Week 4: Testing/Regression

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[1] These slides are heavily influenced by Matt Blackwell, Adam Glynn and Jens Hainmueller.

Where We've Been and Where We're Going...

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- Last Week
  - inference and estimator properties
  - point estimates, confidence intervals

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- Long Run
  - probability $\rightarrow$ inference $\rightarrow$ regression

Questions?

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- The trial included 345 patients with initial systolic blood pressure between 140-159.
- Each subject was assigned to take the drug combination for 16 weeks.
- Systolic blood pressure was measured on each subject before and after the treatment period.

Example

Subject	SBP_{before}	SBP_{after}	Decrease
1			
2			
3			
4			
⋮			
345			

Example

Subject	SBP_{before}	SBP_{after}	Decrease
1	147		
2	153		
3	142		
4	141		
⋮	⋮		
345	155		

Example

Subject	SBP_{before}	SBP_{after}	Decrease
1	147	135	
2	153	122	
3	142	119	
4	141	134	
⋮	⋮	⋮	
345	155	115	

Example

Subject	SBP_{before}	SBP_{after}	Decrease
1	147	135	12
2	153	122	31
3	142	119	23
4	141	134	7
⋮	⋮	⋮	⋮
345	155	115	40

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Question: Should the FDA allow the drug to proceed to the next stage of testing?

The FDA's Decision

We can think of the FDA's problem in terms of two dimensions:

- The true state of the world
- The decision made by the FDA

	Drug works	Drug doesn't work
FDA approves		
FDA doesn't approve		

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- Test Statistic
- Rejection Region

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  Example: The drug does not reduce blood pressure on average ($\mu_{decrease} \leq 0$)

- Alternative Hypothesis: Claim to be tested (research hypothesis)

  Example: The drug does reduce blood pressure on average ($\mu_{decrease} > 0$)

More Examples

Null Hypothesis Examples (H_0):

Alternative Hypothesis Examples (H_a):

More Examples

Null Hypothesis Examples (H_0):

- The drug does not change blood pressure on average ($\mu_{decrease} = 0$)

Alternative Hypothesis Examples (H_a):

- The drug does change blood pressure on average ($\mu_{decrease} \neq 0$)

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FDA approves (reject H_0)	Correct	
FDA doesn't approve (don't reject H_0)		Correct

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FDA approves (reject H_0)	Correct	Type I error
FDA doesn't approve (don't reject H_0)		Correct

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FDA approves (reject H_0)	Correct	Type I error
FDA doesn't approve (don't reject H_0)	Type II error	Correct

Test Statistics, Null Distributions, and Rejection Regions

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Test Statistic: A function of the sample and the null hypothesis value of the parameter. For example:

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Null Distribution: the sampling distribution of the statistic/test statistic assuming that the null is true.

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If we assume that the null hypothesis is true such that $\mu = \mu_0$, then

$$\overline{X} \sim_{approx} N(\mu_0, S^2/n)$$
$$\frac{\overline{X} - \mu_0}{\frac{S}{\sqrt{n}}} \sim_{approx} N(0, 1)$$

α

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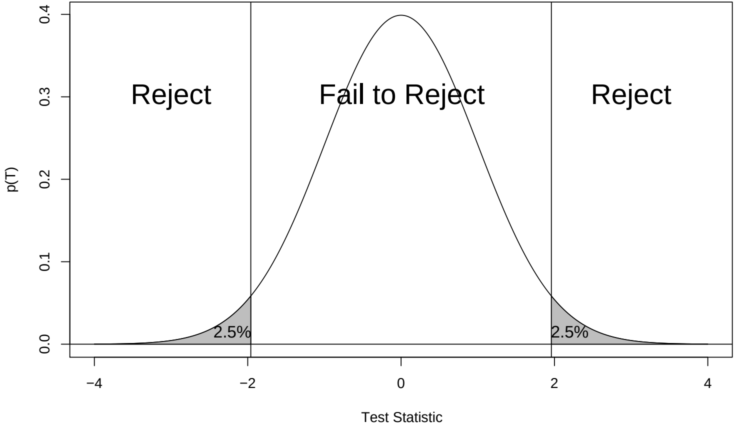
Example: Suppose α =5%, the test statistic is $\frac{\overline{X}-\mu_0}{\frac{S}{\sqrt{n}}}$, the null hypothesis is $H_0 : \mu = \mu_0$, and the alternative hypothesis is $H_a : \mu \neq \mu_0$.

Two-sided rejection region

Two-sided rejection region

Rejection region with $\alpha = .05$, $H_0 : \mu = 0$, $H_A : \mu \neq 0$:



One-sided Rejection Region

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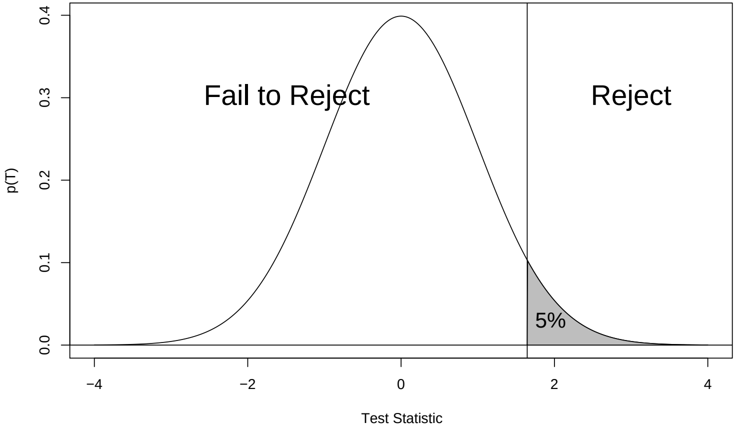
Rejection region with $\alpha = .05$, $H_0 : \mu \leq 0$, $H_A : \mu > 0$:



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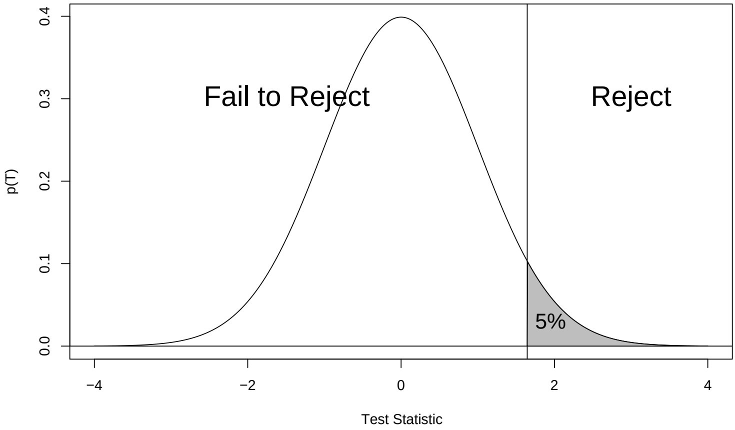
What is the decision?

Rejection Region with $\alpha = .05$



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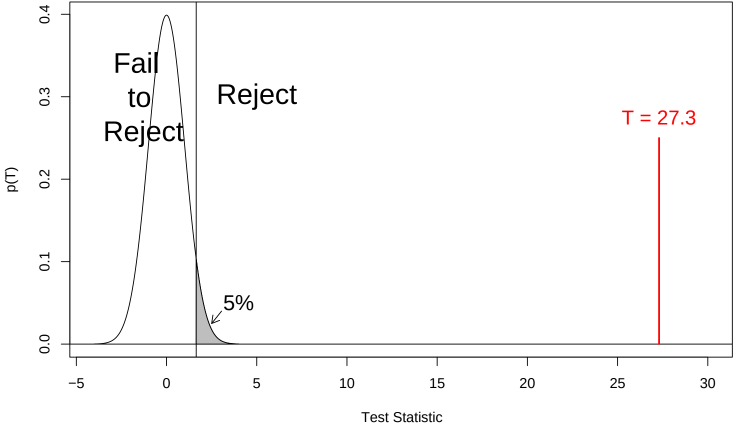

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P-value: Assuming that the null hypothesis is true, the probability of getting something at least as extreme as our observed test statistic, where extreme is defined in terms of the alternative hypothesis.

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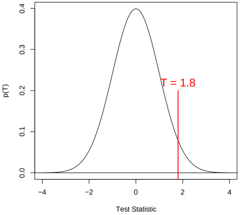

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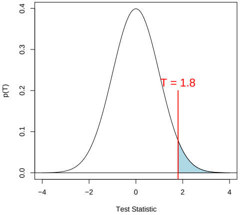

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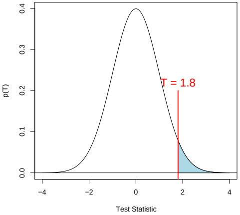

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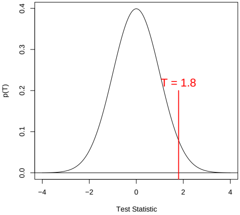

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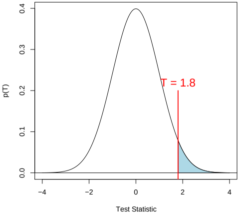

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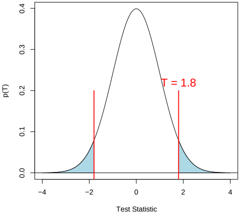

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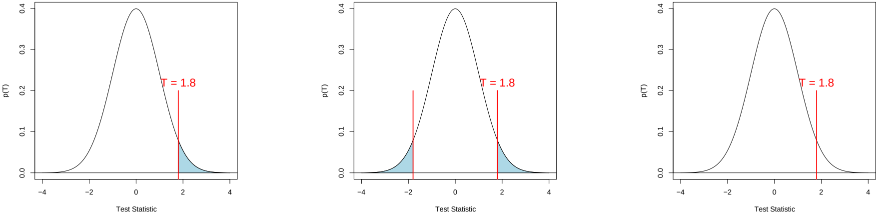
$H_a : \mu > 0$ $\quad$ $H_a : \mu \neq 0$ $\quad$ $H_a : \mu < 0$



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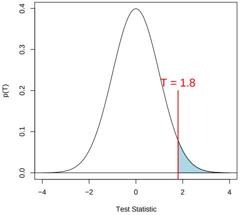

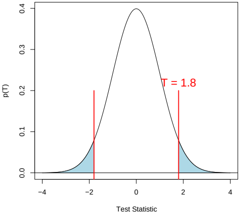

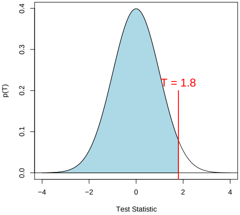

$p = 0.036$	$p = .072$	$p = 0.964$

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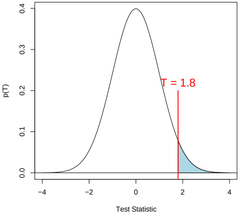

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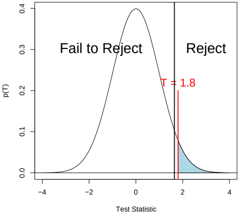

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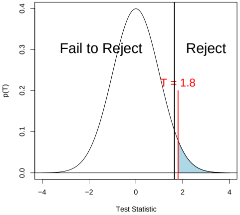

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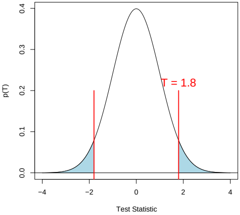

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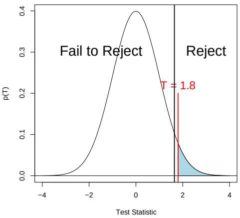

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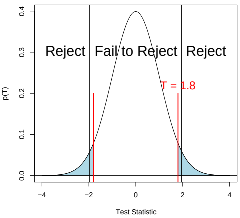

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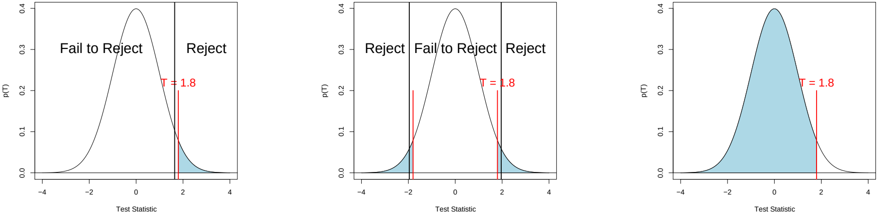

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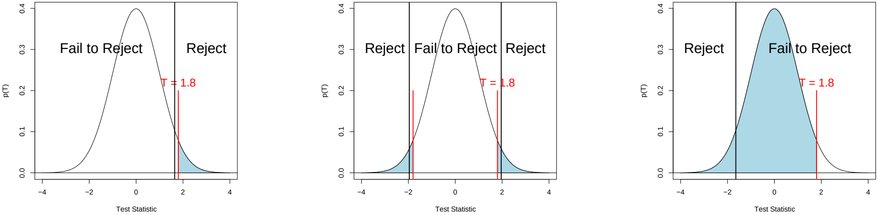

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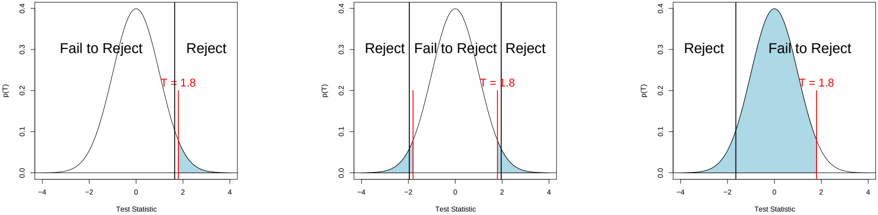


If $p < \alpha$, then the test statistic falls in the rejection region for the α-level test.

Example 1

Recall the drug testing example, where $H_0 : \mu_0 \leq 0$ and $H_a : \mu_0 > 0$:

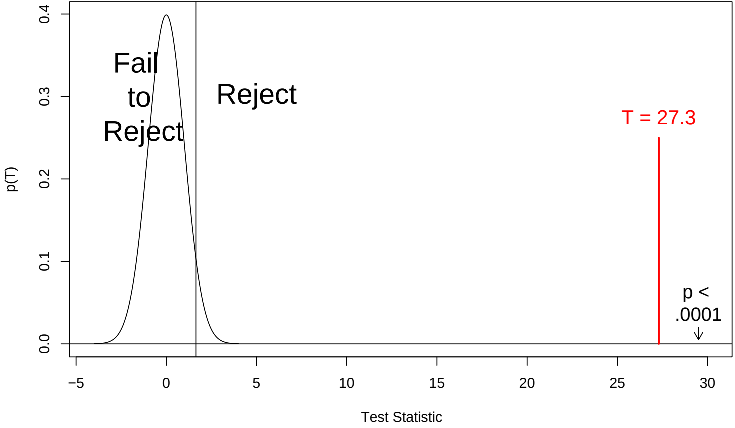
- $\overline{x} = 21.0$
- $s = 14.3$
- $n = 345$

Therefore,

$$T = \frac{21.0 - 0}{\frac{14.3}{\sqrt{345}}} = 27.3$$

What is the probability of observing a test statistic greater than 27.3 if the null is true?

Example 1



p(T)
0.0
0.1
0.2
0.3
0.4
Fail
to
Reject
Reject
T = 27.3
p <
.0001
−5
0
5
10
15
20
25
30
Test Statistic

α Rejection Regions and $1 - \alpha$ CIs

Up to this point, we have defined rejection regions in terms of the test statistic.

In some cases, we can define an equivalent rejection region in terms of the parameter of interest.

For a two-sided, large-sample test, we reject if:

$$\frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}} > z_{\alpha/2} \text{ or } \frac{\overline{X} - \mu_0}{\frac{s}{\sqrt{n}}} < -z_{\alpha/2}$$

α Rejection Regions and $1 - \alpha$ CIs

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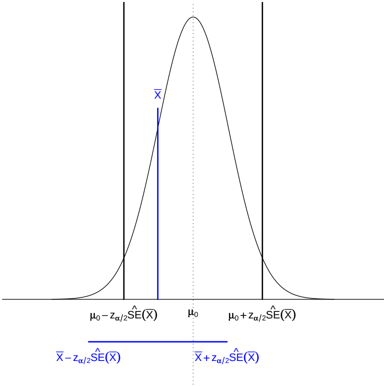
Therefore, we can use the $1 - \alpha$ CI to test the null hypothesis at the α level.



Another interpretation of CIs

The form of the "fail to reject" region of an α-level hypothesis test is:

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So the $1-\alpha$ CI is the set of null hypotheses μ_0 that would not be rejected at the α level.

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Goal: test a hypothesis about the value of a parameter.

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However, creating a decision rule which minimizes both types of errors at the same time is impossible. We therefore need to balance them.

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Now, suppose that we have a statistical model for the probability of convicting and acquitting, conditional on whether the defendant is actually guilty or innocent.

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Hypothesis testing follows an analogous logic, where we want to decide whether to reject (= convict) or fail to reject (= acquit) a null hypothesis (= defendant) using sample data.

Hypothesis Testing: Steps

		Null Hypothesis (H_0)	
		False	True
Decision	Reject	$1-\beta$	α
	Fail to Reject	β	$1-\alpha$

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4. Assuming H_0 is true, derive the null distribution of T (e.g. standard normal)

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5. Using the critical values from a statistical table, evaluate how unusual the observed value of T is under the null hypothesis:
   - If the probability of drawing a T at least as extreme as the observed T is less than α, we reject H_0.
     (e.g. there are too many testimonies against the defendant for her to be innocent, so reject the hypothesis that she was innocent.)
   - Otherwise, we fail to reject H_0.
     (e.g. there is not enough evidence against the defendant, so give her the benefit of the doubt.)

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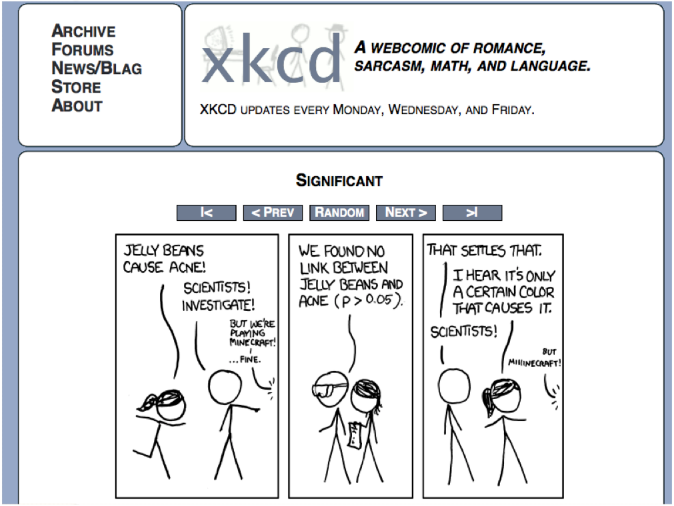
  - practical significance (e.g. a big effect)
  - statistical significance (i.e. we reject the null)
- In large samples even tiny effects will be significant, but the results may not be very important substantively. Always discuss both!

Star Chasing (aka there is an XKCD for everything)



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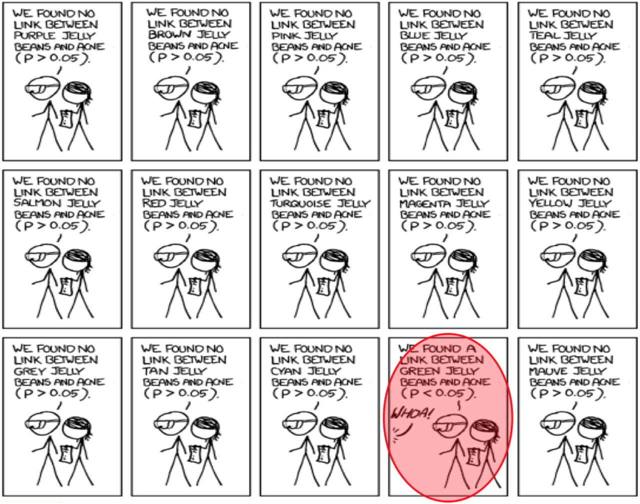

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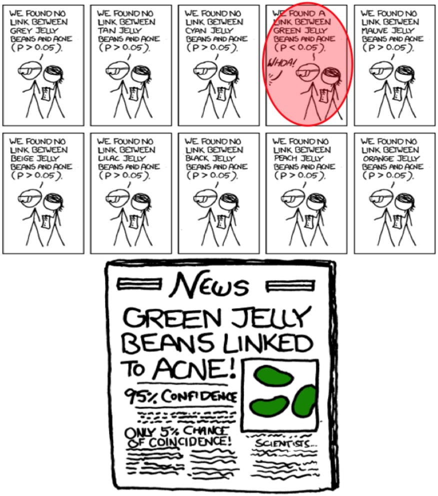

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- Illustration: randomly draw 21 variables, and run a regression of the first variable on the rest.
- By design, no effect of any variable on any other, but when we run the regression:

Multiple Test Example

```
##
## Coefficients:
##               Estimate Std. Error t value Pr(>|t|)
## (Intercept) -0.0280393  0.1138198  -0.246  0.80605
## X2          -0.1503904  0.1121808  -1.341  0.18389
## X3           0.0791578  0.0950278   0.833  0.40736
## X4          -0.0717419  0.1045788  -0.686  0.49472
## X5           0.1720783  0.1140017   1.509  0.13518
## X6           0.0808522  0.1083414   0.746  0.45772
## X7           0.1029129  0.1141562   0.902  0.37006
## X8          -0.3210531  0.1206727  -2.661  0.00945 **
## X9          -0.0531223  0.1079834  -0.492  0.62412
## X10          0.1801045  0.1264427   1.424  0.15827
## X11          0.1663864  0.1109471   1.500  0.13768
## X12          0.0080111  0.1037663   0.077  0.93866
## X13          0.0002117  0.1037845   0.002  0.99838
## X14         -0.0659690  0.1122145  -0.588  0.55829
## X15         -0.1296539  0.1115753  -1.162  0.24872
## X16         -0.0544456  0.1251395  -0.435  0.66469
## X17          0.0043351  0.1120122   0.039  0.96923
## X18         -0.0807963  0.1098525  -0.735  0.46421
## X19         -0.0858057  0.1185529  -0.724  0.47134
## X20         -0.1860057  0.1045602  -1.779  0.07910 .
## X21          0.0021111  0.1081179   0.020  0.98447
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 0.9992 on 79 degrees of freedom
## Multiple R-squared:  0.2009, Adjusted R-squared:  -0.00142
## F-statistic: 0.993 on 20 and 79 DF,  p-value: 0.4797
```

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- Notice that out of 20 variables, one of the variables is significant at the 0.05 level (in fact, at the 0.01 level).
- But this is exactly what we expect: $1/20 = 0.05$ of the tests are false positives at the 0.05 level
- Also note that $2/20 = 0.1$ are significant at the 0.1 level. Totally expected!

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- It remains a heated debate.

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- The level of a test determines how often the researcher is willing to reject a correct null hypothesis.
- Reporting p-values allows the researcher to separate the analysis from the decision.
- There is a close relationship between the results of an α level hypothesis test and the coverage of a $(1-\alpha)\%$ confidence interval.

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- From here on out, we'll be interested in the relationships between variables. How does one variable change as we change the values of another variable? This question will be the bread and butter of the class moving forward.

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  - Does parental incarceration affect intergenerational mobility for child?

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- Generally our goal is to understand how Y varies as a function of X:

$$Y = f(X) + \text{error}$$

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- Joint densities, covariance, and correlation were all ways to summarize the relationship between two variables.
- But these were population quantities and we only have samples, so we may want to estimate these quantities using their sample analogs (plug-in principle or analogy principle)

Scatterplots

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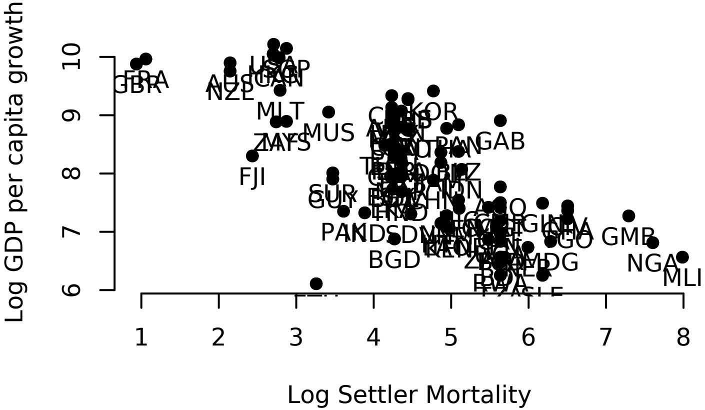

Data from Acemoglu, Johnson and Robinson

Non-linear relationship

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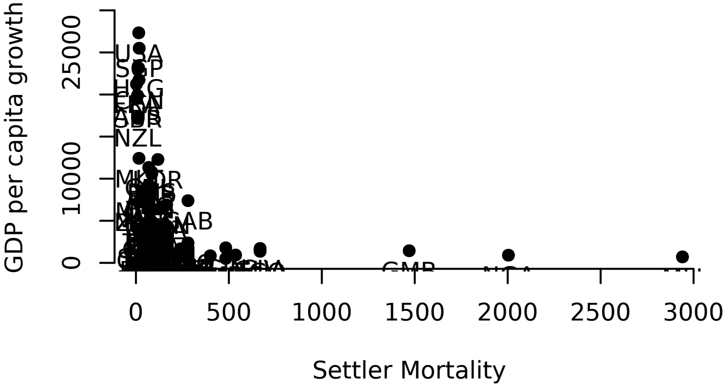

Sample Covariance

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The sample version of population covariance,
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Definition (Sample Covariance)

The **sample covariance** between Y_i and X_i is

$$S_{XY} = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X}_n)(Y_i - \overline{Y}_n)$$

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$$\hat{\rho} = r = \frac{S_{XY}}{S_X S_Y} = \frac{\sum_{i=1}^{n}(X_i - \overline{X}_n)(Y_i - \overline{Y}_n)}{\sqrt{\sum_{i=1}^{n}(X_i - \overline{X}_n)^2 \sum_{i=1}^{n}(Y_i - \overline{Y}_n)^2}}$$

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- Our key goal is to approximate the conditional expectation function $E[Y|X]$, which summarizes how the average of Y varies across all possible levels of X (also called the population regression function)
- Once we have estimated $E[Y|X]$, we can use it for prediction and/or causal inference, depending on what assumptions we are willing to make

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Definition (Conditional Expectation Function)

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is the function that gives the mean of Y at various values of x.

- Note that this is a function of the population distributions. We will want to produce estimates $\widehat{r}(x)$.

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- Note that these are just conditional expectations. Define Y to be the loan amount, $X = 1$ to indicate a man, and $X = 0$ to indicate a woman and then we have:

$$\mu_m = r(1) = E[Y|X = 1]$$
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- Notice here that since X can only take on two values, 0 and 1, then these two conditional means completely summarize the CEF.

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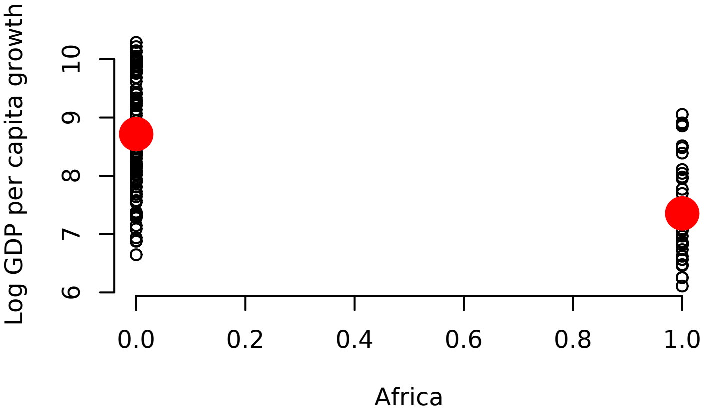
- The sum here $\sum_{i:X_i=1}$ is just summing only over the observations i such that have $X_i = 1$, meaning that i is a man.
- This is very straightforward: estimate the mean of Y conditional on X by just estimating the means within each group of X.

Binary covariate example CEF plot



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- For a given sample dataset, we obtain an estimate of $E[Y|X]$.
- We want to extend the regression idea to the case of multiple X variables, but we will start this week with the simple bivariate case where we have a single X

Fun With Salmon

Bennett, Baird, Miller and Wolford. (2009). "Neural correlates of interspecies perspective taking in the post-mortem Atlantic Salmon: an argument for multiple comparisons correction."

Methods

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(a.k.a. the greatest methods section of all time)

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  "One mature Atlantic Salmon (Salmo salar) participated in the fMRI study. The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning."

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  "One mature Atlantic Salmon (Salmo salar) participated in the fMRI study. The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning."
- Task
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- Subject
  "One mature Atlantic Salmon (Salmo salar) participated in the fMRI study. The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning."
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- Subject
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  "One mature Atlantic Salmon (Salmo salar) participated in the fMRI study. The salmon was approximately 18 inches long, weighed 3.8 lbs, and was not alive at the time of scanning."
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  "The task administered to the salmon involved completing an open-ended mentalizing task. The salmon was shown a series of photographs depicting human individuals in social situations with a specified emotional valence. The salmon was asked to determine what emotion the individual in the photo must have been experiencing."
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  "Stimuli were presented in a block design with each photo presented for 10 seconds followed by 12 seconds of rest.

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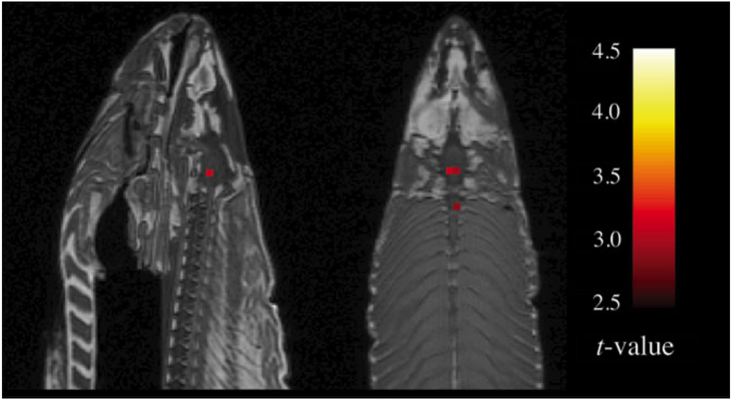
  "The task administered to the salmon involved completing an open-ended mentalizing task. The salmon was shown a series of photographs depicting human individuals in social situations with a specified emotional valence. The salmon was asked to determine what emotion the individual in the photo must have been experiencing."
- Design
  "Stimuli were presented in a block design with each photo presented for 10 seconds followed by 12 seconds of rest.A total of 15 photos were displayed. Total scan time was 5.5 minutes."

Results



"Several active voxels were discovered in a cluster located within the salmon's brain cavity. The size of this cluster was 81 mm^3 with a cluster-level significance of $p = .001$."

Where We've Been and Where We're Going...

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  - point estimates, confidence intervals

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Questions?

Nonparametric Regression with Discrete X

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- We use two variables:
  - Y: income
  - X: educational attainment
- Goal is to characterize the conditional expectation $E[Y|X]$, i.e. how average income varies with education level

Nonparametric Regression with Discrete X

Nonparametric Regression with Discrete X

`educ`: Respondent's education:

- 1. 8 grades or less and no diploma or
- 2. 9-11 grades
- 3. High school diploma or equivalency test
- 4. More than 12 years of schooling, no higher degree
- 5. Junior or community college level degree (AA degrees)
- 6. BA level degrees; 17+ years, no postgraduate degree
- 7. Advanced degree

Nonparametric Regression with Discrete X

`income`: Respondent's family income:

- 1. None or less than $2,999
- 2. $3,000-$4,999
- 3. $5,000-$6,999
- 4. $7,000-$8,999
- 5. $9,000-$9,999
- 6. $10,000-$10,999

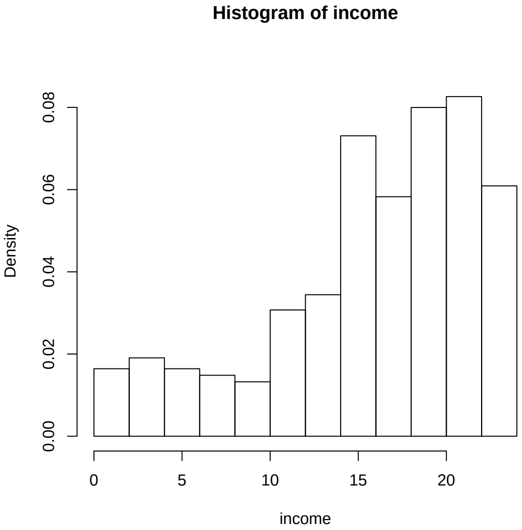
  ⋮

- 17. $35,000-$39,999
- 18. $40,000-$44,999

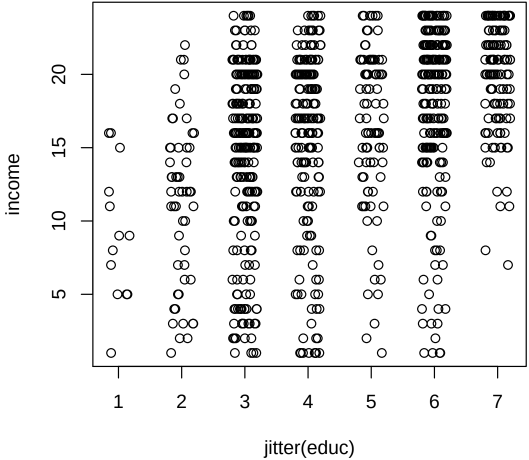
  ⋮

- 23. $90,000-$104,999
- 24. $105,000 and over

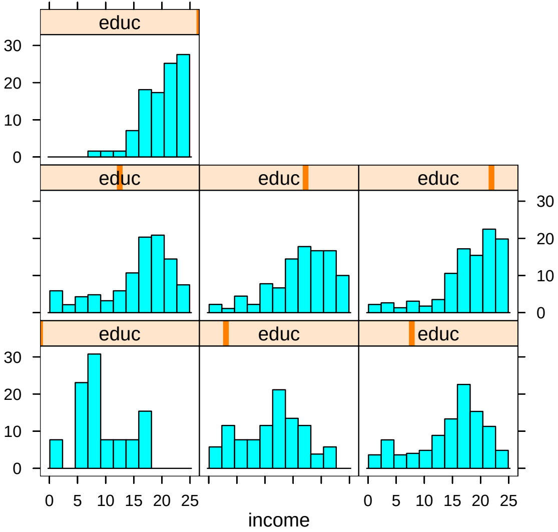
Marginal Distribution of Y



Income and Education



Distribution of income given education $p(y|x)$



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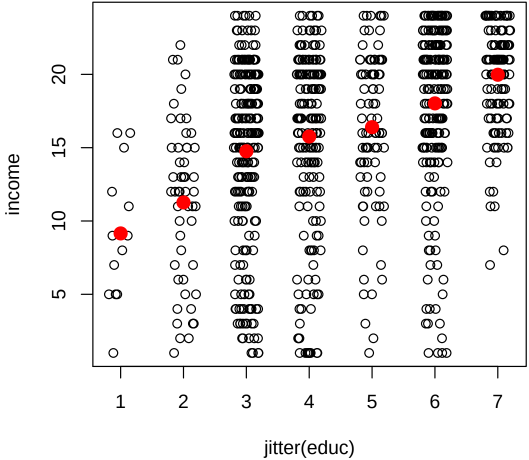
- Hard to decode what is going on in the histograms
- Let's try to find a more parsimonious summary measure: $E[Y|X]$
- Here our X variable education has a small number of levels (7) and there are a reasonable number of observations in each level
- In situations like this we can estimate $E[Y|X = x]$ as the sample mean of Y at each level of $x \in X$ (just like the binary case)

Nonparametric Regression with Discrete X



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- But what do we do when X is continuous and has many values?

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- We're interested in the relationship between:
  - Y: intensity of the peasant rebellion
  - X: inequality of land tenure

Nonparametric Regression with Continuous X

Consider the `Chirot` data:

- Chirot, D. and C. Ragin (1975). The market, tradition and peasant rebellion: The case of Romania. American Sociological Review 40, 428-444

- Peasant Rebellions in Romanian counties in 1907
- Peasants made up 80% of the population
- About 60 % of them owned no land which was mostly concentrated among large landowners

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- Around 11,000 peasants were killed by Romanian military

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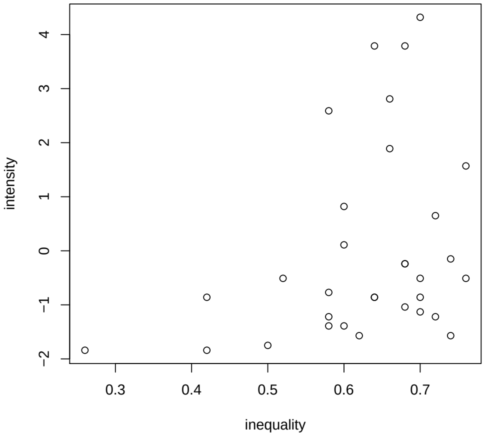

Uniform Kernel Regression: Simple Local Averages

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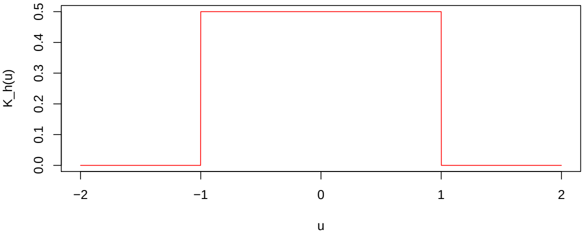

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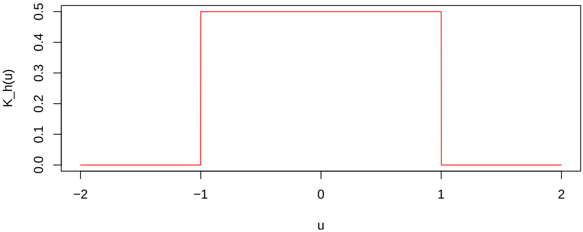

- This gives the uniform kernel regression:

$$\widehat{E}[Y|X = x_0] = \frac{\sum_{i=1}^{N} K_h((X_i - x_0)/h) Y_i}{\sum_{i=1}^{N} K_h((X_i - x_0)/h)} \text{ where } K_h(u) = \frac{1}{2}\mathbf{1}_{\{|u| \leq 1\}}$$

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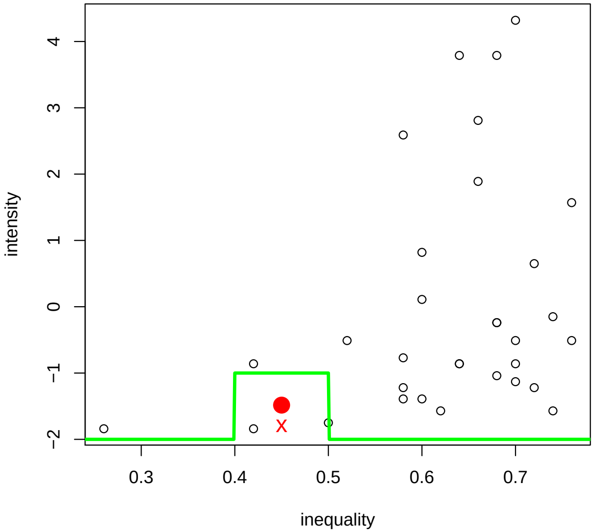

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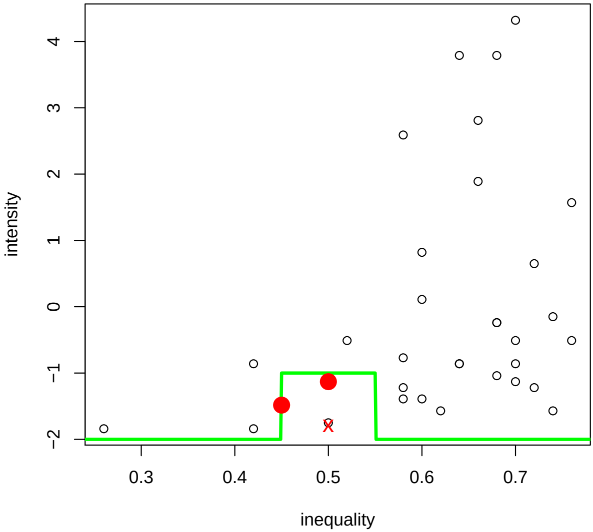

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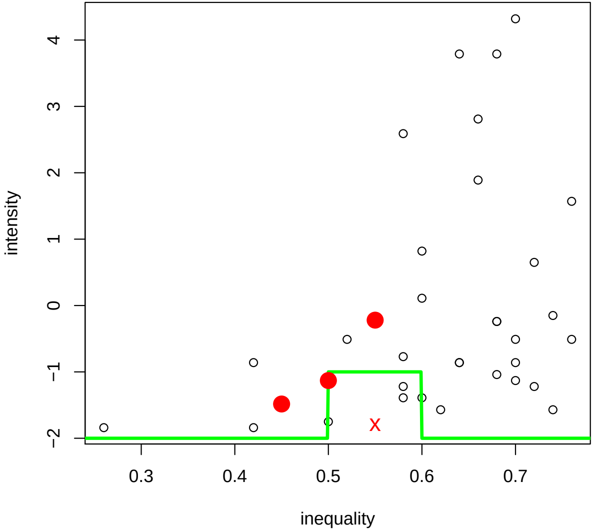

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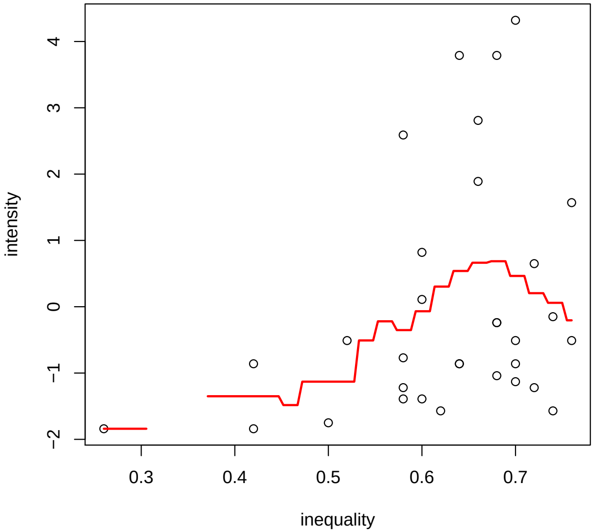
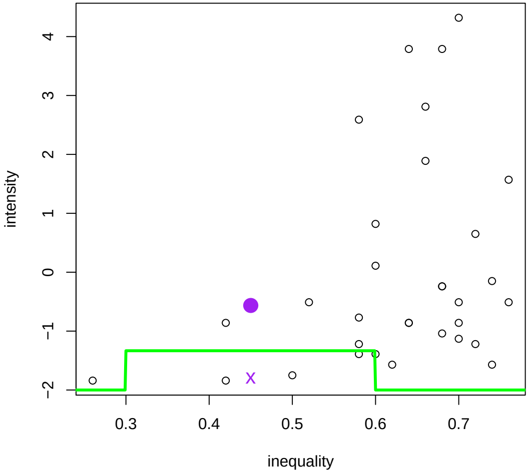


Changing the Bandwidth



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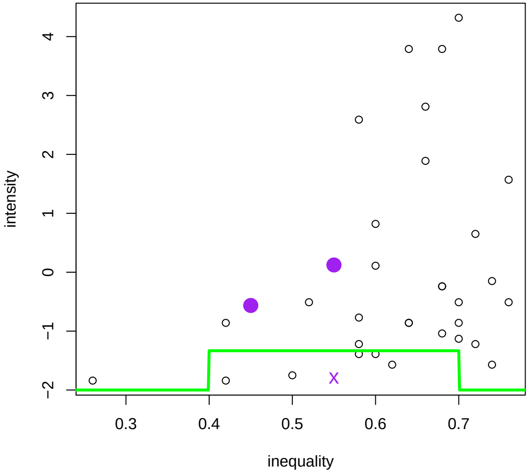

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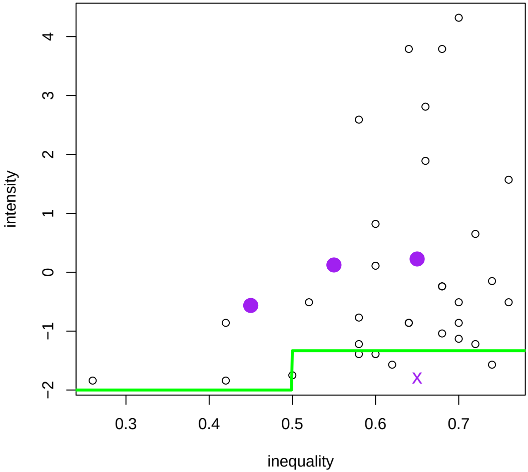

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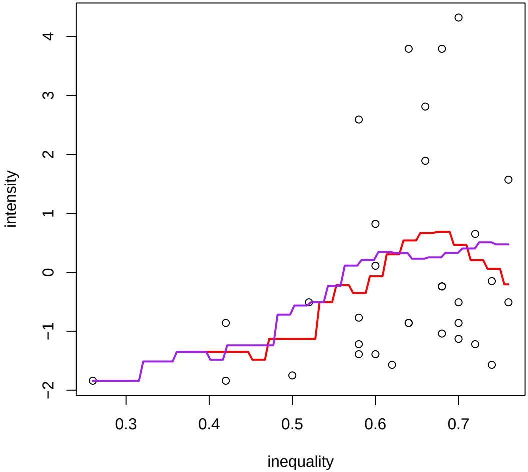

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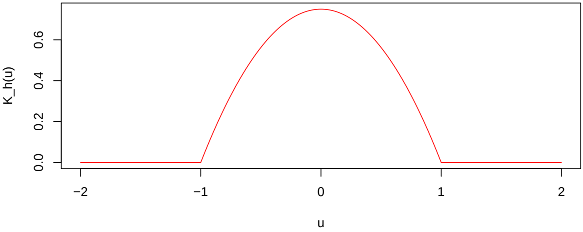

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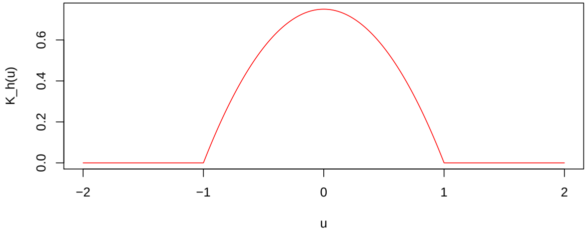

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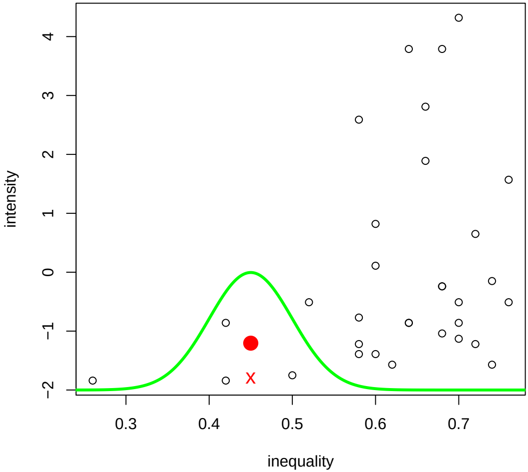

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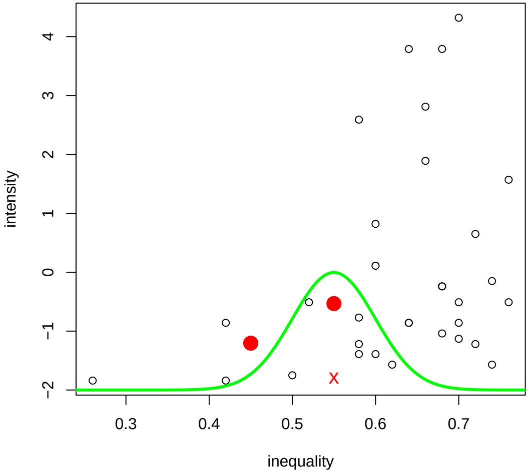

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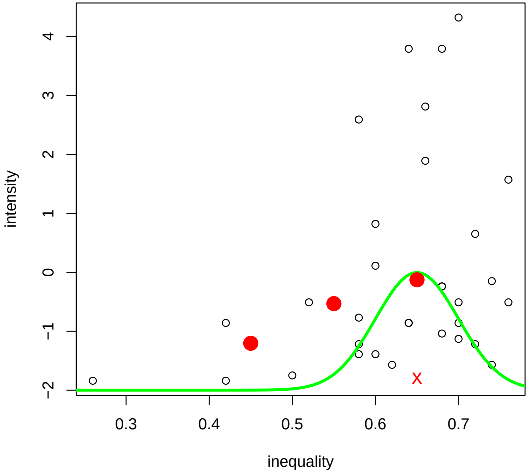

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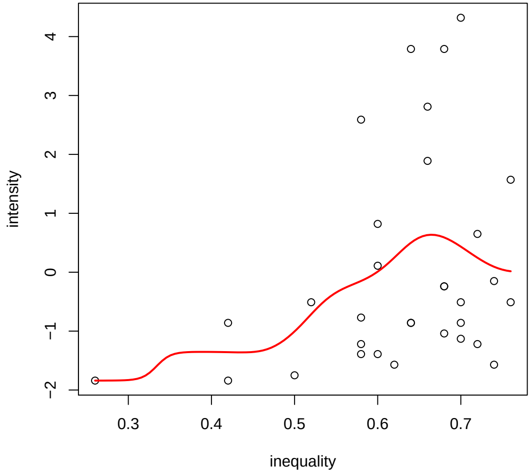

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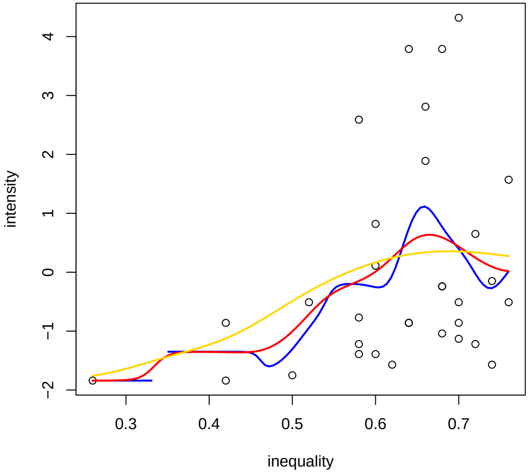

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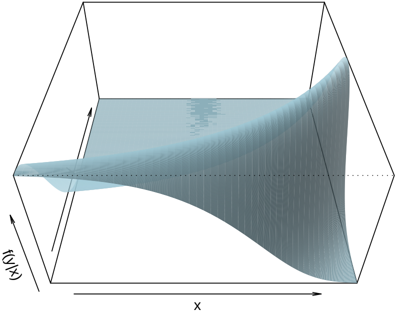

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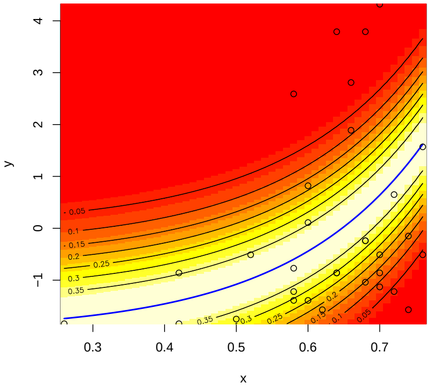

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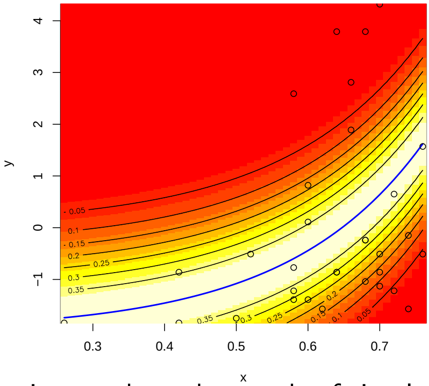
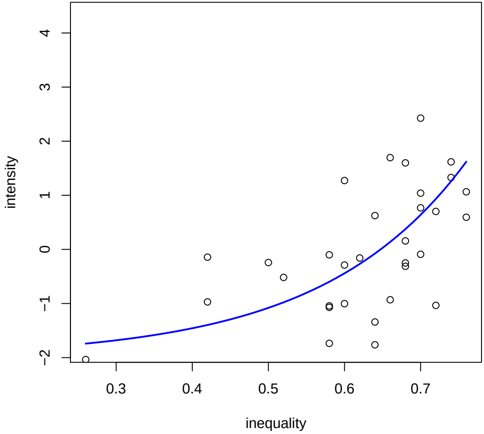


- From this distribution we draw thousands of simulated data sets.

An Example of Simulated Data Set



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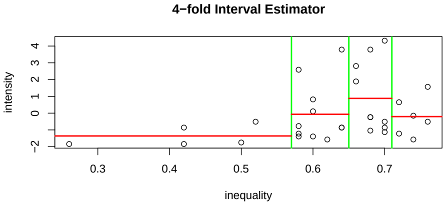

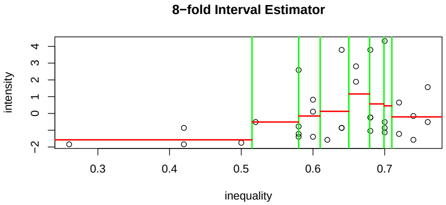

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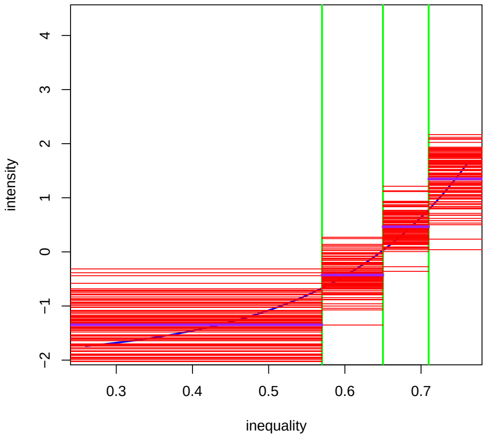
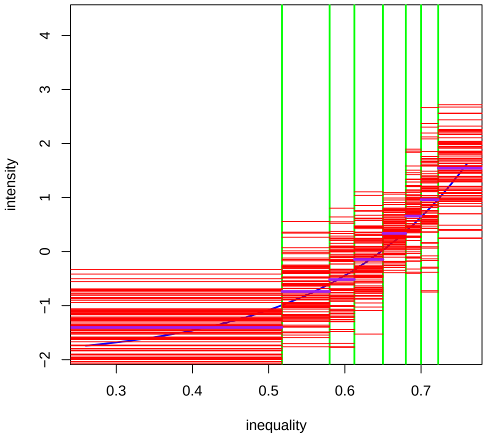


Simulated Distribution of Estimates: 8 Intervals



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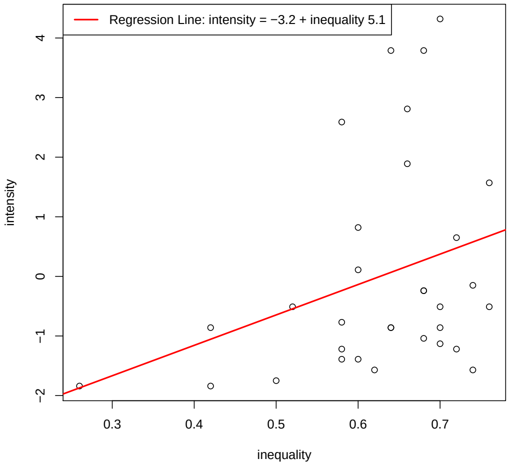

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Linear regression always returns a line regardless of the data.

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- Thus, we can read off the difference in means between two groups as the slope coefficient on a linear regression

Linear CEF with a binary covariate

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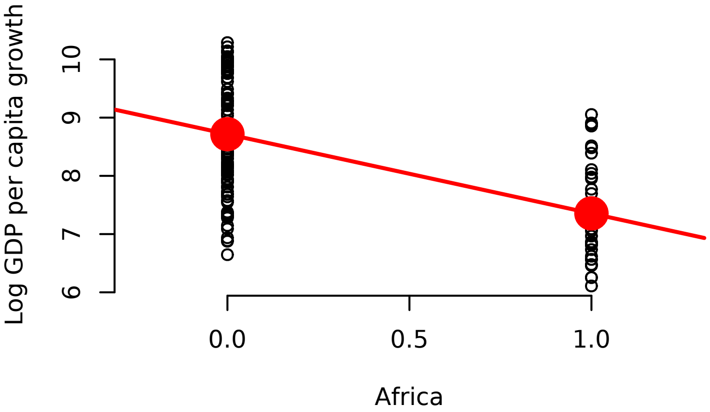

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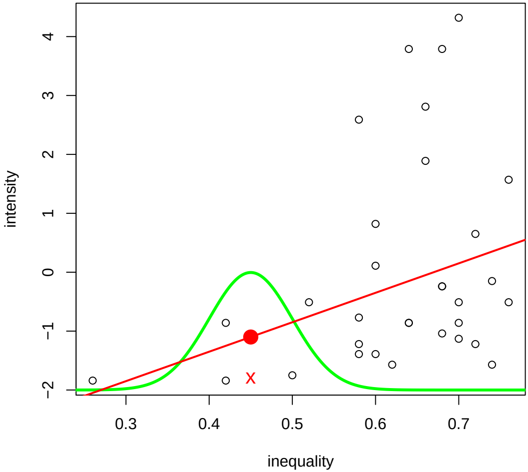
Weighted Local Linear Regressions



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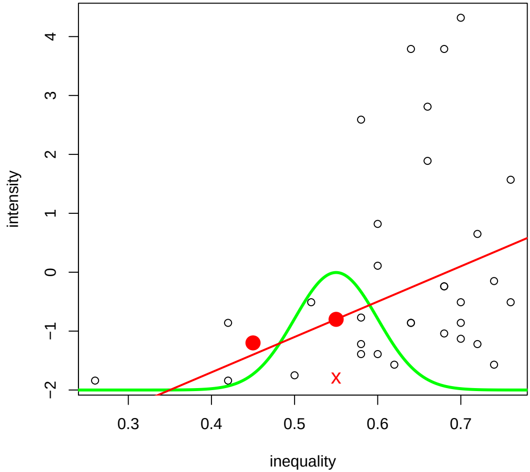

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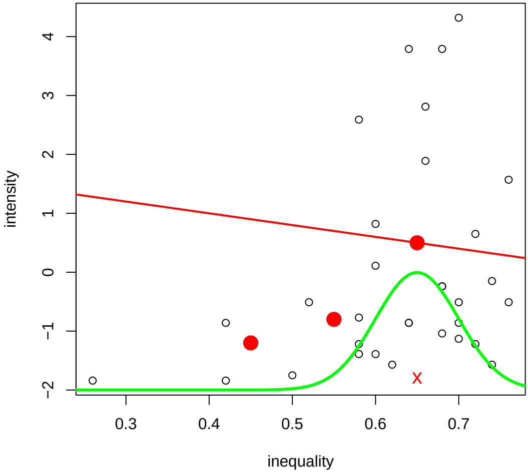

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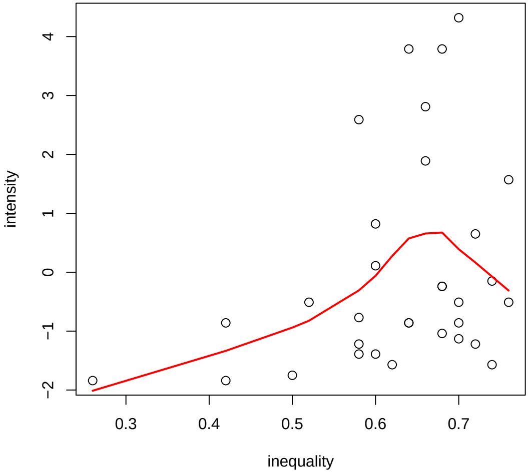

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- Need to estimate them in our samples! But how?

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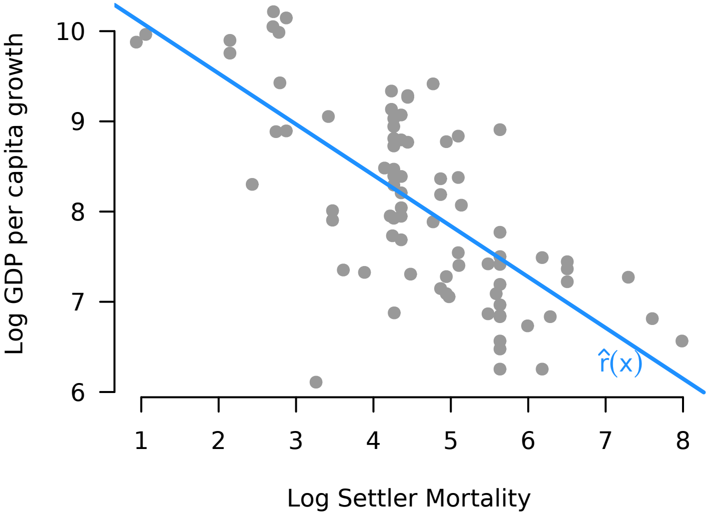
- Now, suppose we have some estimates of the slope, $\hat{\beta}_1$, and the intercept, $\hat{\beta}_0$. Then the fitted or sample regression line is

$$\widehat{r}(x) = \widehat{\beta}_0 + \widehat{\beta}_1 x$$

Fitted linear CEF/regression function



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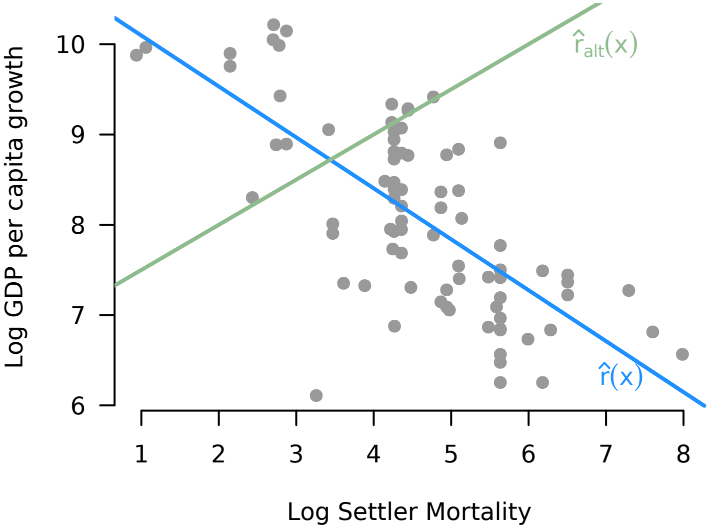

Fitted values and residuals

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Definition (Fitted Value)

A **fitted value** or **predicted value** is the estimated conditional mean of Y_i for a particular observation with independent variable X_i:

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Definition (Residual)

The **residual** is the difference between the actual value of Y_i and the predicted value, $\widehat{Y}_i$:

$$\widehat{u}_i = Y_i - \widehat{Y}_i = Y_i - \widehat{\beta}_0 - \widehat{\beta}_1 X_i$$

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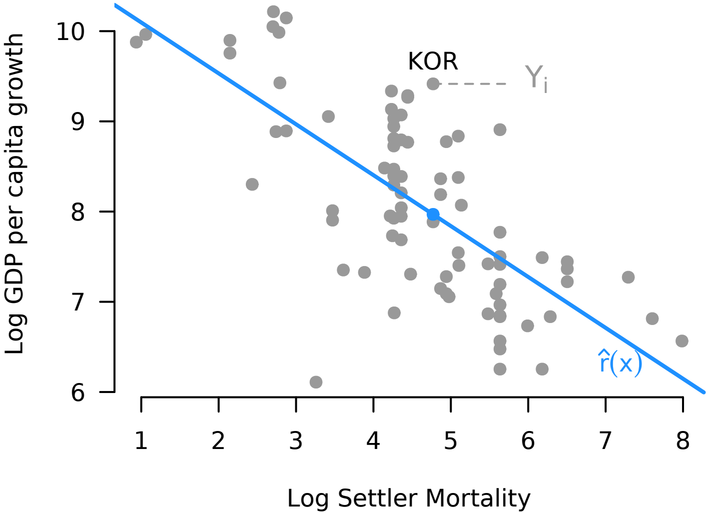

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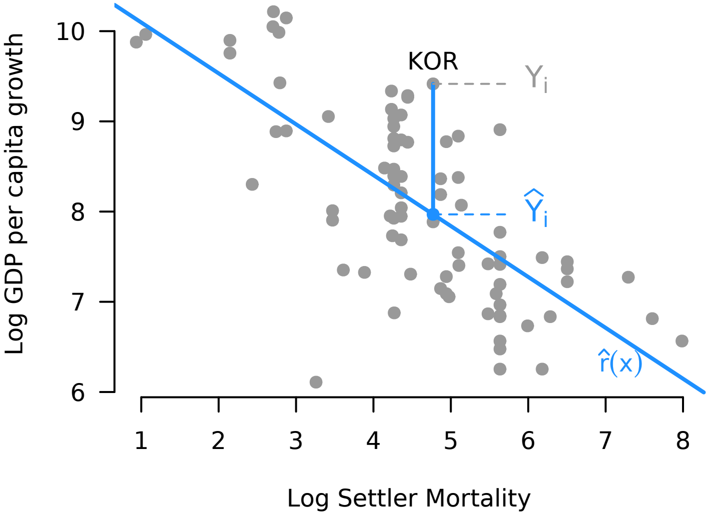

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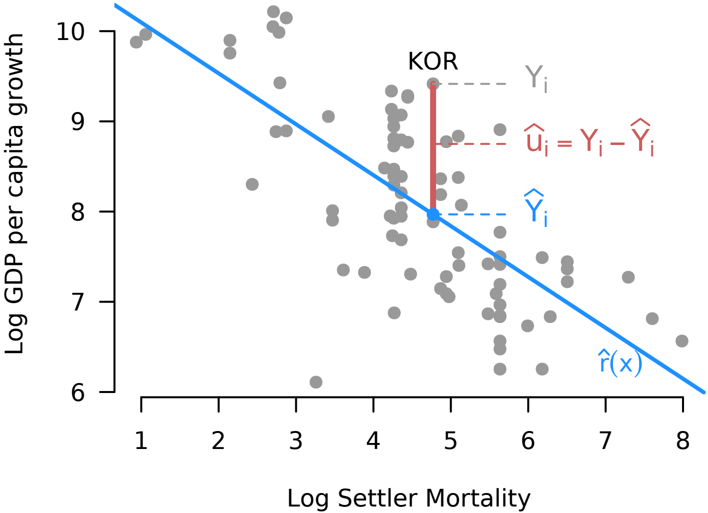
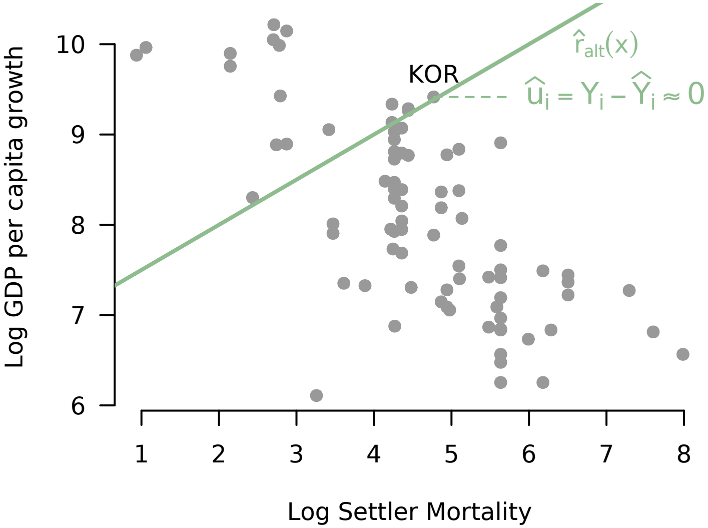


Why not this line?



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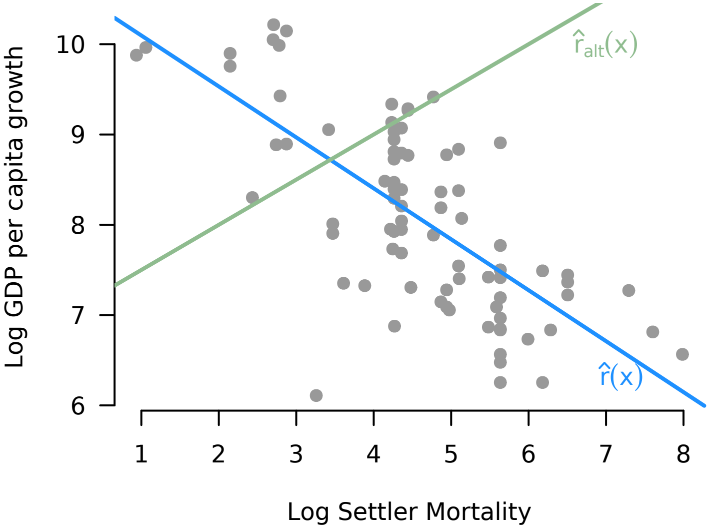
- The smaller the magnitude of the residuals, the better we are doing at predicting Y
- Choose the line that minimizes the residuals

Which is better at minimizing residuals?



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- Sometimes called **ordinary least squares** (OLS)

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- Least squares is optimal in a certain sense that we'll see in the coming weeks

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  - We can make the model more flexible, even in a linear framework (e.g. we can add polynomials, use log transformations, etc.)

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- While the line is defined over all regions of the data we may be concerned about:
  - interpolation
  - extrapolation
  - predicting in ranges of X with sparse data

Which Predictions Do You Trust?

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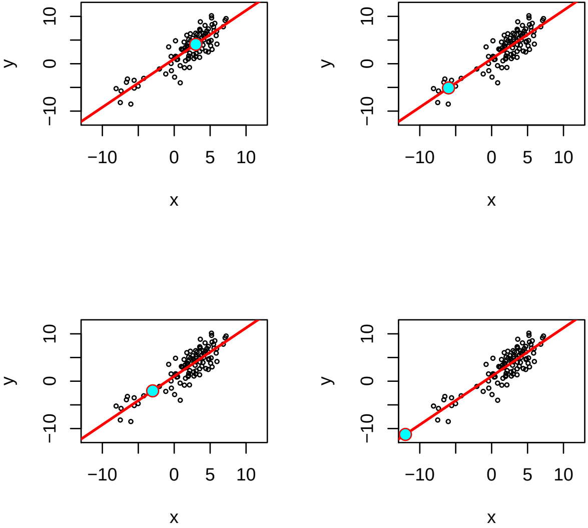

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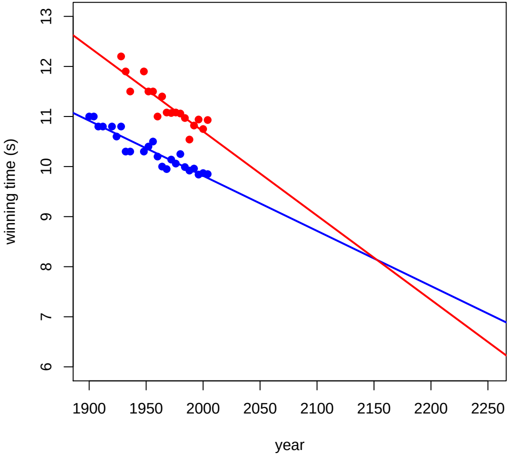
How do the authors make this conclusion?

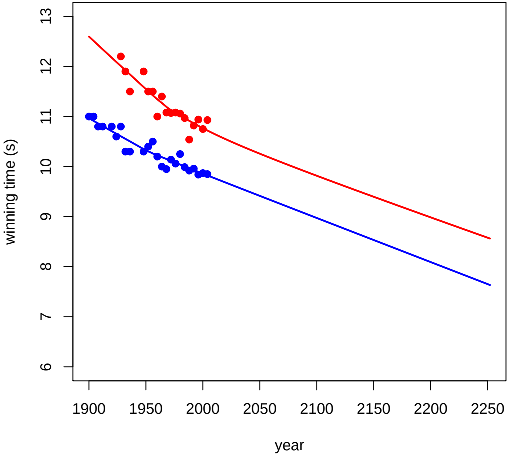
Using data from 1900 to 2004, they fit linear regression models of the winning 100 meter time on year for both men and women. They then use the estimates from these models to extrapolate 152 years into the future.

Tatem et al. Extrapolation



Tatem et al.'s predictions. Men's times are in blue, women's times are in red.

Alternate Models Fit Well, Yield Different Predictions



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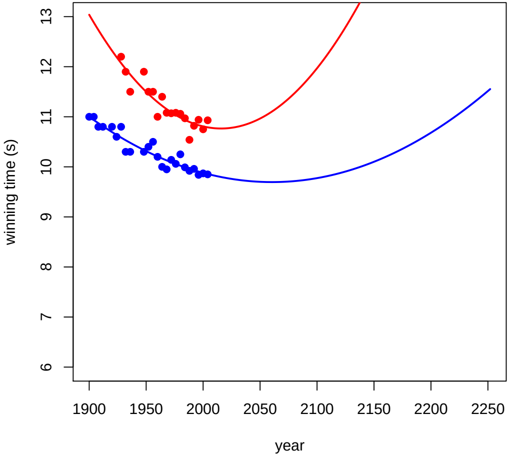

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- Next semester we will talk about how this problem gets much harder in high dimensions

A More Subtle Example

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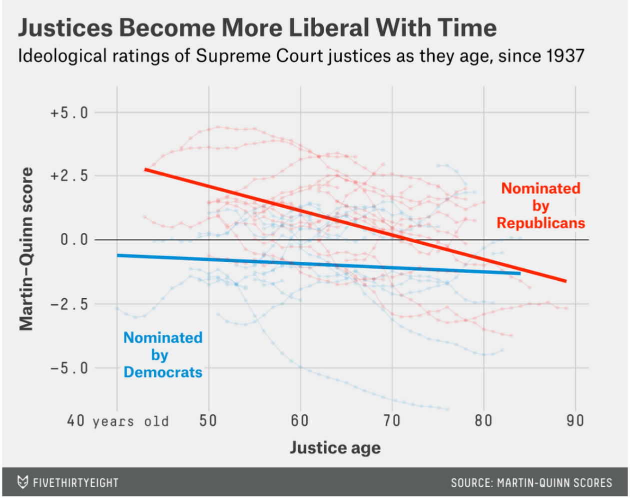

A More Subtle Example

Nate Silver @NateSilver538 · Oct 5
So, basically, John Roberts is going to be Ruth Bader Ginsburg by 2036.
53eig.ht/1Gsl2u6

Supreme Court Justices Get More Liberal As They Get Older
The Supreme Court justices are back from vacation. They've picked up their robes from the cleaners — Alito's had a pesky mustard stain — and are ...

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- Summary: ‘prediction is hard, especially about the future’

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- We will return to this later in the course

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- We will return to this later in the course
- For now, it is safest to treat β as a purely descriptive/predictive quantity

Summary of Today

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- Reading:
  - Fox Chapter 6.1
  - Optional: Imai 4.2

Fun with Linearity

Psychonomic Bulletin & Review
2007, 14 (2), 288-294

Iterated learning: Intergenerational knowledge transmission reveals inductive biases

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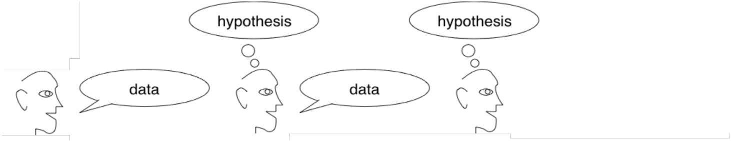
Cultural transmission of information plays a central role in shaping human knowledge. Some of the most complex knowledge that people acquire, such as languages or cultural norms, can only be learned from other people, who themselves learned from previous generations. The prevalence of this process of "iterated learning" as a mode of cultural transmission raises the question of how it affects the information being transmitted. Analyses of iterated learning utilizing the assumption that the learners are Bayesian agents predict that this process should converge to an equilibrium that reflects the inductive biases of the learners. An experiment in iterated function learning with human participants confirmed this prediction, providing insight into the consequences of intergenerational knowledge transmission and a method for discovering the inductive biases that guide human inferences.

Images on following slides courtesy of Tom Griffiths

Fun with Linearity



The Design

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- Each learner sees a set of (x, y) pairs

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- Makes predictions of y for new x values

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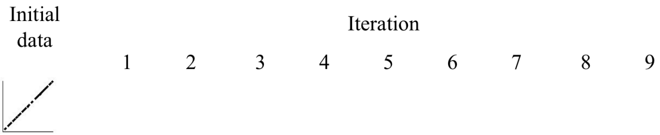
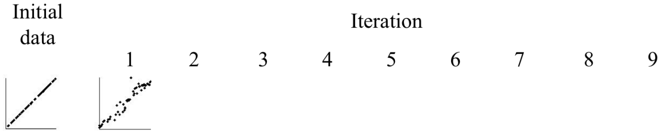
- Each learner sees a set of (x, y) pairs
- Makes predictions of y for new x values
- Predictions are data for the next learner

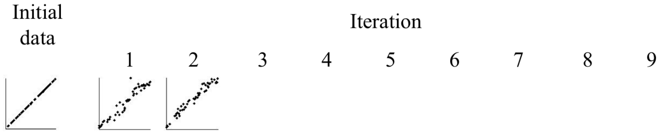
Results



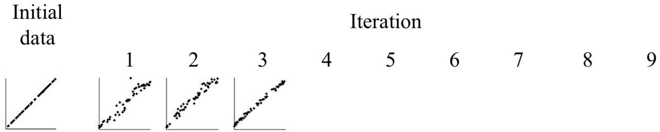
Results



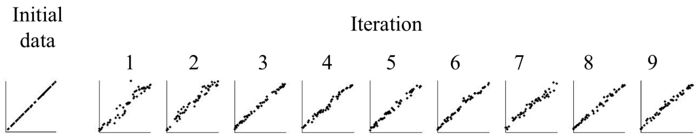
Results



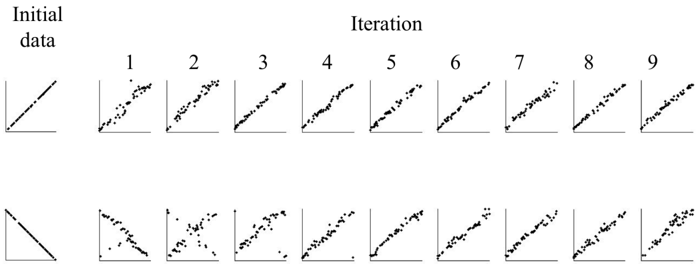
Results



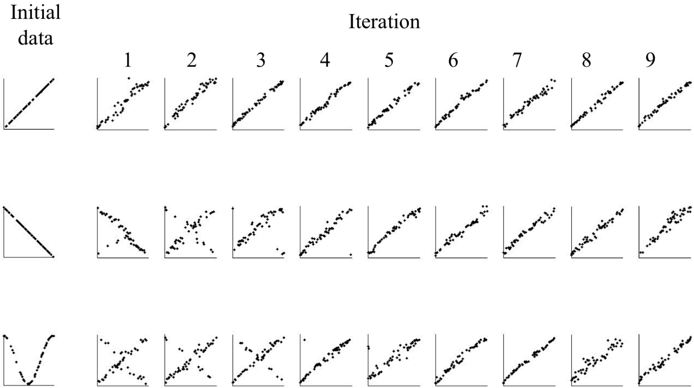
Results



Results



Results



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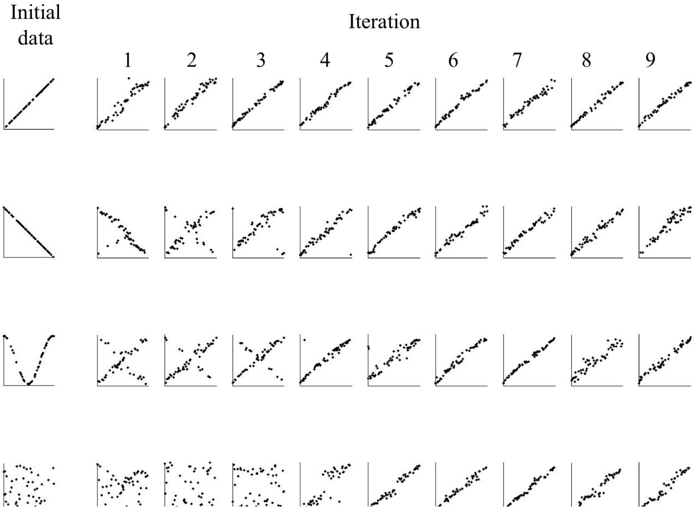

References

Chirot, D. and C. Ragin (1975). The market, tradition and peasant rebellion: The case of Romania. American Sociological Review 40, 428-444

Acemoglu, Daron, Simon Johnson, and James A. Robinson. "The colonial origins of comparative development: An empirical investigation." 2000.