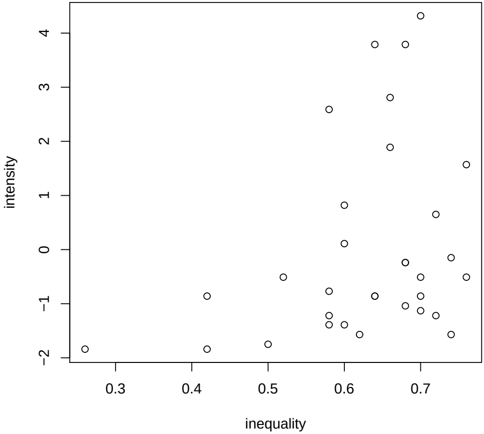
- Chirot, D. and C. Ragin (1975). The market, tradition and peasant rebellion: The case of Romania. American Sociological Review 40, 428-444

- Peasant Rebellions in Romanian counties in 1907
- Peasants made up 80% of the population
- About 60 % of them owned no land which was mostly concentrated among large landowners

- We're interested in the relationship between:
  - Y: intensity of the peasant rebellion
  - X: inequality of land tenure

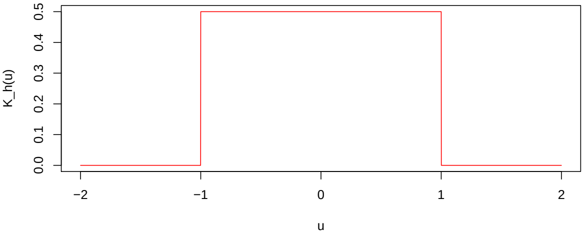
- Around 11,000 peasants were killed by Romanian military

Nonparametric Regression with Continuous X



Uniform Kernel Regression: Simple Local Averages

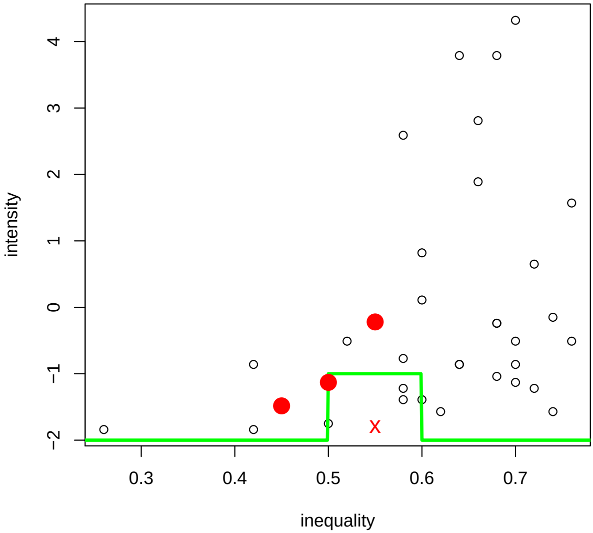
- One approach is to use a moving local average to estimate $E[Y|X]$
- Calculate the average of the observed y points that have x values in the interval $[x_0 - h\, ,\, x_0 + h]$
- $h =$ some positive number (called the bandwidth)
- Uniform kernel: every observation in the interval is equally weighted



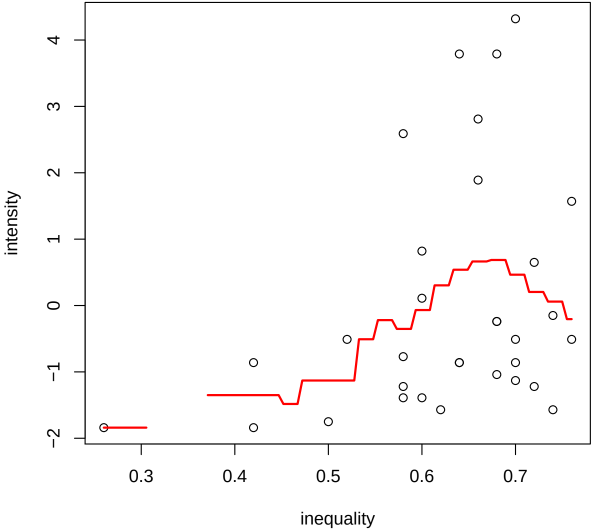
- This gives the uniform kernel regression:

$$\widehat{E}[Y|X = x_0] = \frac{\sum_{i=1}^{N} K_h((X_i - x_0)/h) Y_i}{\sum_{i=1}^{N} K_h((X_i - x_0)/h)} \text{ where } K_h(u) = \frac{1}{2}\,\mathbf{1}_{\{|u| \leq 1\}}$$

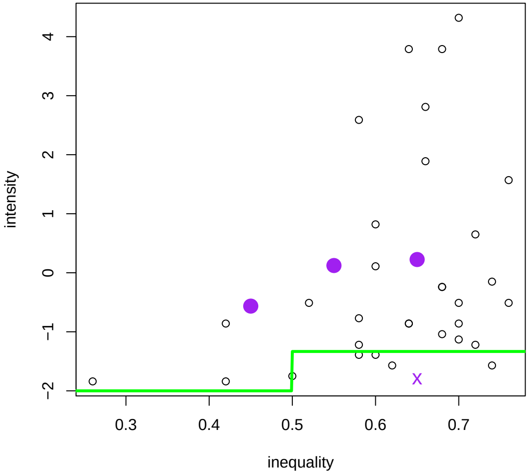
Uniform Kernel Regression: Simple Local Averages



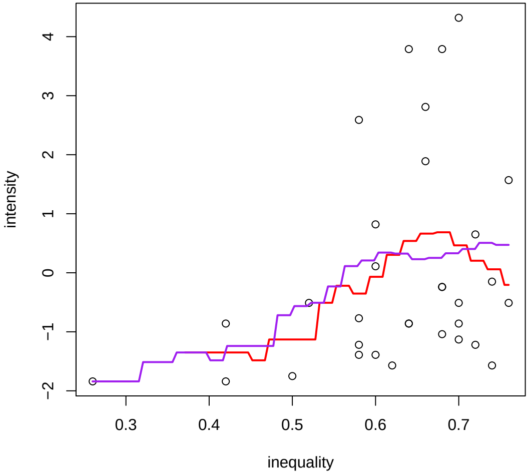
Uniform Kernel Regression: Simple Local Averages



Changing the Bandwidth



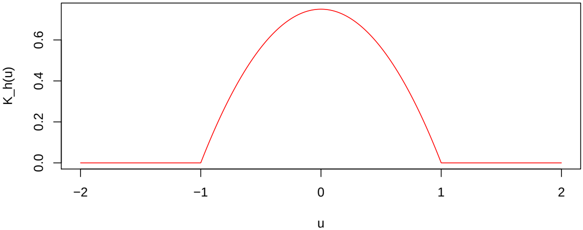
Changing the Bandwidth



Kernel Regression: Weighted Local Averages

- Another approach is to construct weighted local averages
- Data points that are closer to x_0 get more weight than points farther away

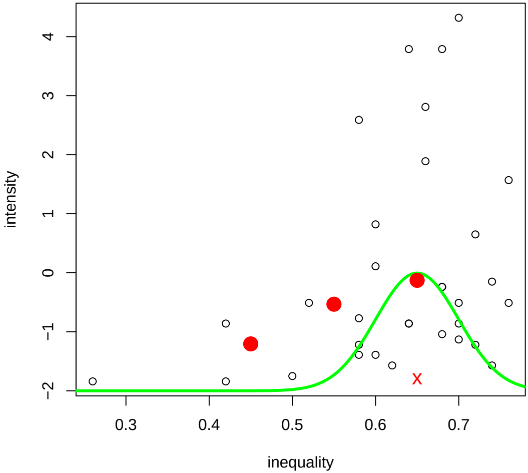
1. decide on a symmetric, non-negative kernel weight function K_h (e.g. Epanechnikov)



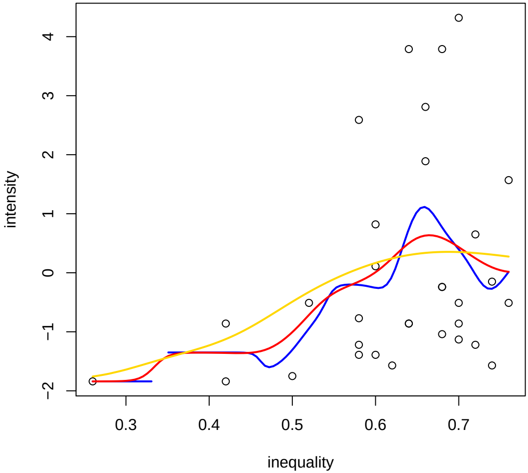
2. compute weighted average of the observed y points that have x values in the bandwidth interval $[x_0 - h\,,\, x_0 + h]$ e.g.

$$\widehat{E}[Y|X = x_0] = \frac{\sum_{i=1}^{N} K_h((X_i - x_0)/h)Y_i}{\sum_{i=1}^{N} K_h((X_i - x_0)/h)} \text{ where } K_h(u) = \frac{3}{4}(1 - u^2)\,\mathbf{1}_{\{|u|\leq 1\}}$$

Kernel Regression: Weighted Local Averages



Changing the Bandwidth



Bias-Variance Tradeoff

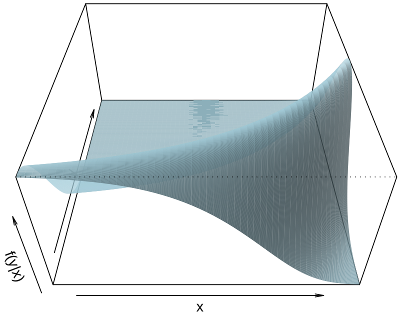
- When choosing an estimator $\widehat{E}[Y|X]$ for $E[Y|X]$, we face a bias-variance tradeoff

- Notice that we can chose models with various levels of flexibility:
  - A very flexible estimator allows the shape of the function to vary (e.g. a kernel regression with a small bandwidth)

  - A very inflexible estimator restricts the shape of the function to a particular form
    (e.g. a kernel regression with a very wide bandwidth)

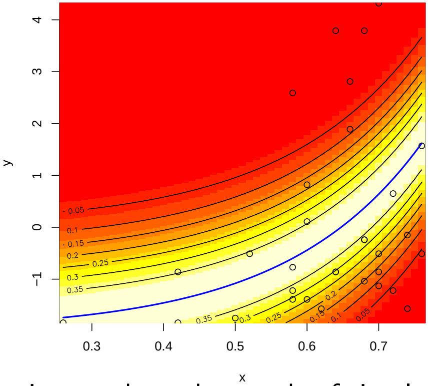
Hypothetical True Distribution

- Let's conduct a simulation experiment to actually see the tradeoff
- Suppose we have the following population distribution:



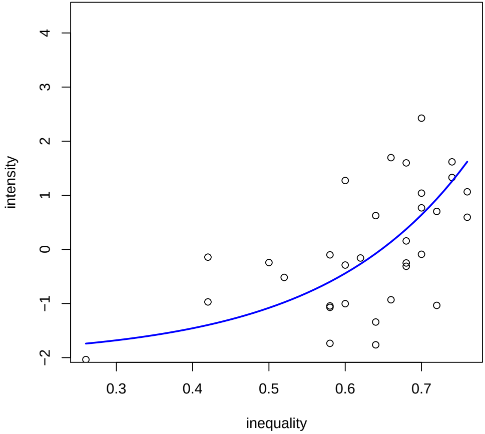
Hypothetical True Distribution

- Another way of representing the same population distribution:



- From this distribution we draw thousands of simulated data sets.

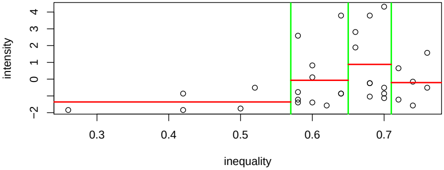
An Example of Simulated Data Set



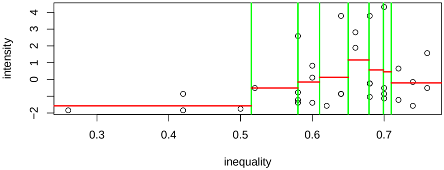
Two Estimators

- For each simulated data, we apply two simple estimators of $E(Y|X)$:
    - Divide X into **4** ranges and take the mean for each
    - Divide X into **8** ranges and take the mean for each
- We then evaluate how well these estimators do in terms of bias and variance.

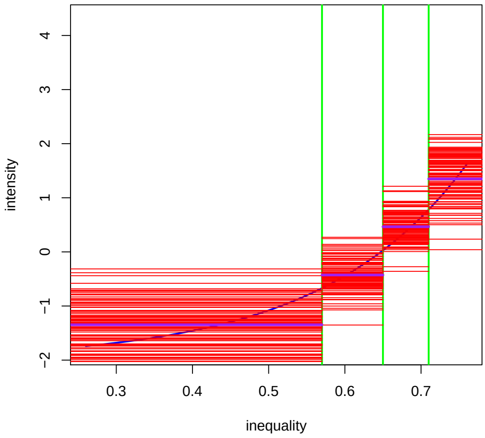
4−fold Interval Estimator



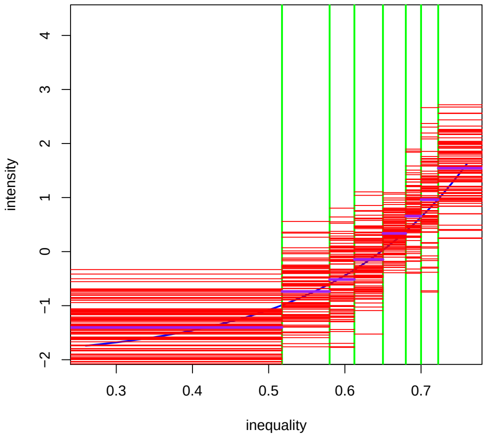
8−fold Interval Estimator



Simulated Distribution of Estimates: 4 Intervals



Simulated Distribution of Estimates: 8 Intervals



Bias-Variance Tradeoff

- A less “flexible” estimator leads to more bias
- A more “flexible” estimator leads to more variance
- As the name suggests, this problem cannot be “fixed”
- If we have more data or fewer variables we can “afford” to use a more flexible estimator

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Parametric Approach: Linear Regression

- Linear regression works by assuming linear parametric form for the conditional expectation function:

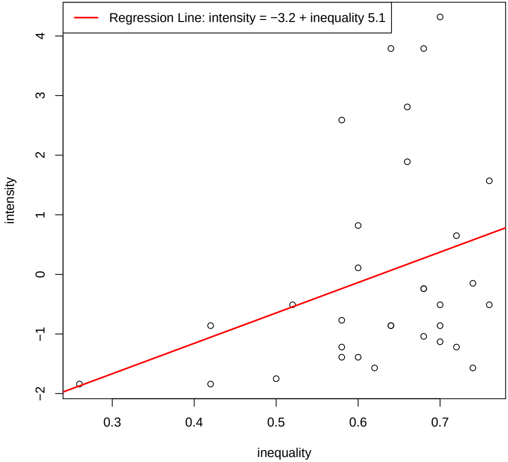
$$E[Y|X] = \beta_0 + X\,\beta_1$$

- Conditional expectation defined by only two coefficients which are estimated from the data:
  - β_0 is called the intercept or constant
  - β_1 is called the slope coefficient

- Notice that the linear functional form imposes a constant slope
- Assumption: Change in $E[Y|X]$ is the same at all values of X

- Geometrically, the linear regression function will look like:
  - A line in cases with a single X variable
  - A plane in cases with two X variables
  - A hyperplane in cases with more than two X variables

Parametric Approach: Linear Regression



Parametric Approach: Linear Regression

Warning: the model won't always be a good fit for the data
(even though it really wants to be)

Figure: 'If I fits, I sits'

Linear regression always returns a line regardless of the data.

Interpretation of the regression slope

- When we model the regression function as a line, we can interpret the parameters of the line in appealing ways:

  1. **Intercept**: the average outcome among units with $X = 0$ is β_0:

$$E[Y|X=0] = r(0) = \beta_0 + \beta_1 0 = \beta_0$$

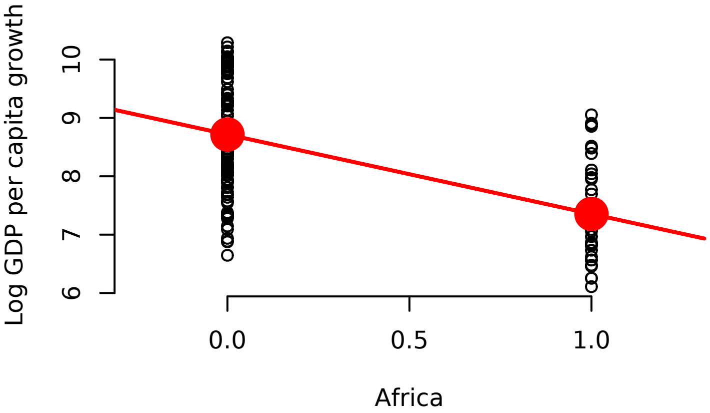
  2. **Slope**: a one-unit change in X is associated with a β_1 change in Y

$$\begin{aligned} E[Y|X=x+1] - E[Y|X=x] &= r(x+1) - r(x) \\ &= (\beta_0 + \beta_1(x+1)) - (\beta_0 + \beta_1 x) \\ &= \beta_0 + \beta_1 x + \beta_1 - \beta_0 - \beta_1 x \\ &= \beta_1 \end{aligned}$$

Linear regression with a binary covariate

- Using the two facts above, it's easy to see that when X is binary, then we have the following:
  1. **Intercept**: $E[Y|X = 0] = \beta_0$
  2. **Slope**: average difference between $X = 1$ group and $X = 0$ group: $\beta_1 = E[Y|X = 1] - E[Y|X = 0]$
- Thus, we can read off the difference in means between two groups as the slope coefficient on a linear regression

Linear CEF with a binary covariate

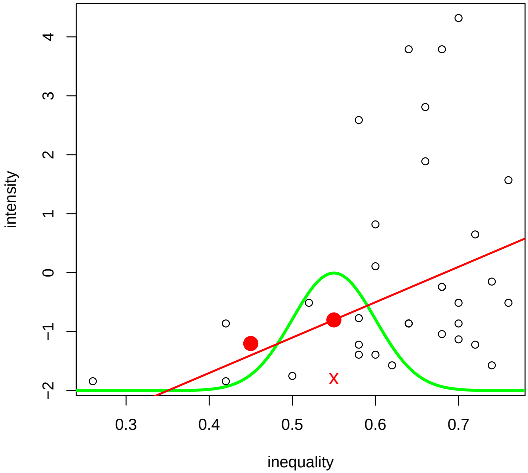


1. Testing: Making Decisions
    - Hypothesis testing
    - Forming rejection regions
    - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
    - Discrete X
    - Continuous X
    - Bias-Variance Tradeoff
7. Linear Regression
    - Combining Linear Regression with Nonparametric Regression
    - Least Squares
8. Interpreting Regression
9. Fun With Linearity

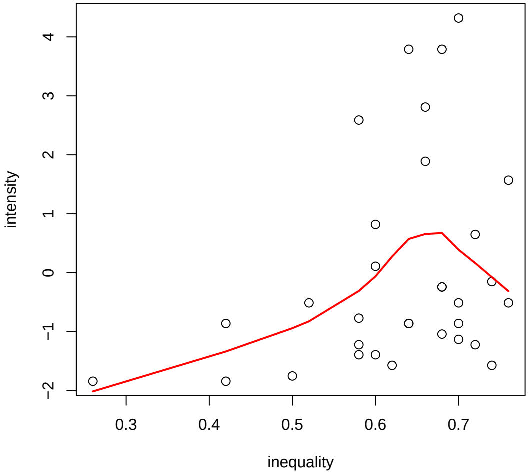
LOESS

- We can combine the nonparametric kernel method idea of using only local data with a parametric model
- Idea: fit a linear regression within each band
- Locally weighted scatterplot smoothing (LOWESS or LOESS):
  1. Pick a subset of the data that falls in the interval $[x - h\,,\, x + h]$
  2. Fit a line to this subset of the data ($=$ local linear regression), weighting the points by their distance to x using a kernel function
  3. Use the fitted regression line to predict the expected value of $E[Y|X = x_0]$

Weighted Local Linear Regressions



Weighted Local Linear Regressions



1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. Interpreting Regression
9. Fun With Linearity

Back up and review

- To review our approach:
  - We wanted to estimate the CEF/regression function $r(x) = E[Y|X = x]$, but it may be too hard to do nonparametrically
  - So we can model it: place restrictions on its functional form.
  - Easiest functional form is a line:

$$r(x) = \beta_0 + \beta_1 x$$

- β_0 and β_1 are population parameters just like μ or σ^2!
- Need to estimate them in our samples! But how?

Simple linear regression model

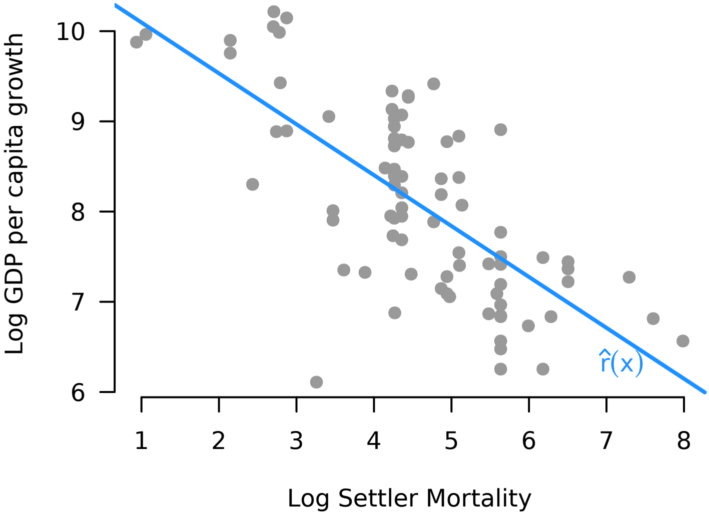
- Let's write our model as:

$$Y_i = r(X_i) + u_i$$
$$Y_i = \beta_0 + \beta_1 X_i + u_i$$

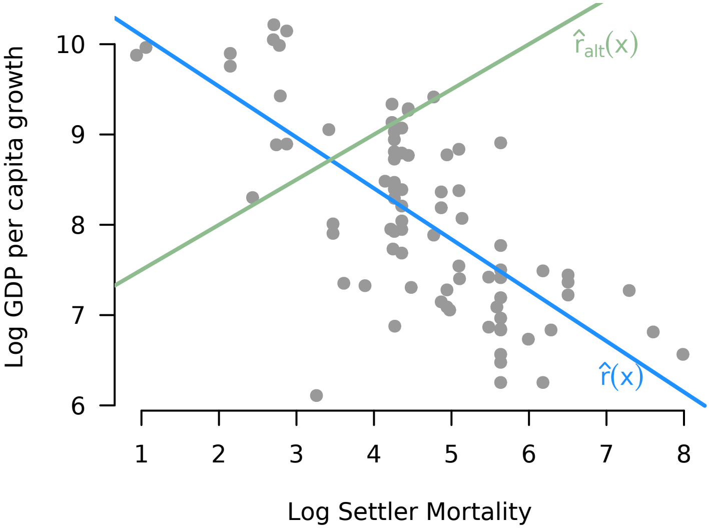
- Now, suppose we have some estimates of the slope, $\hat{\beta}_1$, and the intercept, $\hat{\beta}_0$. Then the fitted or sample regression line is

$$\widehat{r}(x) = \widehat{\beta}_0 + \widehat{\beta}_1 x$$

Fitted linear CEF/regression function



Fitted linear CEF/regression function



Fitted values and residuals

Definition (Fitted Value)

A **fitted value** or **predicted value** is the estimated conditional mean of Y_i for a particular observation with independent variable X_i:

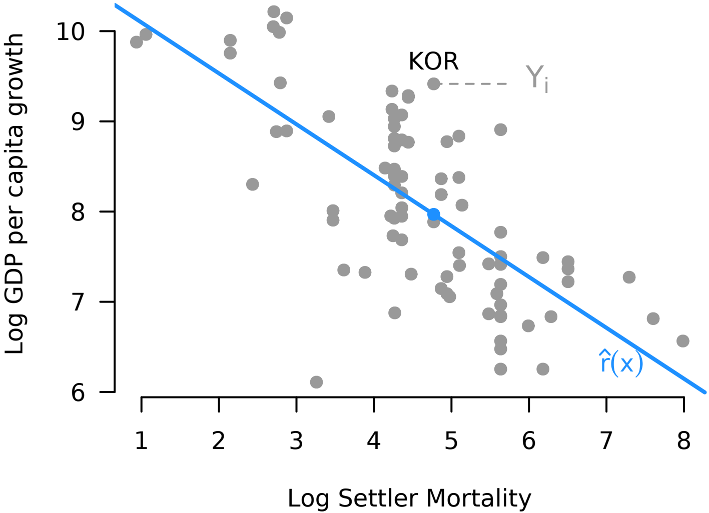
$$\widehat{Y}_i = \widehat{r}(X_i) = \widehat{\beta}_0 + \widehat{\beta}_1 X_i$$

Definition (Residual)

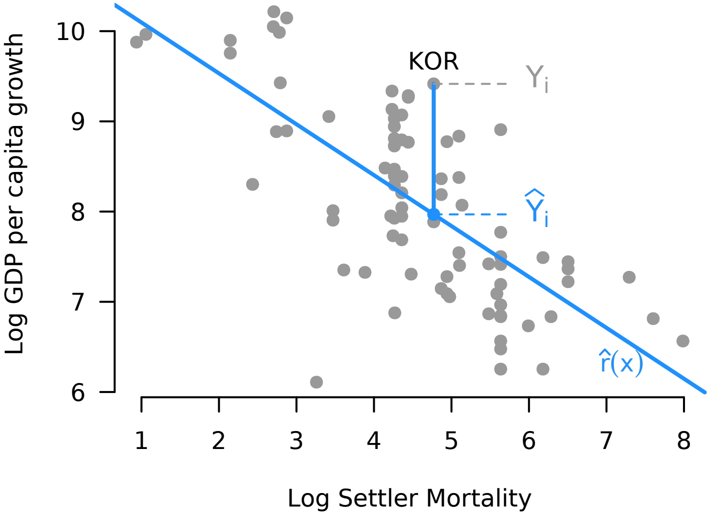
The **residual** is the difference between the actual value of Y_i and the predicted value, $\widehat{Y}_i$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i = Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 X_i$$

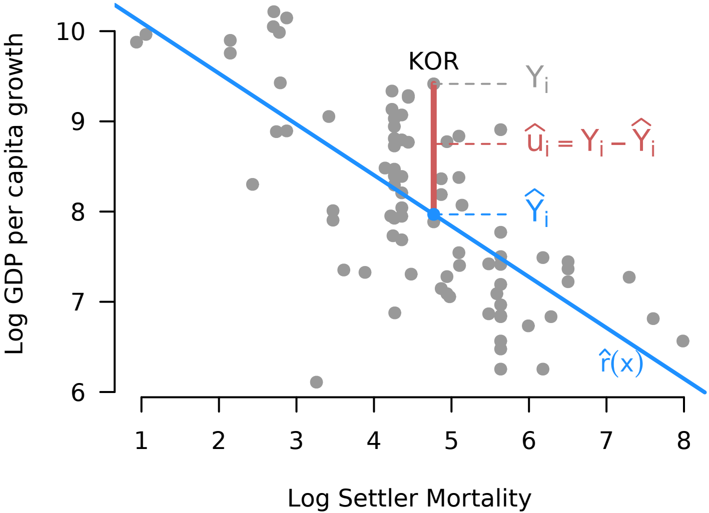
Fitted linear CEF/regression function



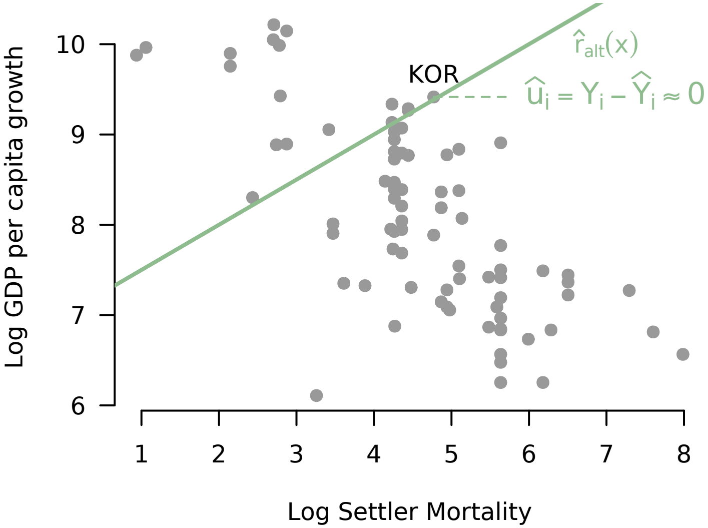
Fitted linear CEF/regression function



Fitted linear CEF/regression function



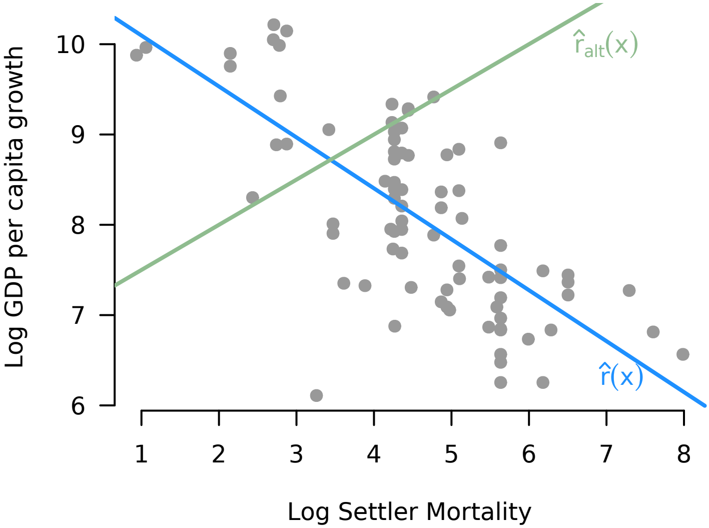
Why not this line?



Minimize the residuals

- The residuals, $\widehat{u}_i = Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 X_i$, tell us how well the line fits the data.
  - Larger magnitude residuals means that points are very far from the line
  - Residuals close to 0 mean points very close to the line
- The smaller the magnitude of the residuals, the better we are doing at predicting Y
- Choose the line that minimizes the residuals

Which is better at minimizing residuals?



Minimizing the residuals

- Let $\widetilde{\beta}_0$ and $\widetilde{\beta}_1$ be possible values of the intercept and slope
- **Least absolute deviations** (LAD) regression:

$$(\widehat{\beta}_0^{LAD}, \widehat{\beta}_1^{LAD}) = \underset{\widetilde{\beta}_0, \widetilde{\beta}_1}{\arg\min} \sum_{i=1}^{n} |Y_i - \widetilde{\beta}_0 - \widetilde{\beta}_1 X_i|$$

- **Least squares** (LS) regression:

$$(\widehat{\beta}_0, \widehat{\beta}_1) = \underset{\widetilde{\beta}_0, \widetilde{\beta}_1}{\arg\min} \sum_{i=1}^{n} (Y_i - \widetilde{\beta}_0 - \widetilde{\beta}_1 X_i)^2$$

- Sometimes called **ordinary least squares** (OLS)

Why least squares?

- Easy to derive the least squares estimator
- Easy to investigate the properties of the least squares estimator
- Least squares is optimal in a certain sense that we'll see in the coming weeks

Linear Regression: Justification

- Linear regression imposes a **strong** assumption on $E[Y|X]$
- Why would we ever want to do this?
  - Theoretical reason to assume linearity
  - Ease of interpretation
  - Bias-variance tradeoff
  - Analytical derivation of sampling distributions (next few weeks)
  - We can make the model more flexible, even in a linear framework (e.g. we can add polynomials, use log transformations, etc.)

1. Testing: Making Decisions
   - Hypothesis testing
   - Forming rejection regions
   - P-values
2. Review: Steps of Hypothesis Testing
3. The Significance of Significance
4. Preview: What is Regression
5. Fun With Salmon
6. Nonparametric Regression
   - Discrete X
   - Continuous X
   - Bias-Variance Tradeoff
7. Linear Regression
   - Combining Linear Regression with Nonparametric Regression
   - Least Squares
8. **Interpreting Regression**
9. Fun With Linearity

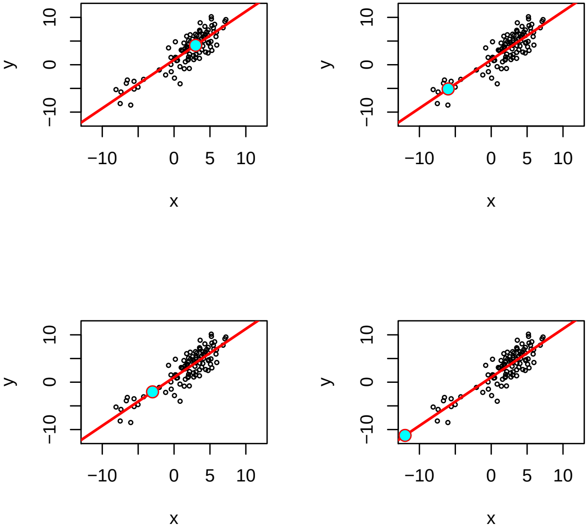
Linear Regression as a Predictive Model

- Linear regression can also be used to predict new observations
- Basic idea:
  - Find estimates $\hat{\beta}_0, \hat{\beta}_1$ of β_0, β_1 based on the in-sample data
  - To find the expected value of Y for an out-of-sample data point with $X = x_{new}$ calculate:

$$\hat{E}[Y|X = x_{new}] = \hat{\beta}_0 + \hat{\beta}_1 x_{new}$$

- While the line is defined over all regions of the data we may be concerned about:
  - interpolation
  - extrapolation
  - predicting in ranges of X with sparse data

Which Predictions Do You Trust?



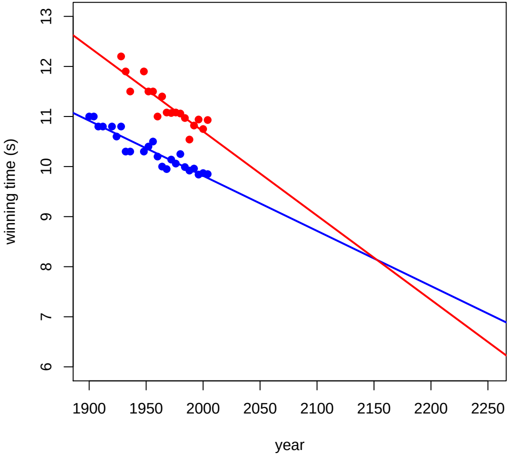
Example: Tatem, et al. Sprinters Data

In a 2004 *Nature* article, Tatem et al. use linear regression to conclude that in the year 2156 the winner of the women's Olympic 100 meter sprint may likely have a faster time than the winner of the men's Olympic 100 meter sprint.

How do the authors make this conclusion?

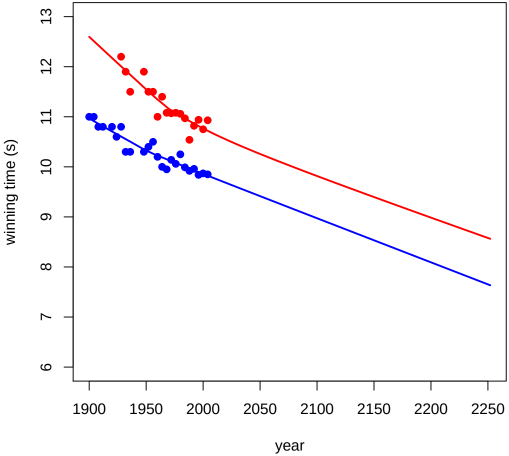
Using data from 1900 to 2004, they fit linear regression models of the winning 100 meter time on year for both men and women. They then use the estimates from these models to extrapolate 152 years into the future.

Tatem et al. Extrapolation

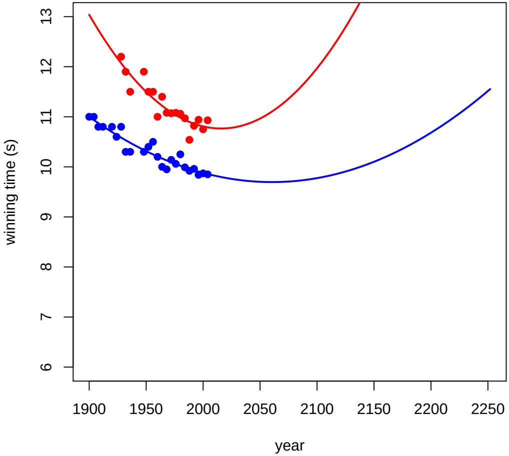


Tatem et al.'s predictions. Men's times are in blue, women's times are in red.

Alternate Models Fit Well, Yield Different Predictions



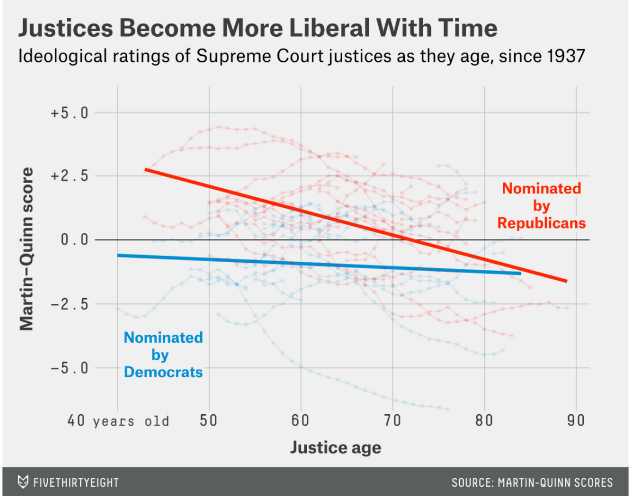
Alternate Models Fit Well, Yield Different Predictions



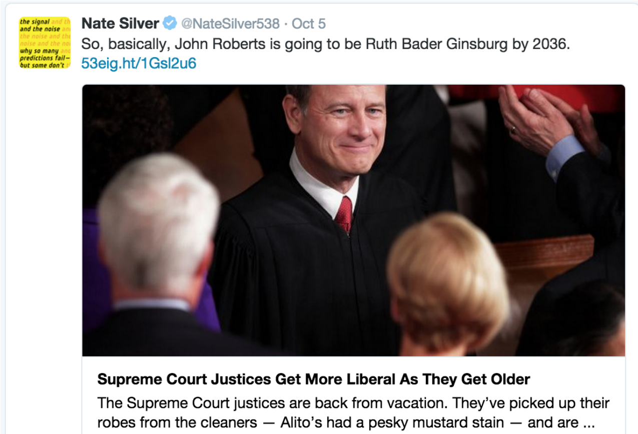
The Trouble with Extrapolation

- The model only gives the best fitting line where we have data, it says little about the shape where there isn't any data.
- We can always ask illogical questions and the model gives answers.
    - For example, when will women finish the sprint in negative time?
- Fundamentally we are assuming that data outside the sample looks like data inside the sample, and the further away it is the less likely that is to hold.
- Next semester we will talk about how this problem gets much harder in high dimensions

A More Subtle Example



A More Subtle Example



Nate Silver @NateSilver538 · Oct 5

So, basically, John Roberts is going to be Ruth Bader Ginsburg by 2036.

53eig.ht/1Gsl2u6

Supreme Court Justices Get More Liberal As They Get Older

The Supreme Court justices are back from vacation. They've picked up their robes from the cleaners — Alito's had a pesky mustard stain — and are ...

Regression as Description/Prediction

- Even for simple problems regression can be challenging
- Always think about where we have data and what we are using to build our claims
- Summary: ‘prediction is hard, especially about the future’

Regression as a Causal Model (A Preview)

- Can regression be also used for causal inference?
- Answer: A very qualified yes
- For example, can we say that a one unit increase in inequality *causes* a 5.2 point increase in intensity?
- To interpret β as a causal effect of X on Y, we need very specific and often unrealistic assumptions:
  1. $E[Y|X]$ is correctly specified as a linear function (linearity)
  2. There are no other variables that affect both X and Y (exogeneity)

  (1) can be relaxed by:
    - ⋆ Using a flexible nonlinear or nonparametric method
    - ⋆ "Preprocessing" data to make analysis robust to misspecification

  (2) can be made plausible by:
    - ⋆ Including carefully-selected control variables in the model
    - ⋆ Choosing a clever research design to rule out confounding
- We will return to this later in the course
- For now, it is safest to treat β as a purely descriptive/predictive quantity

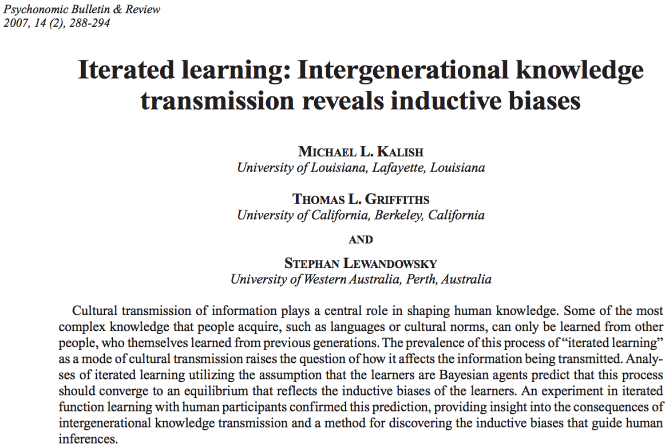
Summary of Today

- Regression is about conditioning
- Regression can be used for description, causation and prediction
- Linear regression is a parametrically restricted form of regression

Next Week

- Basic linear regression
- Properties of OLS
- Reading:
  - Fox Chapter 6.1
  - Optional: Imai 4.2

Fun with Linearity



Psychonomic Bulletin & Review
2007, 14 (2), 288-294

Iterated learning: Intergenerational knowledge transmission reveals inductive biases

MICHAEL L. KALISH
University of Louisiana, Lafayette, Louisiana

THOMAS L. GRIFFITHS
University of California, Berkeley, California

AND

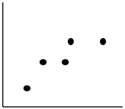
STEPHAN LEWANDOWSKY
University of Western Australia, Perth, Australia

Cultural transmission of information plays a central role in shaping human knowledge. Some of the most complex knowledge that people acquire, such as languages or cultural norms, can only be learned from other people, who themselves learned from previous generations. The prevalence of this process of "iterated learning" as a mode of cultural transmission raises the question of how it affects the information being transmitted. Analyses of iterated learning utilizing the assumption that the learners are Bayesian agents predict that this process should converge to an equilibrium that reflects the inductive biases of the learners. An experiment in iterated function learning with human participants confirmed this prediction, providing insight into the consequences of intergenerational knowledge transmission and a method for discovering the inductive biases that guide human inferences.

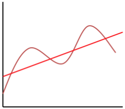
Images on following slides courtesy of Tom Griffiths

The Design

data

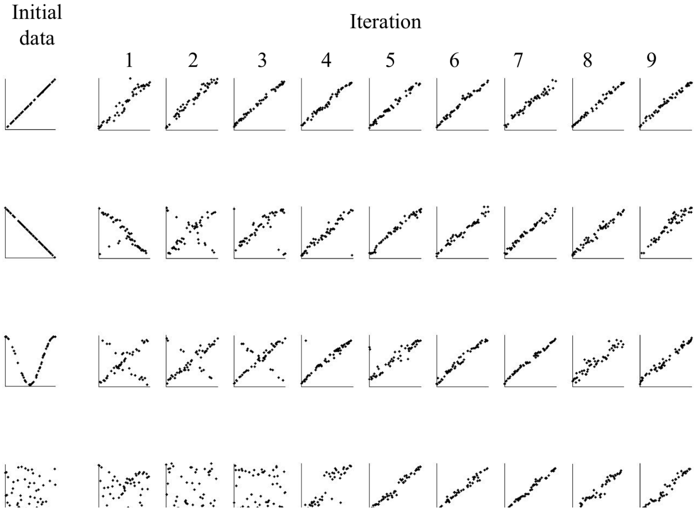


hypotheses



- Each learner sees a set of (x, y) pairs
- Makes predictions of y for new x values
- Predictions are data for the next learner

Results



References

Chirot, D. and C. Ragin (1975). The market, tradition and peasant rebellion: The case of Romania. American Sociological Review 40, 428-444

Acemoglu, Daron, Simon Johnson, and James A. Robinson. “The colonial origins of comparative development: An empirical investigation.” 2000.